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EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR CERTAIN SECOND–ORDER ODES

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Abstract In this paper we have shown conditions of R for the existence of the solution to the problem of the type $-u'' = ue^{\varepsilon u}$ on the interval $(0, R)$ with boundary and other conditions as in $(1-\varepsilon)-(4)$ for various ε.

1. Introduction

We study here about existence and nonexistence of solutions for the following problem:

$$-\frac{d^2u}{dx^2} = ue^{\varepsilon u}, x \in (0, R), \quad R \text{ could be } +\infty \qquad (1-\varepsilon)$$

$$\frac{du}{dx}(0) = 0, \qquad (2)$$

$$u(R) = 0, \qquad (3)$$

$$u \text{ is positive and decreasing on } (0, R). \qquad (4)$$

This is an extension of [JN]. where we considered the specific case $\varepsilon = 1$. These are steady-state problems of the associated semilinear parabolic PDEs

$$\begin{array}{l} u_t - u_{xx} = ue^{\varepsilon u}, \quad (x,t) \in (0, R) \times (0, T). \\ \quad \text{plus suitable initial and boundary conditions.} \end{array} \qquad (5-\varepsilon)$$

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In [J] we have studied the problem $(5-\varepsilon)$ for the case $\varepsilon = 1$ and investigated the condition for the existence of the bounded solution and for the existence of blowup solution as well as blowup behavior of the blowup solution.

In this article we have found the relationship between R and ε by observing the changing behavior of the time length using the elliptic integrals (see [S]).

2. The Case $\varepsilon = 0$.

This case is quite simple. From the problem $(1-0) - u'' = u$, we get the first integral $H(u,v) = \frac{v^2}{2} + G(u) = h$, where $G(u) = \frac{u^2}{2}$ and $v = u'$. Let $h = G(\alpha)$, and let $S(\alpha) = \int_0^\alpha \frac{du}{\sqrt{\alpha^2 - u^2}}$. Let $u(0) = \alpha = G^{-1}(h)$. The problem (1-0) has infinitely many solutions with $R = \frac{\pi}{2}$. If $R \notin (0, \frac{\pi}{2}) \bigcup (\frac{\pi}{2}, +\infty)$, the problem has no solution.

From the description above, we have the following:

COROLLARY 1. *Initial value problem* $-u''(x) = u(x), \quad x \in \mathbb{R}, \quad u'(0) = 0, \quad u(0) > 0$ *has infinitely many symmetric oscillating periodic (hence bounded) solutions with a fixed period* 2π. *In fact, these are all of the solutions.*

Note that if $u(0) = \alpha$, then $u'(\frac{\pi}{2}) = -\alpha$. Also the phase-portraits are filled with the concentric circles (periodic solutions) of the fixed period 2π.

3. The Case $\varepsilon > 0$.

First integral of the problem $(1-\varepsilon)$ with $\varepsilon > 0$ is $H(u,v) = \frac{v^2}{2} + G_\varepsilon(u) = h$, where $G_\varepsilon(u) = \frac{\varepsilon u - 1}{\varepsilon^2} e^{\varepsilon u} + \frac{1}{\varepsilon^2}$ and $v = u'$. $G'_\varepsilon(u) = ue^{\varepsilon u} > 0$ for $u > 0$. Hence the function $G_\varepsilon : [0,\infty) \to [0,\infty)$ has an inverse function. To rewrite the problem $(1-\varepsilon)$ with $\varepsilon > 0$ as a system of the first order equation, setting $v = u'$, we obtain

$$\begin{aligned} u' &= v \\ v' &= -ue^{\varepsilon u}. \end{aligned} \tag{6}$$

Note that $H(0,0) = 0$ and (0.0) is a center. We can show that if $h \geqslant \frac{1}{\varepsilon^2}$, then v approaches to $\pm\sqrt{2\left(h - \frac{1}{\varepsilon^2}\right)}$ as x tends to $\pm\infty$ respectively, so, as u tends to $-\infty$.

Also the curve is a periodic orbit for $0 < h < \frac{1}{\varepsilon^2}$. Let $\alpha = G_\varepsilon^{-1}(h)$.

Then $\frac{du}{dx} = v = -\sqrt{2(G_\varepsilon(\alpha) - G_\varepsilon(u))}$ since $\frac{du}{dx} \leq 0$ by the condition (4). From the equation $(1-\varepsilon), (2), (3)$ and (6), $u(0) = \alpha = G_\varepsilon^{-1}(h)$, $v(0) = 0$, $u(R) = 0$, $v(R) = \sqrt{-2h}$, so

$$\int_{u(0)}^{u(R)} \frac{du}{-\sqrt{2(G_\varepsilon(\alpha) - G_\varepsilon(u))}} = \int_0^R dx = R. \tag{7}$$

Now we define $S_\varepsilon(\alpha) :\equiv \int_0^\alpha \frac{du}{\sqrt{2(G_\varepsilon(\alpha) - G_\varepsilon(u))}}$. Note that the left hand side of (7) equals to $S_\varepsilon(\alpha)$. To prove that the problem $(1-\varepsilon)-(4)$ has a unique positive solution, it is enough to show that there exists a unique $\alpha \in (0, \infty)$ such that $S_\varepsilon(\alpha) = R$. We investigate the properties of $S_\varepsilon(\alpha)$. Of course, $S_\varepsilon(\alpha)$ is continuous on $(0, +\infty)$.

LEMMA 2. $S_\varepsilon(0^+) = \lim_{\alpha \to 0^+} S_\varepsilon(\alpha) = \frac{\pi}{2}^-$.

Proof. Let $0 < u < \alpha$. Then

$$\begin{aligned} G_\varepsilon(\alpha) - G_\varepsilon(u) &= G_\varepsilon(\sqrt{\beta}) - G_\varepsilon(\sqrt{w}) \\ &= G'_\varepsilon(\sqrt{w^*})\frac{1}{2\sqrt{w^*}}(\beta - w) \\ &\qquad \text{(for some } w < w^* = w^*(w.\beta) < \beta) \\ &= e^{\varepsilon\sqrt{w^*}}\frac{(\alpha^2 - u^2)}{2}. \end{aligned}$$

Let $u^* = u^*(u,\alpha) = \sqrt{s^*}$. Then

$$S_\varepsilon(\alpha) = \int_0^\alpha \frac{du}{\sqrt{2(G_\varepsilon(\alpha) - G_\varepsilon(u))}} = \int_0^\alpha \frac{du}{\sqrt{e^{\varepsilon u^*}(\alpha^2 - u^2)}}.$$

and $e^{-\frac{\varepsilon\alpha}{2}} \int_0^\alpha \frac{du}{\sqrt{\alpha^2 - u^2}} \le S_\varepsilon(\alpha) \le \int_0^\alpha \frac{du}{\sqrt{\alpha^2 - u^2}}$. Now, let $\alpha \to 0^+$. Then we obtain the required result.

LEMMA 3. *$S_\varepsilon'(\alpha) < 0$ for all $\alpha > 0$.*

Proof. Note that $S_\varepsilon(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\alpha\cos\theta d\theta}{\sqrt{2(G_\varepsilon(\alpha) - G_\varepsilon(\alpha\sin\theta))}}$. Let $N_\varepsilon(\alpha) = G_\varepsilon(\alpha) - G_\varepsilon(\alpha\sin\theta)$. Then

$$S_\varepsilon'(\alpha) = -\frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} N_\varepsilon(\alpha)^{-\frac{3}{2}} N_\varepsilon'(\alpha)\alpha\cos\theta d\theta + \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} N_\varepsilon(\alpha)^{-\frac{1}{2}} \cos\theta d\theta.$$

So, $2\sqrt{2}S_\varepsilon'(\alpha) = \int_0^{\frac{\pi}{2}} N_\varepsilon^{-\frac{3}{2}}(2N_\varepsilon(\alpha) - N_\varepsilon'(\alpha)\alpha)\cos\theta d\theta.$

$$\begin{aligned} 2N_\varepsilon(\alpha) - N_\varepsilon'(\alpha)\alpha &= 2(G_\varepsilon(\alpha) - G_\varepsilon(\alpha\sin\theta)) \\ &\quad - \alpha(G_\varepsilon'(\alpha) - G_\varepsilon'(\alpha\sin\theta)\sin\theta) \\ &= [2G_\varepsilon(\alpha) - \alpha G_\varepsilon'(\alpha)] \\ &\quad - [2G_\varepsilon(\alpha\sin\theta) - G_\varepsilon'(\alpha\sin\theta)\sin\theta] \\ &\quad \text{letting} \quad \gamma_\varepsilon(x) = 2G_\varepsilon(x) - xG_\varepsilon'(x)) \\ &= \gamma_\varepsilon(\alpha) - \gamma_\varepsilon(\alpha\sin\theta). \end{aligned}$$

$\gamma_\varepsilon'(x) = -\varepsilon x e^{\varepsilon x} < 0.$
Hence γ_ε is strictly decreasing on $(0,\infty)$.
Thus, $2\sqrt{2}S_\varepsilon'(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\gamma_\varepsilon(\alpha) - \gamma_\varepsilon(\alpha\sin\theta)}{G_\varepsilon(\alpha) - G_\varepsilon(\alpha\sin\theta)^{\frac{3}{2}}} \cos\theta d\theta < 0.$

LEMMA 4. $\lim_{\alpha\to\infty} S_\varepsilon(\alpha) = 0.$

Proof. If we rewrite $S'_\varepsilon(\alpha)$,

$$S'_\varepsilon(\alpha) = \frac{1}{\alpha}S_\varepsilon(\alpha) - \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} N_\varepsilon(\alpha)^{-\frac{3}{2}} N'_\varepsilon(\alpha)\alpha\cos\theta d\theta.$$

$$\frac{S'_\varepsilon(\alpha)}{S_\varepsilon(\alpha)} = \frac{1}{\alpha} - \frac{1}{2}\frac{\int_0^{\frac{\pi}{2}} N_\varepsilon(\alpha)^{-\frac{3}{2}} N'_\varepsilon(\alpha)\alpha\cos\theta d\theta}{\int_0^{\frac{\pi}{2}} N_\varepsilon(\alpha)^{-\frac{3}{2}} N_\varepsilon(\alpha)\alpha\cos\theta d\theta}.$$

$$\begin{aligned}\varepsilon N_\varepsilon(\alpha) &= N'_\varepsilon(\alpha) - \frac{1}{\varepsilon}(e^{\varepsilon\alpha} - e^{\varepsilon\alpha\sin\theta}) \\ &\qquad - \alpha\sin\theta e^{\varepsilon\alpha\sin\theta}(1-\sin\theta) \\ &\le N'_\varepsilon(\alpha).\end{aligned}$$

Note that α and ε are positive and $0 \le \theta \le \frac{\pi}{2}$.
Hence

$$\frac{N'_\varepsilon(\alpha)}{N_\varepsilon(\alpha)} \ge \varepsilon, \text{ and } \frac{S'_\varepsilon(\alpha)}{S_\varepsilon(\alpha)} \le \frac{1}{\alpha} - \frac{\varepsilon}{2}.$$

Integrating this differential inequality over 1 to α yields $S_\varepsilon(\alpha) \le \frac{\alpha}{e^{\frac{\varepsilon(\alpha-1)}{2}}} S_\varepsilon(1)$.
From this $S_\varepsilon(\alpha) \to 0$ as $\alpha \to +\infty$.
This completes the proof.

From Lemma 2, Lemma 3 and Lemma 4, we have the following.

THEOREM 5. *The problem* $(1-\varepsilon) - (4)(\varepsilon > 0)$ *has a unique solution if and only if* $0 < R < \frac{\pi}{2}$.

Proof. By Lemmas 1-3, the range of $S_\varepsilon(\alpha)$ on the domain $(0,\infty)$ is $(0, \frac{\pi}{2})$, and from this our claim is clear.

It is obvious to prove the next corollary from the equation and phase-portraits.

COROLLARY 6. *There is a unique solution $u_\varepsilon(x)$ of the problem $u'' + ue^{\varepsilon u} = 0$ $(x \in \mathbb{R})$ satisfying the following :*

(a)$H(u_\varepsilon, u'_\varepsilon) \equiv \frac{1}{\varepsilon^2}$ *on* $\mathbb{R}$.

(b) $\lim_{|x|\to\infty} u_\varepsilon(x) = -\infty$.

(c) $\lim_{|x|\to\infty} u'_\varepsilon(x) = 0$.

(d)For each $h \in (0, \frac{1}{\varepsilon^2})$, *there exists a unique oscillating periodic solution with period*
$T(h)^-$ *converging* 2π *as* $h \to 0^+$.

(e) For each $h \in (\frac{1}{\varepsilon^2}, +\infty)$, *there exists a unique unbounded solution* u *satisfying :*

$$\lim_{|x|\to\infty} u(x) = -\infty \quad \textit{and} \quad \lim_{x\to\pm\infty} u'(x) = \mp\sqrt{2\left(h - \frac{1}{\varepsilon^2}\right)}$$

4. The Case $\varepsilon < 0$.

Situations are almost similar as in §3 except the sign of ε, which is negative here. H, G_ε and S_ε are same as in the previous section.

LEMMA 7. $S_\varepsilon(0^+) := \lim_{\alpha\to 0^+} S_\varepsilon(\alpha) = \frac{\pi}{2}^+$.

Proof. Let $0 < u < \alpha$. Then

$$\begin{aligned} G_\varepsilon(\alpha) - G_\varepsilon(u) &= G_\varepsilon(\sqrt{\beta}) - G_\varepsilon(\sqrt{w}) \\ &= G'_\varepsilon(\sqrt{w^*})\frac{1}{2\sqrt{w^*}}(\beta - w) \\ &\qquad (\text{ for some } \ w < w^* = w^*(w,\beta) < \beta) \\ &= e^{\varepsilon\sqrt{w^*}}\frac{(\alpha^2 - u^2)}{2} \end{aligned}$$

Let $u^* = u^*(u,\alpha) = \sqrt{w^*}$.

Then

$$S_\varepsilon(\alpha) = \int_0^\alpha \frac{du}{\sqrt{2(G_\varepsilon(\alpha) - G_\varepsilon(u))}} = \int_0^\alpha \frac{du}{\sqrt{e^{\varepsilon u^*}(\alpha^2 - u^2)}}.$$

and

$$\int_0^\alpha \frac{du}{\sqrt{\alpha^2 - u^2}} \le S_\varepsilon(\alpha) \le e^{-\frac{\varepsilon\alpha}{2}} \int_0^\alpha \frac{du}{\sqrt{\alpha^2 - u^2}}.$$

Now, let $\alpha \to 0^+$. Then we obtain the required result.

LEMMA 8. *$S'_\varepsilon(\alpha) > 0$ for all $\alpha > 0$.*

Proof. Note that $S_\varepsilon(\alpha) = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\alpha\cos\theta d\theta}{\sqrt{2(G_\varepsilon(\alpha) - G_\varepsilon(\alpha\sin\theta))}}$. Let $N_\varepsilon(\alpha) = G_\varepsilon(\alpha) - G_\varepsilon(\alpha\sin\theta)$. Then

$$\begin{aligned} S'_\varepsilon(\alpha) &= -\frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} N_\varepsilon(\alpha)^{-\frac{3}{2}} N'_\varepsilon(\alpha)\alpha\cos\theta d\theta \\ &\quad + \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} N_\varepsilon(\alpha)^{-\frac{1}{2}} \cos\theta d\theta. \end{aligned}$$

So,

$$2\sqrt{2}S'_\varepsilon(\alpha) = \int_0^{\frac{\pi}{2}} N_\varepsilon^{-\frac{3}{2}}(2N_\varepsilon(\alpha) - N'_\varepsilon(\alpha)\alpha)\cos\theta d\theta.$$

$$\begin{aligned} 2N_\varepsilon(\alpha) - N'_\varepsilon(\alpha)\alpha &= 2(G_\varepsilon(\alpha) - G_\varepsilon(\alpha\sin\theta)) \\ &\quad - \alpha(G'_\varepsilon(\alpha) - G'_\varepsilon(\alpha\sin\theta)\sin\theta) \\ &= [2G_\varepsilon(\alpha) - \alpha G'_\varepsilon(\alpha)] \\ &\quad - [2G_\varepsilon(\alpha\sin\theta) - G'_\varepsilon(\alpha\sin\theta)\sin\theta] \\ &\quad (\text{ letting } \gamma_\varepsilon(x) = 2G_\varepsilon(x) - xG'_\varepsilon(x)) \\ &= \gamma_\varepsilon(\alpha) - \gamma_\varepsilon(\alpha\sin\theta). \end{aligned}$$

$\gamma'_\varepsilon(x) = -\varepsilon x e^{\varepsilon x} > 0$.
Hence γ_ε is strictly increasing on $(0, \infty)$.
Thus,

$$2\sqrt{2}S'_\varepsilon(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\gamma_\varepsilon(\alpha) - \gamma_\varepsilon(\alpha\sin\theta)}{G_\varepsilon(\alpha) - G_\varepsilon(\alpha\sin\theta)^{\frac{3}{2}}} \cos\theta d\theta > 0.$$

LEMMA 9. $\lim_{\alpha\to\infty} S_\varepsilon(\alpha) = \infty$.

Proof. In the proof of Lemma 7,

$$S_\varepsilon(\alpha) = \int_0^\alpha \frac{du}{\sqrt{2(G_\varepsilon(\alpha) - G_\varepsilon(u))}} = \int_0^\alpha \frac{du}{\sqrt{e^{\varepsilon u^*}(\alpha^2 - u^2)}}$$

for some $0 < u < u^* = u^*(u, \alpha) < \alpha$.
Hence

$$S_\varepsilon(\alpha) \geq \int_0^\alpha \frac{e^{-\frac{\varepsilon u}{2}}\, du}{\sqrt{\alpha^2 - u^2}} = \int_0^{\frac{\pi}{2}} e^{-\frac{\varepsilon\alpha \sin\theta}{2}}\, d\theta.$$

Last integral grows without bounds as α tends to infinity.
This completes the proof.

From Lemma 7, Lemma 8 and Lemma 9, we have the following.

THEOREM 10. *The problem* $(1-\varepsilon)-(4)$ $(\varepsilon < 0)$ *has a unique solution if and only if* $\frac{\pi}{2} < R < +\infty$.

Proof. By Lemma 7-9, the range of $S_\varepsilon(\alpha)$ on the domain $(0, \infty)$ is $(\frac{\pi}{2}, +\infty)$, and from this our claim is clear.

COROLLARY 11. *There is a unique solution* $u_\varepsilon(x)$ *of the problem* $u'' + ue^{\varepsilon u} = 0$ $(x \in \mathbb{R})$ *satisfying the following :*

(a) $H(u_\varepsilon, u'_\varepsilon) \equiv \frac{1}{\varepsilon^2}$ on $\mathbb{R}$.

(b) $\lim_{|x|\to\infty} u_\varepsilon(x) = +\infty$.

(c) $\lim_{|x|\to\infty} u'_\varepsilon(x) = 0$.

(d) *For each* $h \in (0, \frac{1}{\varepsilon^2})$, *there exists a unique oscillating periodic solution with period* $T(h)^+$ *converging* 2π *as* $h \to 0^+$.

(e) *For each* $h \in (\frac{1}{\varepsilon^2}, +\infty)$, *there exists a unique unbounded solution* u *satisfying :*

$$\lim_{|x|\to\infty} u(x) = +\infty \quad \text{and} \quad \lim_{x\to\pm\infty} u'(x) = \pm\sqrt{2\left(h - \frac{1}{\varepsilon^2}\right)}$$

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