Exactly Solvable Time-Dependent Problems: Potentials of Monotonously Decreasing Function of Time

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We solve the Schrödinger equation analytically for systems whose potentials have a certain time-dependence (which is monotonously decreasing) and general coordinate-dependences. Only a few time-dependent systems have been reported to be analytically solved whose potentials are constant, linear, and quadratic functions of coordinate with arbitrary time-dependences. From a different perspective, we focus on the time-dependent systems whose potentials are monotonously decreasing functions of time with arbitrary coordinate-dependences. Time-dependent potential of any coordinate-dependence can be handled analytically by transforming it to a time-independent potential of known solutions if its time-dependence is monotonously decreasing. We do this by a unitary transformation of the wavefunction and variable transformations to change the Schrödinger equation to be time-independent in new variables. These variables are then determined by solving a set of simple differential equations. This way we are able to find and to obtain analytical solutions for time-dependent potentials which we mention above.

Key Words : Time-dependent, Eckart, Barrier

Introduction

Time-dependent Schrödinger equations are not generally solvable in closed forms even in one-dimension despite much theoretical attention has been paid.[1] Only a few systems are analytically solved whose time-dependent potentials are constant,[2,3] linear,[4,5] and quadratic[6-12] functions of coordinate. These problems are usually solved by transforming the Schrödinger equations to the time-independent forms either by introducing invariant operators[4,6-8] or by using canonical variables.[5,9-12]

The most famous problem is a time-dependent quadratic Hamiltonian which has various applications in quantum optics such as the motion of ions in Paul trap[10,11] and the degenerate parametric amplifier.[13] Since Lewis has solved it by the invariant operator approach,[6] the time-dependent harmonic oscillator (TDHO) Hamiltonian has been investigated by different methods for different physical problems.[7,8] The time-dependent square barrier,[2] square well,[3] and linear potential models[4,5] have also been widely studied as these problems have applications in atoms or semiconductors under laser field. No analytical solutions have been yet obtained for other time-dependent systems of physical interests than these three types of Hamiltonian.

In the present work, from a different perspective, we focus attention and solve the Schrödinger equation for systems whose potentials have a certain time-dependence (which is monotonously decreasing) and various coordinate-dependences which are physically as interesting as cases mentioned above. Time-dependent potential of any coordinate-dependence can be solved analytically by transforming it to a time-independent potential of known solutions if its time-dependence is monotonously decreasing. We do this by a unitary transformation of the wavefunction and variable transformations to change the Schrödinger equation to be time-independent in new variables. These variables are then determined by solving a set of differential equations which can be easily handled. This way we are able to find exactly solvable time-dependent potentials which have aforementioned properties.

This paper is organized as follows; in Sec. II we introduce a unitary transform of the wavefunction and new variables to simplify the Schrödinger equation. Then in Sec. III we determine the new variables by solving a set of simple auxiliary differential equations and derive the functional form of time-dependent potentials which can be solved exactly. In Sec. IV we apply the result to two systems which are a monotonously decreasing time-dependent harmonic oscillator and an Eckart barrier of monotonoulsy decreasing height and position in time. We summarize and conclude in Sec. V.

Transformation of Schrödinger Equation

One-dimensional Schrödinger equation for a time-dependent potential is given as

$$i\frac{\partial\Psi(x,t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t), \quad (1)$$

where we assume $\hbar$ is equal to one. Using a unitary transformation, we define $\Psi(x,t)$ as

$$\Psi(x,t) = \Phi(x,t)e^{if(x,t)} \quad (2)$$

Inserting Eq. (2) into Eq. (1), we have

$$-\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2} + 2if_x\frac{\partial}{\partial x} + if_{xx} - f_x^2\right)\Phi + V(x,t)\Phi = i\left(\frac{\partial}{\partial t} + if_t\right)\Phi, \quad (3)$$

where the subscripts stand for partial differentiations with respect to corresponding variables as $f_{xx} = \partial^2 f/\partial x^2$, $f_x = \partial f/\partial x$, and $f_t = \partial f/\partial t$ respectively. Here the subscripted notations are adopted only for $f(x,t)$ so that the Schrödinger equation are easily identified in form. With solutions of the following differential equation

$$-\frac{i}{2m}f_{xx} + \frac{1}{2m}f_x^2 = -f_t. \qquad (4)$$

Eq. (3) will be simplified as below;

$$-\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2} + 2if_x\frac{\partial}{\partial x}\right)\Phi + V(x,t)\Phi = i\frac{\partial\Phi}{\partial t} \qquad (5)$$

In order to determine $f(x,t)$, we assume that $f(x,t) = -i\ln g(x,t)$ and insert it into Eq. (4). We then have a differential equation for $g(x,t)$ as follows;

$$g_t = \frac{i}{2m}g_{xx} \qquad (6)$$

The solution of Eq. (6) is given as[14]

$$g(x,t) = \frac{1}{2}\left(\frac{2m}{i\pi t}\right)^{1/2}\int_{-\infty}^{\infty} g(\xi,0)\exp\left[\frac{im(x-\xi)^2}{2t}\right]d\xi, \qquad (7)$$

which shows that $g(x,t)$ can be determined if $g(x,0)$ is known. You may notice from Eq. (7) that $g(x,t)$ is a time-dependent gaussian with t^{-1} dependence regardless of the functional form of $g(x,0)$. We will determine $g(x,t)$ later.

To remove the second term in the left hand side of Eq. (5), we introduce new variables $z = z(x,t)$ and $s = s(t)$ and replace the differential operators in x and t with ones in z and s, we will have Eq. (5) to be given as

$$-\left[\left(\frac{1}{2m}z_{xx} + \frac{i}{m}z_x f_x\right)\frac{\partial}{\partial z} + \frac{1}{2m}z_x^2\frac{\partial^2}{\partial z^2}\right]\Phi + V(x,t)\Phi$$

$$= i\left(z_t\frac{\partial}{\partial z} + \dot{s}\frac{\partial}{\partial s}\right)\Phi \qquad (8)$$

where $\dot{s} = ds/dt$. Comparing the coeffcients of differential operators for both sides of the equality, we have following equations for the variables z and s

$$-\frac{1}{2m}z_{xx} - \frac{i}{m}z_x f_x = iz_t \qquad (9a)$$

$$z_x^2 = \dot{s}. \qquad (9b)$$

which simplfy Eq. (8) as given below;

$$i\frac{\partial\Phi(z,s)}{\partial s} = -\frac{1}{2m}\frac{\partial^2\Phi(z,s)}{\partial z^2} + V(x,t)\dot{s}^{-1}\Phi(z,s) \qquad (10)$$

If the following relation holds

$$V(x,t) = \dot{s}\,U(z(x,t)), \qquad (11)$$

Eq. (10) would become a Schrödinger equation for the time-independent potential $U(z)$ and the solutions will be easily obtained in new variables as

$$\Phi(z,s) = \phi(z)e^{-iEs} \qquad (12)$$

where $\phi(z)$ is the solution of the time-independent Schrödinger equation given as

$$-\frac{1}{2m}\frac{\partial^2\phi}{\partial z^2} + U(z)\phi = E\phi \qquad (13)$$

and $\Psi(x,t)$ will be obtained from Eqs. (2) and (12) as below;

$$\Psi(x,t) = g(x,t)\phi(z(x,t))e^{-iEs(t)}. \qquad (14)$$

We need to determine the transformed variables $z(x,t)$, $s(t)$, and $g(x,t)$ which is equal to $e^{if(x,t)}$ to completely specify $\Psi(x,t)$.

Determination of $z(x,t)$ and $s(t)$

From Eq. (9b), we note that the first term in Eq. (9a) would vanish. Integrating the resulting equation with respect to t, we obtain

$$z = -\frac{1}{m}\int^t \dot{s}^{1/2}f_x dt_1 + \upsilon(x). \qquad (15)$$

If we differentiate Eq. (15) with x and use $z_x = \dot{s}^{1/2}$, we will have an equation for $\dot{s}$ as

$$\dot{s}^{1/2} = -\frac{1}{m}\int^t f_{xx}\dot{s}^{1/2}dt_1. \qquad (16)$$

where $d\upsilon/dx$ disappears because $\dot{s}$ is a function of t only. We note from Eq. (16) that f_{xx} should be a function of t only and we assume that $f_{xx} = 2i\alpha(t)$. Solving Eq. (16) with this assumption, we have $s(t)$ as given below;

$$s(t) = \int^t \exp\left(-\frac{2}{m}\int^{t_1} f_{xx}dt_2\right)dt_1$$

$$= \int^t \exp\left(-\frac{4i}{m}\int^{t_1} \alpha(t_2)dt_2\right)dt_1. \qquad (17)$$

We obtain $z(x,t)$ by integrating $\dot{s}^{1/2}$ (which is evaluated using Eq. (17)) with respect to x since $z_x = \dot{s}^{1/2}$). The variable $z(x,t)$ is then given as

$$z(x,t) = \int^x \exp\left(-\frac{1}{m}\int^t f_{x_1x_1}dt_1\right)dx_1$$

$$= x\exp\left(-\frac{2i}{m}\int^t \alpha(t_1)dt_1\right) + u(t) \qquad (18)$$

where $u(t)$ is found later. As long as f_{xx} is a function of t only Eq. (18) is equivalent to Eq. (15). Since $f_{xx} = -i\frac{\partial}{\partial x}\left(\frac{1}{g}\frac{\partial g}{\partial x}\right)$, $g(x,t)$ should be written as follows;

$$g(x,t) = \exp[-\alpha(t)x^2 + \beta(t)x + \gamma(t)]. \qquad (19)$$

which is a time-dependent gaussian. Inserting $g(\xi,0) = \exp[-\alpha(0)\xi^2 + \beta(0)\xi + \gamma(0)]$ into Eq. (7), we determine $g(x,t)$ as follows;

$$g(x,t) = \sqrt{\frac{m}{2i\alpha_0 t + m}}\exp\left[\left(\frac{m}{2i\alpha_0 t + m}\right)\left(-\alpha_0 x^2 + \beta_0 x + \frac{i\beta_0^2 t}{2m}\right) + \gamma_0\right] \quad (20)$$

where $\alpha_0 = \alpha(0)$, $\beta_0 = \beta(0)$, and $\gamma_0 = \gamma(0)$ respectively. Comparison of Eqs. (19) and (20) leads to $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ as follows:

$$\alpha(t) = \frac{m\alpha_0}{2i\alpha_0 t + m}, \quad (21a)$$

$$\beta(t) = \frac{m\beta_0}{2i\alpha_0 t + m}, \quad (21b)$$

$$\gamma(t) = \gamma_0 + \frac{1}{2}\left[\frac{i\beta_0^2 t}{2i\alpha_0 t + m} + \ln\left(\frac{m}{2i\alpha_0 t + m}\right)\right]. \quad (21c)$$

As we expect from Eq. (7), functions α and β show t^{-1} dependence and they give proper initial values when $t = 0$. To specify $z(x,t)$ completely, we put Eq. (18) into Eq. (9a) and we have $u(t)$ as

$$u(t) = -\frac{1}{m}\int^t \beta(t_1)\exp\left(-\frac{2i}{m}\int^{t_1}\alpha(t_2)dt_2\right)dt_1$$
$$= -i\frac{\beta_0}{2\alpha_0}(2i\alpha_0 t + m)^{-1}. \quad (22)$$

By inserting $u(t)$ and $\alpha(t)$ into Eqs. (17) and (18), we finally have $z(x,t)$ and $s(t)$ as given below;

$$z(x,t) = (2i\alpha_0 t + m)^{-1}\left(x - i\frac{\beta_0}{2\alpha_0}\right) \quad (23)$$

$$s(t) = -\frac{1}{2i\alpha_0}(2i\alpha_0 t + m)^{-1} \quad (24)$$

For systems whose potentials satisfy Eq. (11), we show that $\Psi(x,t)$ is completely determined as given in Eq. (14) where $g(x,t)$, $z(x,t)$, and $s(t)$ are given in Eqs. (20), (23), and (24) respectively. The function $\phi(z)$ is again a solution of the time-independent Schrödinger equation (Eq. (13)). As we note from Eqs. (11), (23), and (24) the time-dependent potentials for which we can solve Schrödinger equation have monotonously decreasing dependence on t. These time-dependent potentials may be derived from the time-independent forms according to Eq. (11) since $s(t)$ and $z(x,t)$ are explicitly obtained. In the following, we present two examples to show how we use Eq. (11) to devise time-dependent potentials.

Examples

Although we can think of many time-dependent potentials which satisfy Eq. (11), we provide a harmonic oscillator and an Eckart barrier which have monotonous dependence on time as examples.

A. Harmonic oscillator, $V(x,t) = 1/2\, k(t)x^2$ **where** $k(t) = k_0(2i\alpha_0 t + m)^{-4}$

This system can be exactly solved by other methods,[10] but we present it to show that the solution obtained in this work is identical to that by those approaches. According to Eq. (11), we have $U(z) = 1/2\, k_0 z^2$ (Eq. (13) and $\beta_0 = 0$) for the potential $V(x,t)$ and have solutions $\Phi_n(z,s)$ (Eq. (12)) given as

$$\Phi_n(z,s) = N_n\exp\left(-\frac{1}{2}a^2z^2 - iE_n s\right)H_n(az) \quad (25)$$

which is obtained as a stationary harmonic oscillator wavefunction in z, and s. N_n, a, and E_n are $\left(\frac{a}{2^n n!\sqrt{\pi}}\right)^{1/2}$, $(mk_0)^{1/4}$, and $\left(n + \frac{1}{2}\right)\sqrt{\frac{k_0}{m}}$ respectively.

H_n is the Hermite polynomial. The final wavefunction $\Psi_n(x,t)$ will become as follows:

$$\Psi_n(x,t) = N_n\exp\left(-\alpha(t)x^2 + \gamma(t) - \frac{1}{2}a^2z^2(x,t) - iE_n s(t)\right) \times H_n(az(x,t)) \quad (26)$$

where $\alpha(t)$, $\gamma(t)$, $z(x,t)$, and $s(t)$ are given in Eqs. (21a), (21c), (23), and (24) respectively.

B. Eckart barrier, $V(x,t) = \frac{V_0}{(2i\alpha_0 t + m)^2}\cosh^{-2}\left(\frac{x}{2i\alpha_0 t + m}\right)$

Time-dependent barrier in general, whether it is simple or realistic, is an important model for the study of condensed phase processes such as dissociation or diffusion on surface.[15] Although analytical studies on time-dependent square barriers have been made, no such work on a more realistic time-dependent Eckart barrier has been done yet. For the first time, we present an analytical solution for an Eckart barrier with monotonously decreasing height and position in time given above.

The corresponding time-independent potential $U(z)$ for $V(x,t)$ will be $U(z) = V_0\cosh^{-2}z$ (with $\beta_0 = 0$) and the solution becomes:

$$\Phi(z,s) = \mathrm{sech}^{-ik} z\, F\left(-ik-\delta, -ik+\delta+1, -ik+1, \frac{1}{2}(1-\tanh z)\right)e^{-iEs} \quad (27)$$

where $k = \sqrt{2mE}$, $\delta = \frac{1}{2}(-1 + \sqrt{1-8mV_0})$, and F is the hypergeometric function. Then $\Psi(x,t)$ will be

$$\Psi(x,t) = e^{-\alpha(t)x^2 + \gamma(t) - iEs(t)}\Phi(z(x,t), s(t)) \quad (28)$$

where $\Phi(z(x,t), s(t))$ is given in Eq. (27), and $\alpha(t)$, $\gamma(t)$, $z(x,t)$, and $s(t)$ are same as the case **A**.

Conclusion

We derive an explicit time-dependent potential for which Schrödinger equation can be exactly solved by a series of transformations for wavefunctions and variables. Schrödinger equation for any potentials of monotonously decreasing functions of time can be transformed to time-independent form in terms of new variables and the solution is obtained as a product of a time-dependent gaussian and a stationary wavefunction for the corresponding time-independent potential in new variables. In this work, we thus extend exactly solvable time-dependent potentials to which have more general coordinate-dependence than those have constant, linear, and quadratic dependence reported before.

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