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APPLICATIONS OF A COLUMN–REDUCED ORTHOGONAL RATIONAL MATIRX FUNCTION

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Abstract Applications of a column-reduced orthogonal rational matrix function to McMillan degrees and Wiener-Hopf factorizations are considered.

1. Introduction

For a subset σ of the complex plane and an $m \times m$ constant matrix V such that $V^T = \alpha V$, $\alpha = \pm 1$, let

$$\omega = (C_-, C_+, A_\pi; A_\zeta, B_+, B_-; \Gamma) \tag{1.1}$$

be an *admissible interpolation data set* [3] of sizes $M \times n_\pi$, $M \times n_\pi$, $n_\pi \times n_\pi$, $n_\zeta \times n_\zeta$, $n_\zeta \times M$, $n_\zeta \times M$, $n_\pi \times n_\zeta$, respectively, for which

$$\hat{\omega} \sim \hat{\omega}^T,$$

i.e., $\hat{\omega}$ is *similar* to $\hat{\omega}^T$, where

$$\hat{w} = (\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; A_\zeta, [B_+, B_-]; \Gamma). \tag{1.2}$$

That is, $\hat{\omega}$ defined by (1.2) is a σ-*admissible Sylvester data set* such that the union of the spectrums $\sigma(A_\pi) \cup \sigma(A_\zeta)$ is a subset of σ, (C_π, A_π) is a *null-kernel pair*, i.e, $\bigcap_{j=0}^{n_\pi - 1} \operatorname{Ker} C_\pi A_\pi^j = \{0\}$, (C_π, A_π)

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is a *null-kernel pair*, i.e. $\bigcap_{j=0}^{n_\pi - 1} \operatorname{Ker} C_\pi A_\pi^j = \{0\}$, (A_ζ, B_ζ) is a *full-range pair*, i.e, $\sum_{j=0}^{n_\zeta - 1} \operatorname{Im} A_\zeta^j B_\zeta = \mathbb{C}^{n_\zeta}$, and Γ satisfies the matrix equation $\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi$, where $C_\pi = [C_+ C_-]$ and $B_\zeta = [B_+, B_-]$. For a given τ, we associate another set of matrices $\tau^T = (-V^{-1}B_\zeta^T, A_\zeta^T; A_\pi^T, C_\pi^T V; \Gamma^T)$. Two σ-admissible data sets $\tau = (C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma)$ and $\tau' = (C'_\pi, A'_\pi; A'_\zeta, B'_\zeta; \Gamma')$ are *similar* if there exist invertible matrices Φ and Ψ such that $C_\pi = C'_\pi \Phi$, $A_\pi = \Phi^{-1} A'_\pi \Phi$, $A_\zeta = \Psi^{-1} A'_\zeta \Psi$, $B_\zeta = \Psi^{-1} B'_\zeta$, and $\Gamma = \Psi^{-1}\Gamma'\Phi$. If we want to emphasize the matrices Φ and Ψ we say that τ is (Φ, Ψ)-similar to τ'. If τ is similar to τ^T, τ is said to be *symmetric* and is $(\Phi, \alpha\Phi^T)$ - similar to τ^T for an invertible matrix Φ (see [7]). For an $M \times M$ rational matrix function $\Theta(z)$ and a Sylvester data set τ, Θ is said to have τ as its $\mathbb{C}$-*null-pole triple* if

$$\Theta P_M = \{C_\pi (zI - A_\pi)^{-1} x + h(z) \mid x \in \mathbb{C}^{n_\pi}, h \in P_M \text{ such that } \sum_{z_0 \in \mathbb{C}} Res_{z=z_0} (zI - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x\},$$

where P_M is the set of polynomials with coefficients in $\mathbb{C}^M$. In [8], Kim proved the following results.

THEOREM 1.1. *If τ is a given σ-admissible Sylvester data set which is similar to τ^T, then there exists an $m \times m$ rational matrix function $\Theta(z)$ for which Θ has τ as its $\mathbb{C}$-null-pole triple, Θ is column reduced at infinity, $\Theta^T(z)V\Theta(z) = P$, $\forall z \in \mathbb{C}_\infty$, where $P = [p_{ij}]$ is an $m \times m$ constant matrix with*

$$p_{ij} = \begin{cases} 1, & 1 \le i \le [\frac{m}{2}], \quad j = m+1-i, \\ \alpha, & [\frac{m}{2}] < i \le m, \quad j = m+1-i, \\ 0, & otherwise, \end{cases}$$

where for a real number s, $[s]$ denotes the largest integer not exceeding s. In this case the column indices of Θ are

$$-\alpha_1, -\alpha_2, \cdots, -\alpha_t, \underbrace{0, \cdots, 0}_{(m-2t) \text{ times}}, \alpha_t, \cdots, \alpha_1,$$

where $\alpha_1 \geq \cdots \geq \alpha_t$ *are the nonzero observability indices of* (C_π, A_π).

A rational matrix function satisfying (1.5) is said to be V-*orthogonal.*

In this paper, we consider some applications of Theorem 1.1 to a nonhomogeneous interpolation problem to find the minimal possible McMillan degree [5] of symmetric interpolants and to Wiener-Hopf factorizations.

2. Applications

THEOREM 2.1. *Let* ω *be defined by (1.1) with the property (1.2) and* F_{min} *be a minimal interpolating function of* ω. *Then the McMillan degree of* F_{min}, *denoted by* $\delta(F_{min})$, *is given by*

$$\delta(F_{min}) = n_\pi + \sum_{j=1}^{M} \kappa_{i_j},$$

where κ_{i_j} *are the column indices of* $\Theta(z)$ *constructed as in Theorem 1.1 with* $\hat{\omega}$ *instead of* τ.

Proof. The theorem can be obtained by applying [5] and [7] to Theorem 1.1.

REMARK 2.2. (a) Finding a minimal interpolant for a given admissible interpolation data set without the extra constraint $\hat{\omega} \sim \hat{\omega}^T$ was studied in [1] and [2]. The first one is concerned with the scalar case and the second addresses about the matrix case.

(b) Since the sum of the observability indices of the pair $(\begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi)$ is equal to n_π , the size of A_π,

$$0 \leq \delta(F_{min}) \leq 2n_\pi.$$

THEOREM 2.3. *If $\Theta(z)$ is an $m \times m$ rational matrix function satisfying*

$$\Theta^T(z)V\Theta(z) = V,$$

then there exists a Wiener-Hopf factorization of $\Theta(z)$ at infinity which is given by

$$\Theta(z) = \Theta_-(z)D(z)\Theta_+(z) \tag{2.1}$$

where

$$D(z) = diag(z^{-\alpha_1}, \cdots, z^{-\alpha_t}, 1, \cdots, 1, z^{\alpha t}, \cdots, z^{\alpha_1})$$

and $\alpha_1 \geq \cdots \geq \alpha_t$ are nonzero observability indices of a $\mathbb{C}$-pole pair of $\Theta(z)$. Moreover

$$\Theta_-^T V\Theta_- = P$$

and

$$\Theta_+^T P\Theta_+ = V,$$

where P is as in Theorem 1.1.

Proof. If we assume τ is a $\mathbb{C}$-null-pole triple of $\Theta(z)$, τ is similar to τ^T. With τ, we construct an $m \times m$ rational matrix function as in Theorem 1.1 so that

$$\begin{aligned} &\Theta_0^T V\Theta_0 = V, \\ &\Theta_0 \text{ has } \tau \text{ as its } \mathbb{C} - \text{null-pole triple}, \\ &\Theta_0 \text{ is column reduced at infinity.} \end{aligned}$$

Then, $\Theta_0(z)$ is factored as

$$\Theta_0(z) = \Theta_-(z)D(z), \tag{2.2}$$

where $\Theta_-(z)$ is biproper,

$$D(z) = \text{diag}(z^{-\alpha_1}, \cdots, z^{-\alpha_t}, 1, \cdots, 1, z^{\alpha t}, \cdots, z^{\alpha_1}),$$

and $\alpha_1 \geq \cdots \geq \alpha_t$ are the nonzero controllability (observability) indices of τ at infinity by the construction of $\Theta_0(z)$ in Theorem 1.1. It can be easily seen that

$$\Theta_-^T V \Theta_- = P \tag{2.3}$$

from (2.2) and the fact that

$$P = \Theta_0^T V \Theta_0 = D^T \Theta_-^T V \Theta_- D$$

and

$$DPD = P. \tag{2.4}$$

To show (2.1), we note that there exists an unimodular matrix function $\Theta_+(z)$ for which

$$\Theta(z) = \Theta_0(z)\Theta_+(z)$$

because $\Theta(z)$ and $\Theta_0(z)$ have the same $\mathbb{C}$-null-pole triple. From the above equality and $\Theta_0^T V \Theta_0 = P$, we have

$$\Theta_+^T P \Theta_+ = V.$$

REMARK 2.4. For the details of Wiener-Hopf factorization of a rational matrix functions, readers are referred to [6]. Wiener -Hopf factorization of a rational matrix function which is colum reduced at infinity but not necessarily V-orthogonal is studied in [4].

References

1. A.C. Antoulas and B.D.O. Anderson, *On the scalar rational interpolation problem*, IMA J. Math. Control and Information **3** (1986), 61–88.
2. A.C. Antoulas, J.A. Ball, J.A. Kang(Kim), and J.C. Willems, *On the solution of the minimal interpolation problem*, Linear Alg. and Appl. **137/138** (1990), 511–573.
3. J.A. Ball, I. Gohberg, and L. Rodman, *Interpolation of Rational Matrix Functions*, Birkhauser OT 45, Basel, 1990.
4. J.A. Ball, M.A. Kaashoek, G. Groenewald, and J. Kim, *Column reduced rational matrix functions with given null-pole data in the complex plane*, Linear Alg. and Appl. **203/204** (1994), 67–110.

5. J.A. Ball, J. Kim, L. Rodman, and M. Verma, *Minimal degree coprime factorizations of rational matrix functions*, Linear Alg. and Appl. **186** (1993), 117–164.
6. H. Bart, I. Gohberg, and M.A. Kaashoek, *Explicit Wiener-Hopf factorization and realization, in Constructive Methods of Wiener-Hopf Factorization (I. Gohberg and M.A. Kaashoek ed.)*, Birkh- auser Verlag OT 21, Basel, 1986, p. 235–316.
7. J. A. Ball and J. Kim, *Bitangential interpolation problems for symmetric rational matrix functions*, Linear Alg. and Appl. **241-243** (1996), 113–152.
8. J. Kim, *A column reduced V-orthogonal rational matrix function with the prescribed null-pole structure in the complex plane*, Preprint, 2002.