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ON T-FUZZY GROUPS

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Abstract. We characterize some properties of t-fuzzy groups and t-fuzzy invariant groups and represent every subgroup S of a group X using the level set of t-fuzzy group constructed from S.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh ([7]). Rosenfeld ([3]) used this concept to formulate the notion of fuzzy groups. Since then, many other fuzzy algebraic concepts based on the Rosenfeld's fuzzy groups were developed. Anthony and Sherwood ([1]) redefined fuzzy groups in terms of t-norm which replaced the minimum operation of Rosenfeld's definition. Some properties of these redefined fuzzy groups, which we call t-fuzzy groups in this paper, have been developed by Sherwood ([5]), Sessa ([4]), Sidky and Mishref ([6]). As a continuation of these studies, we characterize some basic properties of t-fuzzy groups and t-fuzzy invariant groups and represent every subgroup S of X using the level set of t-fuzzy group constructed from S.

2. t-fuzzy groups

Definition 1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B.

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Definition 2. (Definition 1.3 of [4]) A *t-norm* is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying, for each p, q, r, s in [0,1],

(1) $T(0,p) = 0,\ T(p,1) = p$
(2) $T(p,q) \leq T(r,s)$ if $p \leq r$ and $q \leq s$
(3) $T(p,q) = T(q,p)$
(4) $T(p, T(q,r)) = T(T(p,q), r))$

Definition 3. Let S be a groupoid and T be a t-norm. A function $B : S \to [0,1]$ is a *t-fuzzy groupoid* in S iff for every x, y in S, $B(xy) \geq T(B(x), B(y))$. If X is a group, a fuzzy groupoid G is a *t-fuzzy group* in X iff for each $x \in X$, $G(x^{-1}) = G(x)$.

Proposition 4. *Let G be a fuzzy subset in a group X. G is a t-fuzzy group such that $G(e) = 1$ iff $G(xy^{-1}) \geq T(G(x), G(y))$ and $G(e) = 1$.*

Proof. Straightforward. □

Proposition 5. *Let G be a t-fuzzy group in a group X such that $G(a) = 1$. Let $r_a : X \to X$ be a right translation defined by $r_a(x) = xa$ and let $l_a : X \to X$ be a left translation defined by $l_a(x) = ax$. Then $r_a(G) = l_a(G) = G$.*

Proof. $r_a(G)(x) = \sup_{z \in r_a^{-1}(x)} G(z) = G(xa^{-1}) \geq T(G(x), G(a^{-1})) = T(G(x), G(a)) = G(x) = G(xa^{-1}a) \geq T(G(xa^{-1}), G(a)) = G(xa^{-1}) = r_a(G)(x)$. Thus $r_a(G)(x) \geq G(x) \geq r_a(G)(x)$. That is, $r_a(G) = G$. Similarly we may show $l_a(G) = G$. □

For fuzzy sets U, V in a set X, $U \circ V$ has been defined in most articles by

$$(U \circ V)(x) = \begin{cases} \sup\limits_{ab=x} \min(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$

We generalize this in the following definition and develop some properties of t-fuzzy groups and t-fuzzy invariant groups.

Definition 6. Let X be a set and let U, V be two fuzzy sets in X. $U \circ V$ is defined by

$$(U \circ V)(x) = \begin{cases} \sup\limits_{ab=x} T(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$

PROPOSITION 7. *Let A, B be fuzzy sets in a set X and let x_p, y_q be fuzzy points in X. Then*

(1) $x_p \circ y_q = (xy)_{T(p,q)}$.
(2) $A \circ B = \bigcup_{x_p \in A, y_q \in B} x_p \circ y_q$, *where*

$$(x_p \circ y_q)(z) = \sup_{cd=z} T(x_p(c), y_q(d)).$$

Proof. (1) $(x_p \circ y_q)(xy) = \sup_{ab=xy} T(x_p(a), y_q(b)) = T(x_p(x), y_q(y)) = T(p, q)$. If $z \neq xy$,

$$(x_p \circ y_q)(z) = \sup_{ab=z} T(x_p(a), y_q(b)) \leq \max[T(p, 0), T(0, q)] = 0,$$

that is, $(x_p \circ y_q)(z) = 0$. Thus $x_p \circ y_q = (xy)_{T(p,q)}$.
(2) If $x_p \in A$ and $y_q \in B$, then $A(s) \geq x_p(s)$ and $B(t) \geq y_q(t)$. Thus

$$\begin{aligned}
(A \circ B)(z) &= \sup_{st=z} T(A(s), B(t)) \\
&\geq \sup_{st=z} \sup_{x_p \in A, y_q \in B} T(x_p(s), y_q(t)) \\
&= \sup_{x_p \in A, y_q \in B} \sup_{st=z} T(x_p(s), y_q(t)) \\
&= \sup_{x_p \in A, y_q \in B} (x_p \circ y_q)(z) \\
&= (\bigcup_{x_p \in A, y_q \in B} x_p \circ y_q)(z).
\end{aligned}$$

Thus $A \circ B \subseteq \cup\, x_p \circ y_q$. Since $s_{A(s)} \in A$ and $t_{B(t)} \in B$,

$$\begin{aligned}
(\bigcup_{x_p \in A, y_q \in B} x_p \circ y_q)(z) &= \sup_{x_p \in A, y_q \in B} \sup_{st=z} T(x_p(s), y_q(t)) \\
&\geq \sup_{st=z} T(s_{A(s)}(s), t_{B(t)}(t)) \\
&= \sup_{st=z} T(A(s), B(t)) = (A \circ B)(z). \qquad \square
\end{aligned}$$

PROPOSITION 8. *Let X be a set. Then*

(1) *If X is associative, commutative, respectively, then so is $\circ$.*
(2) *If X has a unit e, then $A \circ e_p = e_p \circ A$ for a fuzzy set A in X and the fuzzy singleton e_1 is a unit of the operation $\circ$, that is, $A \circ e_1 = A = e_1 \circ A$*

Proof. (1) Suppose X is associative. Then

$$\begin{aligned}[(A\circ B)\circ C](z) &= \sup_{ab=z} T[\sup_{pq=a} T(A(p),B(q)),C(b)] \\ &= \sup_{(pq)b=z} T[T(A(p),B(q)),C(b)] \\ &= \sup_{(pq)b=z} T[A(p),T(B(q),C(b))] \\ &= \sup_{pr=z} T[A(p),\sup_{qb=r} T(B(q),C(b))] \\ &= \sup_{pr=z} T[A(p),(B\circ C)(r)] = [A\circ(B\circ C)](z).\end{aligned}$$

Suppose X is commutative. Then $(A\circ B)(x) = \sup_{yz=x} T(A(y),B(z)) = \sup_{zy=x} T(B(z),A(y)) = (B\circ A)(x)$.
(2) $(e_1\circ A)(x) = T(e_1(e),A(x)) = A(x)$ and $(A\circ e_1)(x) = T(A(x),e_1(e)) = A(x)$.

$$\begin{aligned}(A\circ e_p)(x) &= T(A(x),e_p(e)) = T(e_p(e),A(x)) \\ &= \sup_{yz=x} T(e_p(y),A(z)) = (e_p\circ A)(x).\end{aligned}$$ □

Theorem 9. *Let A be an non-empty fuzzy set of a groupoid X. Then the following are equivalent.*

(1) *A is a t-fuzzy groupoid.*
(2) *For any $x_p, y_q \in A$, $x_p\circ y_q \in A$.*
(3) *$A\circ A \subseteq A$.*

Proof. (1) → (2). Suppose that $A(xy) \geq T(A(x),A(y))$. By Proposition 7,

$$(x_p\circ y_q)(z) = [(xy)_{T(p,q)}](z) = \begin{cases} T(p,q), & \text{if } z = xy \\ 0, & \text{if } z \neq xy.\end{cases}$$

Let $x_p, y_q \in A$. Then $A(x) \geq p$ and $A(y) \geq q$. If $z = xy$, $A(z) = A(xy) \geq T(A(x),A(y)) \geq T(p,q) = (x_p\circ y_q)(z)$, and hence $x_p\circ y_q \in A$. If $z \neq xy$, $A(z) \geq (x_p\circ y_q)(z) = 0$, and hence $x_p\circ y_q \in A$.

(2) $\rightarrow$ (3). Suppose that for any $x_p, y_q \in A$, $x_p \circ y_q \in A$. By Proposition 7,

$$(A \circ A)(z) = [\bigcup_{x_p \in A, y_q \in A} x_p \circ y_q](z) = \sup_{x_p \in A, y_q \in A} (x_p \circ y_q)(z) \leq A(z).$$

(3) $\rightarrow$ (1). Suppose $A \circ A \subseteq A$. Then $A(xy) \geq (A \circ A)(xy) = \sup_{ab=xy} T(A(a), A(b)) \geq T(A(x), A(y))$. Thus A is a t-fuzzy groupoid. $\square$

Definition 10. Let A be a t-fuzzy subgroup of a set X. A is called a *t-fuzzy invariant (or normal) subgroup* of X if $A(xy) = A(yx)$ for all $x, y \in X$.

Theorem 11. *Let A be a t-fuzzy invariant subgroup of an associative set X. Then*

(1) *For a fuzzy set B of X, $A \circ B = B \circ A$.*
(2) *If B is a t-fuzzy subgroup of X, so is $B \circ A$.*

Proof. (1)

$$\begin{aligned}(A \circ B)(x) &= \sup_{yz=x} T(A(y), B(z)) = \sup_{xz^{-1}z=x} T(A(xz^{-1}), B(z)) \\ &= \sup_{zz^{-1}xz^{-1}z=x} T(B(z), A(z^{-1}x)) = \sup_{zz^{-1}x=x} T(B(z), A(z^{-1}x)) \\ &= \sup_{zy=x} T(B(z), A(y)) = (B \circ A)(x).\end{aligned}$$

(2) By Theorem 9 and part (1) of this theorem, $(B \circ A) \circ (B \circ A) = B \circ (A \circ B) \circ A = B \circ (B \circ A) \circ A = (B \circ B) \circ (A \circ A) \subseteq B \circ A$. $(B \circ A)(x^{-1}) = \sup_{yz=x^{-1}} T(B(y), A(z)) = \sup_{z^{-1}y^{-1}=x} T(A(z^{-1}), B(y^{-1})) = (A \circ B)(x)$. Since $A \circ B = B \circ A$, $(B \circ A)(x^{-1}) = (B \circ A)(x)$. $\square$

Proposition 12. *If A is a t-fuzzy invariant subgroup of a group X such that $A(e) = 1$, then $X_A = \{x \in X : A(x) = A(e)\}$ is a normal subgroup of X.*

Proof. It is easy to see that X_A is a subgroup of X. Let $g \in X$ and $h \in X_A$. Then $A(h) = 1$. Since A is a t-fuzzy invariant subgroup, $A(ghg^{-1}) = A(hg^{-1}g) = A(h) = 1$, and hence $ghg^{-1} \in X_A$. Thus X_A is a normal subgroup of a group X. $\square$

Proposition 13. *If A is a t-fuzzy invariant subgroup of X and B is a fuzzy set in X, then $h^{-1}(h(B)) = X_A \circ B$, where $h : X \to X/X_A$ is a natural homomorphism.*

Proof.

$$[h^{-1}(h(B))](x) = h(B)(h(x)) = \sup_{y \in h^{-1}(h(x))} B(y) = \sup_{yX_A = xX_A} B(y) = \sup_{xy^{-1} \in X_A} B(y),$$

$$(X_A \circ B)(x) = \sup_{zy=x} T(X_A(z), B(y)) = \sup_{z \in X_A, zy=x} T(1, B(y)) = \sup_{xy^{-1} \in X_A} B(y).$$

Thus $h^{-1}(h(B)) = X_A \circ B$. □

Definition 14. Let B be a fuzzy set in a set X and f be a map defined on X. Then B is called *f-invariant* if, for all $x, y \in X$, $f(x) = f(y)$ implies $B(x) = B(y)$.

Theorem 15. *Let N be a normal subgroup of a group X and let G be a t-fuzzy group in X such that $G(x) = 1$ for all $x \in N$. Let $\phi : X \to X/N$ be a canonical homomorphism. Then G is ϕ-invariant and $\phi(G)$ is a t-fuzzy group in X/N.*

Proof. Suppose $\phi(x) = \phi(y)$. Then $xN = yN$, that is, $xy^{-1} \in N$. $G(x) = G(xy^{-1}y) \geq T(G(xy^{-1}), G(y)) = T(1, G(y)) = G(y)$. $G(y) = G(yx^{-1}x) \geq T(G(yx^{-1}), G(x)) = T(G(xy^{-1}), G(x)) = T(1, G(x)) = G(x)$. Thus $G(x) = G(y)$, that is, G is ϕ-invariant. Since G is ϕ-invariant, $\phi(G)(xNyN) = \phi(G)(xyN) = \sup_{z \in \phi^{-1}(xyN)} G(z) = G(xy)$, $\phi(G)(xN) = \sup_{z \in \phi^{-1}(xN)} G(z) = G(x)$, and $\phi(G)(yN) = \sup_{z \in \phi^{-1}(yN)} G(z) = G(y)$. Thus

$$\phi(G)(xNyN) = G(xy) \geq T(G(x), G(y)) = T(\phi(G)(xN), \phi(G)(yN)),$$

$$\phi(G)((xN)^{-1}) = \phi(G)(x^{-1}N) = \sup_{z \in \phi^{-1}(x^{-1}N)} G(z) = G(x^{-1}) = G(x) = \phi(G)(xN).$$

Thus $\phi(G) = G/N$ is a t-fuzzy group. □

PROPOSITION 16. *Let S be a fuzzy set in a group X. If $S_t = \{x \in X : S(x) \geq t\}$ is a subgroup of X for all $t > 0$, then S is a t-fuzzy group in X.*

Proof. Let $S(x) = t_1$ and $S(y) = t_2$ with $t_1 \leq t_2$. Then $x \in S_{t_1}$, $y \in S_{t_2}$, and $S_{t_2} \subset S_{t_1}$. Thus $y \in S_{t_1}$. Since $x, y \in S_{t_1}$ and S_{t_1} is a subgroup, $xy \in S_{t_1}$, and hence $S(xy) \geq t_1$. Since $T(S(x), S(y)) = T(t_1, t_2) \leq T(t_1, 1) = t_1$, $S(xy) \geq t_1 \geq T(S(x), S(y))$. Let $S(z) = t$. Then $z \in S_t$. Since S_t is a subgroup, $z^{-1} \in S_t$, that is, $S(z^{-1}) \geq t$. Thus $S(z^{-1}) \geq S(z)$. Similarly, we may show $S(z) \geq S(z^{-1})$. Hence S is a t-fuzzy group. $\square$

THEOREM 17. *Let S be a subgroup of a group X and let H be a fuzzy set in X defined by*

$$H(x) = \begin{cases} p & \text{if } x \in S \\ 0 & \text{if } x \in X - S. \end{cases}$$

Then H is a t-fuzzy group and every subgroup S of a group X can be represented as $S = H_p = \{x \in X : H(x) = p\}$, where $0 < p$.

Proof. Let $x, y, z \in X$.
(i) Suppose $x, y \in S$.
Then $xy \in S$, and hence $H(x) = p$, $H(y) = p$, and $H(xy) = p$. Since $T(H(x), H(y)) = T(p, p) \leq T(p, 1) = p$, $H(xy) = p \geq T(H(x), H(y))$. If $z \in S$, then $z^{-1} \in S$, and hence $H(z^{-1}) = p = H(z)$. If $z \notin S$, then $z^{-1} \notin S$, and hence $H(z^{-1}) = H(z) = 0$. Thus H is a t-fuzzy group.
(ii) Suppose $x \in S$, $y \notin S$, and $xy \in S$.
Then $H(x) = p$, $H(y) = 0$, and $H(xy) = p$. Since $T(H(x), H(y)) = T(p, 0) \leq T(p, 1) = p$, $H(xy) = p \geq T(H(x), H(y))$. We may show $H(z^{-1}) = H(z)$ for all $z \in X$ as shown in part (i). Thus H is a t-fuzzy group.
(iii) Suppose $x \in S$, $y \notin S$, and $xy \notin S$.
Then $H(x) = p$, $H(y) = 0$, and $H(xy) = 0$. Since $T(p, 0) = 0$, $T(H(x), H(y)) = T(p, 0) = 0$. Thus $H(xy) = T(H(x), H(y))$. We may show $H(z^{-1}) = H(z)$ for all $z \in X$ as shown in part (i). Thus H is a t-fuzzy group.
(iv) Suppose $x \notin S$ and $y \notin S$.

Then $H(x) = 0$ and $H(y) = 0$. If $xy \in S$, then $H(xy) = p \geq T(H(x), H(y)) = T(0,0)$, and hence $H(xy) \geq T(H(x), H(y))$. If $xy \notin S$, then $H(xy) = T(H(x), H(y)) = 0$. We may show $H(z^{-1}) = H(z)$ for all $z \in X$ as shown in part (i). Thus H is a t-fuzzy group.

From (i), (ii), (iii), and (iv), H is a t-fuzzy group in X. Let $\alpha \in H_p$. Then $H(\alpha) = p > 0$, and hence $\alpha \in S$. Thus $H_p \subseteq S$. Let $\beta \in S$. Then $H(\beta) = p$, and hence $\beta \in H_p$. Thus $S \subseteq H_p$. Hence $S = H_p$. $\square$

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