Kangweon-Kyungki Math. Jour. **9** (2001), No. 2, pp. 83–98

ENTROPY NUMBERS OF (R, W)-NUCLEAR OPERATORS ACTING BETWEEN BANACH SPACES OF CERTAIN WEAK TYPE

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Abstract. We characterize (r, w)-nuclear operators acting from a Banach space whose dual has weak type q into a Banach space of weak type p by the asymptotic behaviour of their entropy numbers.

1. Introduction

The theory of the so-called entropy numbers was introduced by A. Pietsch [12]. Afterwards a lot of results concerning the behaviour of entropy numbers of certain classes of operators were established (cf. [4], [6], [10]). We only remind of diagonal operators acting between Lorentz sequence spaces as well as embedding maps between Besov function spaces.

B. Carl [4] characterized diagonal operators acting between Lorentz sequence spaces by their entropy numbers.

In [5] B. Carl considered operators $S : \ell_q \to X$ admitting a factorization through ℓ_1, $S : \ell_q \xrightarrow{D} \ell_1 \xrightarrow{B} X$, where D is a diagonal operator generated by a sequence belonging to a Lorentz sequence space and B is an arbitrary bounded operator. He characterized these operators in terms of their entropy numbers under the hypothesis that X is a Banach space of type p. By using this result, he [6] showed that the sequence of entropy numbers of r-nuclear operators acting from a Banach space L_p into a Banach space L_q belongs to the Lorentz sequence space.

In [10] T. Kühn dealt with operators $T : X \to \ell_q$ factorizing through ℓ_∞, $T : X \xrightarrow{B} \ell_\infty \xrightarrow{D} \ell_q$, where B is an arbitrary bounded operator

Received November 7, 2000.
2000 Mathematics Subject Classification: 47B06.
Key words and phrases: entropy numbers, (r, w)-nuclear operators.
This research is supported by the Dongguk University research fund

and D is a diagonal operator generated by a sequence belonging to a Lorentz sequence space. He estimated the asymptotic behaviour of the entropy numbers of these operators under the assumption that the dual of a Banach space X has type p. And then he applied this result to r-nuclear operators acting from a Banach space whose dual has type q into a Banach space of type p.

In this paper we determine the "degree of compactness" of (r, w)-nuclear operators acting from a Banach space whose dual has weak type q into a Banach space of weak type p by means of entropy numbers. Here, we present Defant and Junge's approach [7].

2. Definitions and Notation

We present some of the definitions and notation to be used. Throughout this paper X and Y denote Banach spaces.

Let (A_0, A_1) be a couple of quasi-Banach spaces. We consider the functional $K(t, a, A_0, A_1) = K(t, a) = \inf\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} : a_0 \in A_0, a_1 \in A_1, a = a_0 + a_1\}$ on $A_0 + A_1$. If $0 < \theta < 1$ and $0 < q \leq \infty$ then the real interpolation space $(A_0, A_1)_{\theta,q}$ consists of all elements $a \in A_0 + A_1$ which have a finite quasi-norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \|a\|_{\theta,q} = \begin{cases} \left(\int_0^\infty [t^{-\theta}K(t,a)]^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup_t [t^{-\theta}K(t,a)] & \text{if } q = \infty. \end{cases}$$

Notation. (1) The dual of a Banach space X is denoted by X^*.

(2) For $1 < p < \infty$, the conjugate of p is denoted by p', i.e., $1/p + 1/p' = 1$.

(3) The closed unit ball of a Banach space X is denoted by B_X.

(4) $\mathcal{B}(X, Y)$ denotes the set of all bounded linear operators from X into Y.

(5) $\mathcal{F}(X, Y)$ denotes the set of all finite rank operators from X into Y.

(6) The dual operator of an operator T is denoted by T^*.

(7) $\mathrm{Vol}(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^n$.

For every operator $T \in \mathcal{B}(X, Y)$ the n-th outer entropy number $e_n(T)$ is defined to be the infimum of all $\epsilon \geq 0$ such that there are elements $y_1, \cdots, y_q \in Y$ with $q \leq 2^{n-1}$ and $T(B_X) \subseteq \bigcup_{i=1}^{q}\{y_i + \epsilon B_Y\}$.

The n-th approximation number of $T \in \mathcal{B}(X, Y)$ is defined by

$$a_n(T) = \inf\{\|T - L\| : L \in \mathcal{F}(X, Y),\ \operatorname{rank}(L) < n\}.$$

The n-th Gel'fand number of $T \in \mathcal{B}(X, Y)$ is defined by

$$c_n(T) = \inf\{\|TJ_M^X\| : \operatorname{codim}(M) < n\},$$

where J_M^X denotes the canonical injection from the subspace M into X.

The n-th Kolmogorov number of $T \in \mathcal{B}(X, Y)$ is defined by

$$d_n(T) = \inf\{\|Q_N^Y T\| : \dim(N) < \infty\},$$

where Q_N^Y denotes the canonical surjection from Y onto the quotient space Y/N.

The n-th Weyl number of $T \in \mathcal{B}(X, Y)$ is defined by

$$x_n(T) = \sup\{a_n(TU) : U \in \mathcal{B}(\ell_2, X),\ \|U\| \le 1\}.$$

The n-th Grothendieck number of $T \in \mathcal{B}(X, Y)$ is defined by

$$\Gamma_n(T) = \sup\{\,|\det(\langle Tx_i, y_j^*\rangle)|^{1/n} : (x_k)_{k=1}^n \subset B_X,\ (y_k^*)_{k=1}^n \subset B_{Y^*}\}.$$

The n-th volume number of $T \in \mathcal{B}(X, Y)$ is defined by

$$v_n(T) = \sup\Big\{\Big(\tfrac{\operatorname{Vol}(T(B_M))}{\operatorname{Vol}(B_N)}\Big)^{1/n} : M \subset X,\ T(M) \subset N \subset Y,\ \dim M = \dim N = n\Big\}.$$

The n-th volume ratio number of $T \in \mathcal{B}(X, Y)$ is defined by

$$vr_n(T) = \sup\Big\{\Big(\tfrac{\operatorname{Vol}(Q_N^Y(T(B_X)))}{\operatorname{Vol}(B_{Y/N})}\Big)^{1/n} : N \subset Y,\ \operatorname{codim} N = n\Big\}.$$

For an operator $U \in \mathcal{B}(\ell_2^n, X)$, we define

$$\ell(U) = \Big(\int_{\mathbb{R}^n} \|Ux\|^2 \, d\gamma_n(x)\Big)^{1/2},$$

where γ_n is the canonical Gaussian probability measure on $\mathbb{R}^n$.

For an operator $V \in \mathcal{B}(X, \ell_2^n)$, we set

$$\ell^*(V) = \sup\{\,|\operatorname{tr}(VU)| : U \in \mathcal{B}(\ell_2^n, X),\ \ell(U) \le 1\}.$$

A Banach space X is called a weak type p space if there is a constant C such that for all n and all operators $V \in \mathcal{B}(X, \ell_2^n)$, we have $\sup_k k^{1/p'} a_k(V) \le C \cdot \ell^*(V)$. The smallest constant C for which this holds will be denoted by $wT_p(X)$.

A Banach space X is called K-convex if there is a constant C such that for every n and every operator $V \in \ell^*(X, \ell_2^n)$, we have $\ell(V^*) \le C \cdot \ell^*(V)$. In this case, we define the K-convexity constant as $K(X) = \inf C$, where the infimum is taken over all constants C satisfying the above inequality.

If $x = (\xi_i)$ is a bounded sequence then we put $s_n(x) = \inf\{\sigma \ge 0 : \operatorname{card}(i : |\xi_i| \ge \sigma) < n\}$. $\big(s_n(x)\big)$ is called the non-increasing rearrangement of x. Let $0 < r < \infty$ and $0 < w \le \infty$. Then the Lorentz sequence space $\ell_{r,w}$ consists of all sequences $x = (\xi_i)$ having a finite quasi-norm

$$\|x\|_{r,w} = \begin{cases} \left(\sum_{n=1}^{\infty} [n^{1/r-1/w}\, s_n(x)]^w\right)^{1/w} & \text{if } 0 < w < \infty, \\ \sup_n [n^{1/r} s_n(x)] & \text{if } w = \infty. \end{cases}$$

For $s \in \{e, a, c, d, x, v, vr\}$, an operator $T \in \mathcal{B}(X, Y)$ is said to be of s-type $\ell_{r,w}$ if $\big(s_n(T)\big) \in \ell_{r,w}$. The set of these operators is denoted by $\mathcal{L}^{(s)}_{r,w}(X, Y)$. For $T \in \mathcal{L}^{(s)}_{r,w}(X, Y)$, we define $\|T \,|\, \mathcal{L}^{(s)}_{r,w}\| = \|(s_n(T))\|_{r,w}$ [13].

Let $0 < r < 1$ and $0 < w \leq \infty$. An operator $T \in \mathcal{B}(X, Y)$ is said to be (r, w)-nuclear if it can be written in the form $T = \sum_{i=1}^{\infty} \tau_i x_i^* \bigotimes y_i$ with (x_i^*) in B_{X^*}, (y_i) in B_Y and $(\tau_i) \in \ell_{r,w}$. The set of these operators is denoted by $T \in \mathcal{N}_{r,w}(X, Y)$. For $T \in \mathcal{N}_{r,w}(X, Y)$, we define a quasi-norm

$$\nu_{r,w}(T) = \begin{cases} \inf\left(\sum_{n=1}^{\infty} [n^{1/r-1/w} \tau_n]^w\right)^{1/w} & \text{if } 0 < w < \infty, \\ \inf\left(\sup_n [n^{1/r} \tau_n]\right) & \text{if } w = \infty, \end{cases}$$

where the infimum is taken over all (r, w)-nuclear representations such that $\tau_1 \geq \tau_2 \geq \cdots \geq 0$.

3. Results

By using Carl's proof [5] and the generalized Carl-Maurey inequality, we estimate the entropy quasi-norm of operators acting from ℓ_1^m into a Banach space of weak type p.

LEMMA 1. *Let X be of weak type p, $1 \leq p < 2$, and $S \in \mathcal{B}(\ell_1^m, X)$. Then there exists a constant $C > 0$ such that* $\sup_{1 \leq k < \infty} k^{1/s} e_k(S) \leq C\, \|S\|\, m^{1/s-1/p'}$ *for $s < p'$ and $m = 1, 2, \cdots$*

Proof. We invoke Carl-Maurey inequality [8] to infer that there exists a constant $C \geq 0$ such that

$$\begin{aligned} \sup_{1 \leq k \leq m} k^{1/s} e_k(S) &\leq C\, \|S\| \sup_{1 \leq k \leq m} k^{1/s-1/p'} [1 + \ln(\frac{m}{k})]^{1/p'} \\ &\leq C_0\, \|S\|\, m^{1/s-1/p'} \qquad \text{for } \ s < p'. \end{aligned}$$

Now we estimate $\sup_{k>m} k^{1/s}e_k(S)$. Let I_m denote the identity operator on ℓ_1^m. Using the multiplicativity of the entropy numbers we obtain that

$$\sup_{k>m} k^{1/s}e_k(S) = \sup_{k\ge 1} (m+k)^{1/s}\, e_{m+k}(S)$$
$$\le \sup_{k\ge 1} (m+k)^{1/s}\, e_m(S)\, e_k(I_m) \le e_m(S) \sup_{k\ge 1} 2^{1/s}(m^{1/s}+k^{1/s}) e_k(I_m)$$
$$\le 2^{1/s}m^{1/s}e_m(S) + 2^{1/s}e_m(S)\sup_{k\ge 1} k^{1/s}e_k(I_m).$$

Applying proposition 12.1.13 of [12] we derive that

$$\sup_{k\ge 1} k^{1/s}e_k(I_m) \le \Big(\sum_{k=1}^{\infty} e_k^s(I_m)\Big)^{1/s} \le 4\Big(\sum_{k=1}^{\infty}(2^{-(k-1)/2m})^s\Big)^{1/s}$$
$$\le 4\,\frac{1}{(1-2^{-s/2m})^{1/s}} \le 4\,\frac{2^{1/2m}}{(2^{s/2m}-1)^{1/s}}$$
$$\le 4\,\frac{2^{1/2m}}{(s/2m)^{1/s}\,(\ln 2)^{1/s}} \le 8\,\frac{2^{1/s}m^{1/s}}{(s\ln 2)^{1/s}}.$$

It takes another appeal to Carl-Maurey inequality [8] to yield that

$$\sup_{k>m} k^{1/s}e_k(S) \le e_m(S)\, m^{1/s}\, 2^{1/s}\, \Big[1+\frac{8\cdot 2^{1/s}}{(s\ln 2)^{1/s}}\Big]$$
$$\le C_1\,\|S\|\, m^{1/s-1/p'}\, 2^{1/s}\,\Big[1+\frac{8\cdot 2^{1/s}}{(s\ln 2)^{1/s}}\Big] \le C_2\,\|S\|\, m^{1/s-1/p'}.$$

Combining this with the above estimate we see that

$$\sup_{1\le k<\infty} k^{1/s}e_k(S) \le \sup_{1\le k\le m} k^{1/s}e_k(S) + \sup_{k>m} k^{1/s}e_k(S)$$
$$\le C_3\,\|S\|\, m^{1/s-1/p'} \qquad \text{for } s<p'.$$

□

Applying this lemma and Carl's proof [4], [5] we intend to characterize operators of the form SD_σ, where $D_\sigma : \ell_q \to \ell_1$ is a diagonal operator generated by a sequence belonging to a Lorentz sequence space and $S : \ell_1 \to X$ is an arbitrary operator with the image in a Banach space of weak type p, in terms of entropy numbers.

Proposition 1. *Let X be of weak type p, $1 \le p < 2$, $S \in \mathcal{B}(\ell_1, X)$ and let $D_\sigma \in \mathcal{B}(\ell_1, \ell_1)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r,t}$, where $0 < r < \infty$ and $0 < t \le \infty$. Then $SD_\sigma \in \mathcal{L}^{(e)}_{s,t}(\ell_1, X)$ for $1/s = 1/r + 1/p'$.*

Proof. First we shall show that if $\sigma \in \ell_{r,\infty}$ then $SD_\sigma \in \mathcal{L}^{(e)}_{s,\infty}(\ell_1, X)$, where $1/s = 1/r + 1/p'$ $(*)$.
There is no loss in assuming that $|\sigma_1| \ge |\sigma_2| \ge \cdots \ge 0$. We define canonical operators $J_k \in \mathcal{B}(\ell_1^{2^k}, \ell_1)$ and $Q_k \in \mathcal{B}(\ell_1, \ell_1^{2^k})$ by

$$J_k(\xi_1, \cdots, \xi_{2^k}) = (0, \cdots, 0, \xi_1, \cdots, \xi_{2^k}, 0, \cdots),$$
$$Q_k(\xi_1, \cdots, \xi_k, \cdots) = (\xi_{2^k}, \cdots, \xi_{2^{k+1}-1}),$$

respectively, for $k \ge 0$. Let $M_k \in \mathcal{B}(\ell_1^{2^k}, \ell_1^{2^k})$ be the operator defined by $M_k(\eta_1, \cdots, \eta_{2^k}) = (\sigma_{2^k}\, \eta_1, \cdots, \sigma_{2^{k+1}-1}\, \eta_{2^k})$ for $k \ge 0$. Then $D_\sigma = \sum_{k=0}^{\infty} J_k M_k Q_k$ and so $SD_\sigma = \sum_{k=0}^{\infty} SJ_k M_k Q_k$. Taking account of the fact that $\mathcal{L}^{(e)}_{s,\infty}$ admits an equivalent α-norm and using lemma 1 we obtain

$$\begin{aligned}
\| \sum_{k=0}^{m-1} SJ_k M_k Q_k \,|\, \mathcal{L}^{(e)}_{s,\infty} \| &\le C_0 \, (\sum_{k=0}^{m-1} \|SJ_k M_k Q_k \,|\, \mathcal{L}^{(e)}_{s,\infty}\|^\alpha)^{1/\alpha} \\
&\le C_0 \, (\sum_{k=0}^{m-1} \|SJ_k \,|\, \mathcal{L}^{(e)}_{s,\infty}\|^\alpha \, \|M_k\|^\alpha \, \|Q_k\|^\alpha)^{1/\alpha} \\
&\le C_0 \, (\sum_{k=0}^{m-1} \|SJ_k \,|\, \mathcal{L}^{(e)}_{s,\infty}\|^\alpha \, |\sigma_{2^k}|^\alpha)^{1/\alpha} \\
&\le C_1 \, (\sum_{k=0}^{m-1} \|SJ_k\|^\alpha \, 2^{\alpha k(1/s - 1/p')} \, |\sigma_{2^k}|^\alpha)^{1/\alpha} \\
&\le C_1 \, \|S\| \, (\sum_{k=0}^{m-1} 2^{\alpha k(1/s - 1/p')} \, |\sigma_{2^k}|^\alpha)^{1/\alpha} \qquad \text{for } s < p'.
\end{aligned}$$

We assume that $\sigma = (\sigma_i) \in \ell_{r,\infty}$. Then we get

$$\|\sum_{k=0}^{m-1} SJ_kM_kQ_k \,|\, \mathcal{L}_{s,\infty}^{(e)}\| \le C_2\,\|S\|\,(\sum_{k=0}^{m-1} 2^{\alpha k(1/s-1/p')}2^{-\alpha k/r})^{1/\alpha}$$
$$\le C_3\,\|S\|\,2^{m(1/s-1/p'-1/r)} \qquad \text{for } 1/s > 1/r+1/p'.$$

This yields $e_{2^{m-1}}(\sum_{k=0}^{m-1} SJ_kM_kQ_k) \le C_4\,\|S\|\,2^{-m(1/r+1/p')}$. In order to estimate $\|\sum_{k=m}^{\infty} SJ_kM_kQ_k \,|\, \mathcal{L}_{s,\infty}^{(e)}\|$ we choose s such that $1/p' < 1/s < 1/r+1/p'$. By arguing similarly as above, we establish the following estimation

$$\|\sum_{k=m}^{\infty} SJ_kM_kQ_k \,|\, \mathcal{L}_{s,\infty}^{(e)}\| \le C_5\,\|S\|\,(\sum_{k=m}^{\infty} 2^{\alpha k(1/s-1/r-1/p')})^{1/\alpha}$$
$$\le C_5\,\|S\|\,2^{m(1/s-1/r-1/p')}\cdot(\sum_{k=0}^{\infty} 2^{\alpha k(1/s-1/r-1/p')})^{1/\alpha}$$
$$\le C_6\,\|S\|\,2^{m(1/s-1/r-1/p')}.$$

This implies $e_{2^{m-1}}(\sum_{k=m}^{\infty} SJ_kM_kQ_k) \le C_7\,\|S\|\,2^{-m(1/r+1/p')}$. From the additivity of the entropy numbers it follows that

$$e_{2^m}(SD_\sigma) \le e_{2^{m-1}}(\sum_{k=0}^{m-1} SJ_kM_kQ_k) + e_{2^{m-1}}(\sum_{k=m}^{\infty} SJ_kM_kQ_k)$$
$$\le C_8\,\|S\|\,2^{-m(1/r+1/p')}.$$

If n is a natural number we take m so that $2^m \le n < 2^{m+1}$. An appeal to the monotonicity of the entropy numbers establishes that $e_n(SD_\sigma) \le e_{2^m}(SD_\sigma) \le C_9\|S\|n^{-1/r-1/p'}$. Hence $SD_\sigma \in \mathcal{L}_{s,\infty}^{(e)}(\ell_1, X)$ for $1/s = 1/r+1/p'$, which verifies $(*)$.

Now we use real interpolation to derive the desired assertion. Given r with $0<r<\infty$ we can find r_0, r_1 and θ such that $0<r_0<r_1<\infty$, $0<\theta<1$ and $1/r = (1-\theta)/r_0+\theta/r_1$. We consider the operator $\mathcal{T}$ transforming every sequence σ into the composition operator SD_σ. By $(*)$, $\mathcal{T}: \ell_{r_i,\infty} \to \mathcal{L}_{s_i,\infty}^{(e)}(\ell_1, X)$, where $1/s_i = 1/r_i + 1/p'$, $i=0,1$, are both bounded linear operators. Since $1/r = (1-\theta)/r_0+\theta/r_1$, theorem 5.3.1. of [1] tells us that $(\ell_{r_0,\infty}, \ell_{r_1,\infty})_{\theta,t} = \ell_{r,t}$. The well-known interpolation formula concerning entropy ideals allows us to obtain $\big(\mathcal{L}^{(e)}_{s_0,\infty}(\ell_1, X), \mathcal{L}^{(e)}_{s_1,\infty}(\ell_1, X)\big)_{\theta,t} \subseteq \mathcal{L}^{(e)}_{s,t}(\ell_1, X)$, where $1/s = (1-\theta)/s_0 + \theta/s_1 = 1/r + 1/p'$. We apply the interpolation theorem [1] to deduce that $\mathcal{T} : \ell_{r,t} \to \mathcal{L}^{(e)}_{s,t}(\ell_1, X)$ is also bounded. This ends the proof of the proposition. □

Proposition 2. *Let X be of weak type p, $1 \le p < 2$, $S \in \mathcal{B}(\ell_1, X)$ and let $D_\sigma \in \mathcal{B}(\ell_q, \ell_1)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r,t}$, where $0 < r < \infty$, $0 < t \le \infty$, $1 \le q \le \infty$ and $1/r + 1/q > 1$. Then $SD_\sigma \in \mathcal{L}^{(e)}_{s,t}(\ell_q, X)$ provided that $1/s = 1/r + 1/q - 1/p$.*

Proof. The assumptions on q and r guarantee that we can choose r_0 and r_1 such that $0 < r_0, r_1 < \infty$, $1/r = 1/r_0 + 1/r_1$ and $1/r_1 + 1/q > 1$. Then we can split $\sigma = \tau \circ \mu$ with $\mu \in \ell_{r_1,\infty}$ and $\tau \in \ell_{r_0,t}$. As a result the operator $D_\sigma \in \mathcal{B}(\ell_q, \ell_1)$ can be factorized with diagonal operators D_μ and D_τ as $D_\sigma : \ell_q \xrightarrow{D_\mu} \ell_1 \xrightarrow{D_\tau} \ell_1$. Since $1/r_1 > \max(1-1/q, 0)$, we use a result of Carl [4] to see that $D_\mu \in \mathcal{L}^{(e)}_{s_1,\infty}(\ell_q, \ell_1)$ for $1/s_1 = 1/r_1 + 1/q - 1$. An appeal to proposition 1 reveals that $SD_\tau \in \mathcal{L}^{(e)}_{s_0,t}(\ell_1, X)$ for $1/s_0 = 1/r_0 + 1/p'$. Using the multiplication theorem for the entropy ideals we derive that $SD_\sigma = SD_\tau D_\mu \in \mathcal{L}^{(e)}_{s_0,t} \circ \mathcal{L}^{(e)}_{s_1,\infty}(\ell_q, X) \subseteq \mathcal{L}^{(e)}_{s,t}(\ell_q, X)$ for $1/s = 1/s_0 + 1/s_1 = 1/r + 1/q - 1/p$. □

In the next proposition we describe the dual situation.

Proposition 3. *Let X^* be of weak type p, $1 \le p < 2$, $R \in \mathcal{B}(X, \ell_\infty)$ and let $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_q)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r,t}$, where $1 \le q \le \infty$, $0 < r < q \le \infty$ and $0 < t \le \infty$. Then $D_\sigma R \in \mathcal{L}^{(e)}_{s,t}(X, \ell_q)$ for $1/s = 1/r + 1/q' - 1/p$.*

Proof. We start with the case $1 < q < \infty$. For $R \in \mathcal{B}(X, \ell_\infty)$, we denote the restriction of R^* to ℓ_1 by S. Since X^* is of weak type p and $1/r + 1/q' > 1$, it follows from proposition 2 that $SD_\sigma \in \mathcal{L}^{(e)}_{s,t}(\ell_{q'}, X^*)$ with $1/s = 1/r + 1/q' - 1/p$. The K-convexity of $\ell_{q'}$ enables us to invoke a result due to Bourgain, Pajor,Szarek and Tomczak-Jaegermann [2] to get that there exists a constant $C \ge 0$ such that

$\|D_\sigma R \,|\, \mathcal{L}^{(e)}_{s,t}\| \;=\; \|D_\sigma S^*|_X \,|\, \mathcal{L}^{(e)}_{s,t}\| \;\le\; \|D_\sigma S^* \,|\, \mathcal{L}^{(e)}_{s,t}\| \;\le\; C\,\|SD_\sigma \,|\, \mathcal{L}^{(e)}_{s,t}\|$. Consequently $D_\sigma R \in \mathcal{L}^{(e)}_{s,t}(X, \ell_q)$ for $1/s = 1/r + 1/q' - 1/p$.

Now we deal with the case $q = 1$ or $q = \infty$. Given r with $0 < r < q \le \infty$, we choose $r_0,\ r_1$ and u such that $0 < r_0, r_1 < \infty,\ 1 < u < \infty,\ 1/r_0 + 1/u > 1/q,\ 1/r_1 > 1/u$ and $1/r = 1/r_0 + 1/r_1$. Thus we split $\sigma = \mu \circ \tau$ with $\tau \in \ell_{r_1,t}$ and $\mu \in \ell_{r_0,\infty}$ and hence the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_q)$ is factorized with diagonal operators D_τ and D_μ as $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_u \xrightarrow{D_\mu} \ell_q$. Since $1/r_0 > \max(1/q - 1/u, 0)$, we take account of a result due to Carl [4] to conclude that $D_\mu \in \mathcal{L}^{(e)}_{s_0,\infty}(\ell_u, \ell_q)$ with $1/s_0 = 1/r_0 + 1/u - 1/q$. Applying the result of the preceding case to the operator $D_\tau \in \mathcal{B}(\ell_\infty, \ell_u)$ we have $D_\tau R \in \mathcal{L}^{(e)}_{s_1,t}(X, \ell_u)$ with $1/s_1 = 1/r_1 + 1/u' - 1/p$. The multiplication theorem for the entropy ideals assures us that $D_\sigma R = D_\mu D_\tau R \in \mathcal{L}^{(e)}_{s_0,\infty} \circ \mathcal{L}^{(e)}_{s_1,t}(X, \ell_q) \subseteq \mathcal{L}^{(e)}_{s,t}(X, \ell_q)$ whenever $1/s = 1/s_0 + 1/s_1 = 1/r + 1/q' - 1/p$. □

Propositions 2 and 3 permit us to give a description of (r, w)-nuclear operators acting from a Banach space whose dual has weak type q into a Banach space of weak type p, $1 \le p,\ q < 2$, in terms of their entropy numbers.

THEOREM 1. *Let X^* be of weak type q and Y of weak type p, $1 \le p,\ q < 2$. Then $\mathcal{N}_{r,w}(X, Y) \subset \mathcal{L}^{(e)}_{s,w}(X, Y)$ provided that $0 < r < 1,\ 0 < w \le \infty$ and $1/s = 1 + 1/r - 1/p - 1/q$.*

Proof. We divide the proof into two steps.
Step 1. The first step is to verify the following assertion : If $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_r,\ 0 < r < 1$, and $S \in \mathcal{B}(\ell_1, Y)$ then $SD_\sigma R \in \mathcal{L}^{(e)}_{s,r}(X, Y)$ with $1/s = 1 + 1/r - 1/p - 1/q$.
Since $1/r = 1/2r + 1/2r$, for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \tau$ with $\tau \in \ell_{2r}$. Therefore the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ admits a factorization $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_2 \xrightarrow{D_\tau} \ell_1$, where D_τ is a diagonal operator induced by a sequence $\tau \in \ell_{2r}$. Since Y is of weak type p and $1/2r + 1/2 > 1$, we apply proposition 2 to deduce that $SD_\tau \in \mathcal{L}^{(e)}_{s_0,2r}(\ell_2, Y)$ for $1/s_0 = 1/2r + 1/2 - 1/p$. As X^* is of weak type q and $0 < 2r < 2$, we invoke proposition 3 to produce $D_\tau R \in \mathcal{L}^{(e)}_{s_1,2r}(X, \ell_2)$ with $1/s_1 = 1/2r + 1/2 -$

$1/q$. The multiplication theorem for the entropy ideals informs us that $SD_\sigma R = SD_\tau D_\tau R \in \mathcal{L}^{(e)}_{s_0,2r} \circ \mathcal{L}^{(e)}_{s_1,2r}(X,Y) \subseteq \mathcal{L}^{(e)}_{s,r}(X,Y) \subseteq \mathcal{L}^{(e)}_{s,\infty}(X,Y)$ with $1/s = 1/s_0 + 1/s_1 = 1 + 1/r - 1/p - 1/q$.
Step 2. We improve the result of the preceding step by real interpolation. Given r with $0 < r < 1$ we can find r_0, r_1 and θ such that $0 < r_0 < r_1 < 1$, $0 < \theta < 1$ and $1/r = (1-\theta)/r_0 + \theta/r_1$. Let $\mathcal{T}$ be the operator transforming every sequence σ into the composition operator $SD_\sigma R$. By Step 1, $\mathcal{T} : \ell_{r_i} \to \mathcal{L}^{(e)}_{s_i,\infty}(X,Y)$, where $1/s_i = 1+1/r_i-1/p-1/q$, $i = 0,1$, are both bounded linear operators. Since $1/r = (1-\theta)/r_0+\theta/r_1$, we have $(\ell_{r_0}, \ell_{r_i})_{\theta,w} = \ell_{r,w}$ with the help of theorem 5.3.1. of [1]. The well-known interpolation formula concerning entropy ideals asserts that $\big(\mathcal{L}^{(e)}_{s_0,\infty}(X,Y), \mathcal{L}^{(e)}_{s_1,\infty}(X,Y)\big)_{\theta,w} \subseteq \mathcal{L}^{(e)}_{s,w}(X,Y)$, where $1/s = (1-\theta)/s_0 + \theta/s_1 = 1 + 1/r - 1/p - 1/q$. An appeal to the interpolation theorem [1] establishes that $\mathcal{T} : \ell_{r,w} \to \mathcal{L}^{(e)}_{s,w}(X,Y)$ is also bounded.

Now we select any $T \in \mathcal{N}_{r,w}(X,Y)$. It is known that T can be represented as $T = SD_\sigma R$, where $R \in \mathcal{B}(X,\ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty,\ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r,w}$ and $S \in \mathcal{B}(\ell_1,Y)$. This leads us to have that $T = SD_\sigma R \in \mathcal{L}^{(e)}_{s,w}(X,Y)$. This proves the desired inclusion. □

Naturally the question arises : Does the above theorem remain valid even when $p = 2$ or $q = 2$? This is answered by the theorems stated below. By applying a known fact concerning the relationship between entropy numbers and Kolmogorov numbers, together with estimates for the Kolmogorov numbers in terms of the Weyl numbers, we improve proposition 2 in Banach spaces of weak type 2.

Proposition 4. *Let X be of weak type 2, $S \in \mathcal{B}(\ell_1,X)$ and let $D_\sigma \in \mathcal{B}(\ell_q,\ell_1)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r,t}$, where $1 \le q \le \infty$, $0 < r < \min(2,q')$ and $0 < t \le \infty$. Then $SD_\sigma \in \mathcal{L}^{(e)}_{s,t}(\ell_q,X)$ with $1/s = 1/r + 1/q - 1/2$.*

Proof. First we shall show that if $\sigma \in \ell_r$ then $SD_\sigma \in \mathcal{L}^{(e)}_{s,\infty}(\ell_q,X)$, where $1/s = 1/r + 1/q - 1/2$. $(*)$
Given r with $0 < r < \min(2,q')$, we can pick r_1 and r_2 such that $0 < r_1 < 2$, $1/r_2 > \max(1/2 - 1/q, 0)$ and $1/r = 1/r_1 + 1/r_2$. Hence

for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \mu$ with $\mu \in \ell_{r_2}$ and $\tau \in \ell_{r_1}$. Accordingly the operator $D_\sigma \in \mathcal{B}(\ell_q, \ell_1)$ can be factorized with diagonal operators D_μ and D_τ as $D_\sigma : \ell_q \xrightarrow{D_\mu} \ell_2 \xrightarrow{D_\tau} \ell_1$. Since $1/r_2 > \max(1/2 - 1/q, 0)$, we invoke a result of Carl [4] to deduce that $D_\mu \in \mathcal{L}^{(e)}_{s_2,r_2}(\ell_q,\ell_2) \subset \mathcal{L}^{(e)}_{s_2,\infty}(\ell_q,\ell_2)$ for $1/s_2 = 1/r_2 + 1/q - 1/2$. Using a result due to Carl [3] and making use of theorem 11.7.7. of [12] we get that $\|SD_\tau \,|\, \mathcal{L}^{(e)}_{r_1,\infty}\| \le C_0 \|SD_\tau \,|\, \mathcal{L}^{(d)}_{r_1,\infty}\| = C_0 \|(SD_\tau)^* \,|\, \mathcal{L}^{(c)}_{r_1,\infty}\|$. Since X is of weak type 2, it follows that X is K-convex and hence X^* is K-convex. This enables us to use a result of Pajor and Tomczak-Jaegermann [11] to see that

$$\begin{aligned}\|(SD_\tau)^* \,|\, \mathcal{L}^{(c)}_{r_1,\infty}\| &\le C_1\, K(X^*)^{2/r_1}\, \|(SD_\tau)^* \,|\, \mathcal{L}^{(vr)}_{r_1,\infty}\| \\ &\le C_1\, K(X^*)^{2/r_1}\, \|SD_\tau \,|\, \mathcal{L}^{(v)}_{r_1,\infty}\|.\end{aligned}$$

Now we estimate the volume numbers in terms of the Weyl numbers. Let H be an n-dimensional subspace of ℓ_2. By setting $E = \operatorname{ran}(SD_\tau i_H)$, where $i_H : H \to \ell_2$ is the natural injection, we have $\dim E \le n$. Then Lewis' theorem [14] asserts that there is an isomorphism $u : \ell_2^n \to E$ and an operator $v : X \to \ell_2^n$ such that $v|_E = u^{-1}$ and $\ell(u) = \ell^*(v) \le n^{1/2}$. Using Geiss' theorem [9] and a result due to Pajor and Tomczak-Jaegermann [14], together with the multiplicativity of the volume numbers, we obtain the following :

$$\begin{aligned} v_n(SD_\tau i_H) &= v_n(uvSD_\tau i_H) \le v_n(u)\, v_n(vSD_\tau i_H) \\ &\le 4\, e_n(u) \Big(\prod_{k=1}^{n} a_k(vSD_\tau i_H)\Big)^{1/n} \\ &\le 4\, C_2\, n^{-1/2}\, \ell(u)\, n^{-(1/r_1+1/2)} \sup_k k^{1/r_1+1/2}\, x_k(v^*(SD_\tau i_H)^*) \\ &\le 4\, C_3\, n^{-(1/r_1+1/2)}\, \|v \,|\, \mathcal{L}^{(a)}_{2,\infty}\| \cdot \|(SD_\tau)^* \,|\, \mathcal{L}^{(x)}_{r_1,\infty}\| \\ &\le 4\, C_3\, n^{-(1/r_1+1/2)}\, wT_2(X)\ \ell^*(v)\, \|(SD_\tau)^* \,|\, \mathcal{L}^{(x)}_{r_1,\infty}\| \\ &\le 4\, C_3\, n^{-1/r_1}\, wT_2(X)\, \|(SD_\tau)^* \,|\, \mathcal{L}^{(x)}_{r_1,\infty}\| \end{aligned}$$

This gives that $\|SD_\tau \,|\, \mathcal{L}^{(v)}_{r_1,\infty}\| \le 4\, C_3\, wT_2(X)\, \|(SD_\tau)^* \,|\, \mathcal{L}^{(x)}_{r_1,\infty}\|$. From a result of Lubitz [13], we know that $\|(SD_\tau)^* \,|\, \mathcal{L}^{(x)}_{r_1,\infty}\| \le C_4\, \|S\| \cdot$

$\|\tau\|_{r_1,\infty} \le C_4 \|S\| \cdot \|\tau\|_{r_1}$. Combining the above inequalities we arrive at $\|SD_\tau \,|\, \mathcal{L}^{(e)}_{r_1,\infty}\| \le C\, K(X^*)^{2/r_1}\, wT_2(X)\, \|S\| \cdot \|\tau\|_{r_1}$, that is $SD_\tau \in \mathcal{L}^{(e)}_{r_1,\infty}(\ell_2, X)$. An appeal to the multiplication theorem for the entropy ideals ensures that $SD_\sigma = SD_\tau D_\mu \in \mathcal{L}^{(e)}_{r_1,\infty} \circ \mathcal{L}^{(e)}_{s_2,\infty}(\ell_q, X) \subseteq \mathcal{L}^{(e)}_{s,\infty}(\ell_q, X)$ for $1/s = 1/r_1 + 1/s_2 = 1/r + 1/q - 1/2$, which proves $(*)$.

Next we apply real interpolation to derive the required assertion. Given r with $0 < r < \min(2, q')$, we can select $r_0,\ r_1$ and θ such that $0 < r_0 < r_1 < \min(2, q'),\ 0 < \theta < 1$ and $1/r = (1-\theta)/r_0 + \theta/r_1$. Let $\mathcal{T}$ be the operator assigning to every sequence σ the composition operator SD_σ. By $(*)$, $\mathcal{T} : \ell_{r_i} \to \mathcal{L}^{(e)}_{s_i,\infty}(\ell_q, X)$, where $1/s_i = 1/r_i + 1/q - 1/2,\ i = 0, 1$, are both bounded linear operators. Since $1/r = (1-\theta)/r_0 + \theta/r_1$, we have $(\ell_{r_0}, \ell_{r_1})_{\theta,t} = \ell_{r,t}$ in view of theorem 5.3.1. of [1]. The well-known interpolation formula concerning entropy ideals tells us that $\big(\mathcal{L}^{(e)}_{s_0,\infty}(\ell_q, X), \mathcal{L}^{(e)}_{s_1,\infty}(\ell_q, X)\big)_{\theta,t} \subseteq \mathcal{L}^{(e)}_{s,t}(\ell_q, X)$, where $1/s = (1-\theta)/s_0 + \theta/s_1 = 1/r + 1/q - 1/2$. Applying the interpolation theorem [1] we draw that $\mathcal{T} : \ell_{r,t} \to \mathcal{L}^{(e)}_{s,t}(\ell_q, X)$ is also bounded. This completes the proof of the proposition. □

The following proposition shows a corresponding dual formulation for a Banach space whose dual has weak type 2.

Proposition 5. *Let X^* be of weak type 2, $R \in \mathcal{B}(X, \ell_\infty)$ and let $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_q)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r,t}$, where $1 \le q \le \infty,\ 0 < r < \min(2, q)$ and $0 < t \le \infty$. Then $D_\sigma R \in \mathcal{L}^{(e)}_{s,t}(X, \ell_q)$ for $1/s = 1/r - 1/q + 1/2$.*

Proof. We proceed in the same way as in the proof of proposition 3. We first deal with the case $1 < q < \infty$. For $R \in \mathcal{B}(X, \ell_\infty)$, we define $R^*|_{\ell_1} = S \in \mathcal{B}(\ell_1, X^*)$. As X^* is of weak type 2 and $0 < r < \min(2, q)$, we summon up proposition 4 to conclude that $SD_\sigma \in \mathcal{L}^{(e)}_{s,t}(\ell_{q'}, X^*)$ with $1/s = 1/r + 1/q' - 1/2$. Since $\ell_{q'}$ is K-convex, we use a result due to Bourgain, Pajor, Szarek and Tomczak-Jaegermann [2] to obtain that there exists a constant $C \ge 0$ such that $\|D_\sigma R \,|\, \mathcal{L}^{(e)}_{s,t}\| = \|D_\sigma S^*|_X \,|\, \mathcal{L}^{(e)}_{s,t}\| \le \|D_\sigma S^* \,|\, \mathcal{L}^{(e)}_{s,t}\| \le C\, \|SD_\sigma \,|\, \mathcal{L}^{(e)}_{s,t}\|$. This implies $D_\sigma R \in \mathcal{L}^{(e)}_{s,t}(X, \ell_q)$ for $1/s = 1/r - 1/q + 1/2$.

Next we treat the case $q = 1$ or $q = \infty$. Given r with $0 < r < \min(2, q)$, we find $r_0,\ r_1$ and u such that $0 < r_0 < \infty,\ 1 < u <$

∞, $0 < r_1 < \min(2,u)$, $1/r_0 + 1/u > 1/q$ and $1/r = 1/r_0 + 1/r_1$. Therefore we split $\sigma = \mu \circ \tau$ with $\tau \in \ell_{r_1,t}$ and $\mu \in \ell_{r_0,\infty}$ and so the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_q)$ is factorized with diagonal operators D_τ and D_μ as $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_u \xrightarrow{D_\mu} \ell_q$. Since $1/r_0 > \max(1/q - 1/u, 0)$, we invoke a result of Carl [4] to infer that $D_\mu \in \mathcal{L}^{(e)}_{s_0,\infty}(\ell_u, \ell_q)$ with $1/s_0 = 1/r_0 + 1/u - 1/q$. Applying the result of the preceding case to the operator $D_\tau \in \mathcal{B}(\ell_\infty, \ell_u)$ we get $D_\tau R \in \mathcal{L}^{(e)}_{s_1,t}(X, \ell_u)$ with $1/s_1 = 1/r_1 - 1/u + 1/2$. Thanks to the multiplication theorem for the entropy ideals, we have $D_\sigma R = D_\mu D_\tau R \in \mathcal{L}^{(e)}_{s_0,\infty} \circ \mathcal{L}^{(e)}_{s_1,t}(X, \ell_q) \subseteq \mathcal{L}^{(e)}_{s,t}(X, \ell_q)$ for $1/s = 1/s_0 + 1/s_1 = 1/r - 1/q + 1/2$. □

We are now in a position to extend theorem 1 to the case $p = 2$ or $q = 2$.

Theorem 2. *Let X^* be of weak type 2 and Y of weak type p, $1 \le p < 2$. Then $\mathcal{N}_{r,w}(X,Y) \subset \mathcal{L}^{(e)}_{s,w}(X,Y)$ provided that $0 < r < 1$, $0 < w \le \infty$ and $1/s = 1/2 + 1/r - 1/p$.*

Proof. We divide the proof into two steps.
Step 1. The first step is to verify the following assertion : If $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_r$, $0 < r < 1$, and $S \in \mathcal{B}(\ell_1, Y)$ then $SD_\sigma R \in \mathcal{L}^{(e)}_{s,r}(X,Y)$ with $1/s = 1/2 + 1/r - 1/p$.
Since $1/r = 1/2r + 1/2r$, for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \tau$ with $\tau \in \ell_{2r}$. Consequently the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ admits a factorization $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_2 \xrightarrow{D_\tau} \ell_1$, where D_τ is a diagonal operator generated by a sequence $\tau \in \ell_{2r}$. As Y is of weak type p and $1/2r + 1/2 > 1$, it follows from proposition 2 that $SD_\tau \in \mathcal{L}^{(e)}_{s_1,2r}(\ell_2, Y)$ for $1/s_1 = 1/2r + 1/2 - 1/p$. Since X^* is of weak type 2 and $0 < 2r < \min(2,2)$, we have $D_\tau R \in \mathcal{L}^{(e)}_{2r}(X, \ell_2)$ by way of proposition 5. Appealing to the multiplication theorem for the entropy ideals we derive that $SD_\sigma R = SD_\tau D_\tau R \in \mathcal{L}^{(e)}_{s_1,2r} \circ \mathcal{L}^{(e)}_{2r}(X,Y) \subseteq \mathcal{L}^{(e)}_{s,r}(X,Y) \subseteq \mathcal{L}^{(e)}_{s,\infty}(X,Y)$ with $1/s = 1/s_1 + 1/2r = 1/2 + 1/r - 1/p$.
Step 2. We improve the result of the preceding step by real interpolation. Given r with $0 < r < 1$ we can select r_0, r_1 and θ such that $0 < r_0 < r_1 < 1$, $0 < \theta < 1$ and $1/r = (1-\theta)/r_0 + \theta/r_1$. We

consider the operator $\mathcal{T}$ transforming every sequence σ into the composition operator $SD_\sigma R$. By step 1, $\mathcal{T} : \ell_{r_i} \to \mathcal{L}^{(e)}_{s_i,\infty}(X,Y)$, where $1/s_i = 1/2 + 1/r_i - 1/p$, $i = 0, 1$, are both bounded linear operators. Applying the interpolation formulas given in the proof of theorem 1 we deduce that $\mathcal{T} : \ell_{r,w} \to \mathcal{L}^{(e)}_{s,w}(X,Y)$ is also bounded, where $1/s = (1-\theta)/s_0 + \theta/s_1 = 1/2 + 1/r - 1/p$.

Now we take any $T \in \mathcal{N}_{r,w}(X,Y)$. It is known that T can be represented as $T = SD_\sigma R$, where $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r,w}$ and $S \in \mathcal{B}(\ell_1, Y)$. This allows us to have that $T = SD_\sigma R \in \mathcal{L}^{(e)}_{s,w}(X,Y)$. This proves the required inclusion. □

Theorem 3. *Let X^* be of weak type q, $1 \le q < 2$, and Y of weak type 2. Then $\mathcal{N}_{r,w}(X,Y) \subset \mathcal{L}^{(e)}_{s,w}(X,Y)$ for $0 < r < 1$, $0 < w \le \infty$ and $1/s = 1/2 + 1/r - 1/q$.*

Proof. Suppose that $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_r$, $0 < r < 1$, and $S \in \mathcal{B}(\ell_1, Y)$. Since $1/r = 1/2r + 1/2r$, for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \tau$ with $\tau \in \ell_{2r}$. As a result the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ admits a factorization $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_2 \xrightarrow{D_\tau} \ell_1$, where D_τ is a diagonal operator induced by a sequence $\tau \in \ell_{2r}$. Since Y is of weak type 2 and $0 < 2r < \min(2,2)$, we have $SD_\tau \in \mathcal{L}^{(e)}_{2r}(\ell_2, Y)$ with the aid of proposition 4. Since X^* is of weak type q and $0 < 2r < 2$, an appeal to proposition 3 reveals that $D_\tau R \in \mathcal{L}^{(e)}_{s_1,2r}(X, \ell_2)$ for $1/s_1 = 1/2r + 1/2 - 1/q$. We apply the multiplication theorem for the entropy ideals to produce that $SD_\sigma R = SD_\tau D_\tau R \in \mathcal{L}^{(e)}_{2r} \circ \mathcal{L}^{(e)}_{s_1,2r}(X,Y) \subseteq \mathcal{L}^{(e)}_{s,t}(X,Y) \subseteq \mathcal{L}^{(e)}_{s,\infty}(X,Y)$ with $1/s = 1/2r + 1/s_1 = 1/2 + 1/r - 1/q$. The required inclusion can be carried out by arguing exactly as in the second part of the proof of theorem 2. □

Theorem 4. *Let X^* be of weak type 2 and Y of weak type 2. Then $\mathcal{N}_{r,w}(X,Y) \subset \mathcal{L}^{(e)}_{r,w}(X,Y)$ where $0 < r < 1$ and $0 < w \le \infty$.*

Proof. Assume that $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_r$, $0 < r < 1$, and $S \in \mathcal{B}(\ell_1, Y)$. Since $1/r = 1/2r + 1/2r$, for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \tau$

with $\tau \in \ell_{2r}$. Accordingly the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ admits a factorization $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_2 \xrightarrow{D_\tau} \ell_1$, where D_τ is a diagonal operator generated by a sequence $\tau \in \ell_{2r}$. Since Y is of weak type 2 and $0 < 2r < \min(2, 2)$, it follows from proposition 4 that $SD_\tau \in \mathcal{L}_{2r}^{(e)}(\ell_2, Y)$. As X^* is of weak type 2 and $0 < 2r < \min(2, 2)$, proposition 5 steps in to assure that $D_\tau R \in \mathcal{L}_{2r}^{(e)}(X, \ell_2)$. Application of the multiplication theorem for the entropy ideals leads to $SD_\sigma R = SD_\tau D_\tau R \in \mathcal{L}_{2r}^{(e)} \circ \mathcal{L}_{2r}^{(e)}(X, Y) \subseteq \mathcal{L}_r^{(e)}(X, Y) \subseteq \mathcal{L}_{r,\infty}^{(e)}(X, Y)$. The remaining assertions are established by arguing exactly as in the second part of the proof of theorem 2. □

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