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THE EXISTENCE OF SOLUTIONS OF LINEAR MULTIVARIABLE SYSTEMS IN DESCRIPTOR FROM FORM

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Abstract The solutions of a homogeneous system in state space form

$$\dot{x} = Ax$$

are to the form $x = e^{At}x_0$ and the solutions of an inhomogeneous system

$$\dot{x} = Ax(t) + f(t)$$

are to the form $x = e^{At}x_0 + \int_0^t e^{A(t-\tau)}f(\tau)d\tau$. In this note we show that the solution of descriptor systems under some conditions exists, and is unique, moreover it is interesting to know the solutions of descriptor system are schematically like the solutions as in the state space form. Also we will give some algorithms to compute these solutions.

1. Introduction

We study a linear multivariable inhomogeneous system in descriptor form:

$$E\dot{x} = Ax(t) + f(t), \qquad x(0) = x_0 \tag{1.1}$$

where, x and f are functions of time with values in $\mathcal{X} = \Re^n$ and E and A are $n \times n$ real, constant matrices. Here we have followed the works of Wonham[2], Fletcher and Aasaraai[1].

In first step we study the homogeneous system:

$$E\dot{x} = Ax(t), \qquad x(0) = x_0 \tag{1.2}$$

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DEFINITION 1.1. A subspace $\mathcal{V}$ will be said to be (A, E)-invariant if $A\mathcal{V} \subseteq E\mathcal{V}$.

LEMMA 1.1. *Suppose $x(t)$ satisfies (1.2) and let $\mathcal{V}$ be the smallest subspace of $\mathcal{X}$ such that $x(t) \in \mathcal{V}$ for all $t > 0$. Then $\mathcal{V}$ is an (A, E)-invariant subspace, i.e. $A\mathcal{V} \subseteq E\mathcal{V}$. [1]*

LEMMA 1.2. *Define $\mathcal{I} = \mathcal{I}(A, E; \mathcal{X})$ as the class of (A, E)-invariant subspaces of $\mathcal{X}$. Then $\mathcal{I}$ is closed under the operation of subspace addition, and $\mathcal{I}$ contains a supremal element $\mathcal{V}^* = sup\, \mathcal{I}$.*

Proof. The closedness of $\mathcal{I}$ is obvious, and since $\mathcal{X}$ is finite-dimensional there is an element $\mathcal{V}^* \in \mathcal{I}$ of greatest dimension. If $\mathcal{V} \in \mathcal{I}$ we have that $\mathcal{V} + \mathcal{V}^* \in \mathcal{I}$ and so

$$dim(\mathcal{V}^*) \geq dim(\mathcal{V} + \mathcal{V}^*) \geq dim(\mathcal{V}^*);$$

that is, $\mathcal{V}^* = \mathcal{V} + \mathcal{V}^*$, hence, $\mathcal{V} \subseteq \mathcal{V}^*$ and so $\mathcal{V}^*$ is supremal.

2. The existence of solutions of homogeneous systems

THEOREM 2.1. *Let $\mathcal{V}^* = sup\mathcal{I}$. Then:*

(i) *There exists a linear transformation $\mathcal{X} \to \mathcal{X}$ with matrix L such that $L\mathcal{V}^* \subseteq \mathcal{V}^*$ and*

$$Av = ELv \text{ for } v \in \mathcal{V}^* \tag{2.1}$$

(ii) *The homogeneous system (1.2) has a solution if and only if $x(0) \in \mathcal{V}^*$. Moreover any solution $x(t)$ lies in $\mathcal{V}^*$ for all $t > 0$.*

(iii) *There is a unique solution of (1.2) for $x_0 \in \mathcal{V}^*$ if and only if $ker E \cap \mathcal{V}^* = 0$.*

Then a solution of (1.2) for $x_0 \in \mathcal{V}^*$ is

$$x(t) = e^{Lt}x_0 \tag{2.2}$$

Proof. To establish the existence of L we let $v_1, ..., v_r$ be a basis of $\mathcal{V}^*$. Then, since $A\mathcal{V}^* \subseteq E\mathcal{V}^*$, we must have $Av_i = Ew_i$ where $w_i \in \mathcal{V}^*(i = 1, ..., r)$. Define L by $Lv_i = w_i(i = 1, ..., r)$ and define L arbitrarily on the remainder of a basis of $\mathcal{X}$. Clearly now $L\mathcal{V}^* \subseteq \mathcal{V}^*$ moreover $e^{Lt}\mathcal{V}^* \subseteq \mathcal{V}^*$. This completes the proof of part (i) and with it the "if" part of (ii). Conversely, suppose $x(t)$ satisfies (1.2) with $x(t) \in \mathcal{X}$ for all $t \geq 0$. Let $\mathcal{V}$ be the smallest subspace of $\mathcal{X}$ such that $x(t) \in \mathcal{V}$ for all $t \geq 0$; we show that $\mathcal{V} \in \mathcal{I}$. If $v \in \mathcal{V}$ then $v = x(t_1) + x(t_2) + \ldots x(t_k)$ for some $t_1, t_2, \ldots, t_k$ and so

$$Av = E\dot{x}(t_1) + E\dot{x}(t_2) + \ldots E\dot{x}(t_k).$$

Thus, to show that $A\mathcal{V} \subseteq E\mathcal{V}$, it is sufficient to prove that $\dot{x}(t) \in \mathcal{V}$ for all $t \geq 0$. But

$$\dot{x}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \tag{2.3}$$

(with $h > 0$ if $t = 0$) where, by definition, $x(t+h), x(t) \in \mathcal{V}$. Moreover, $\mathcal{V}$ is closed since $\mathcal{X}$ is finite-dimensional, so proceeding to the limit in (2.3) does not leave $\mathcal{V}$. Thus $x(t) \in \mathcal{V}^*$ for all $t \geq 0$ so $x(0) \in \mathcal{V}^*$, as required. To prove part (iii) suppose $ker E \cap \mathcal{V}^* = 0$. Then the equation $E\dot{x} = Ax$ can be written

$$E(\dot{x} - Lx) - 0$$

and moreover $\dot{x} - Lx \in \mathcal{V}^*$ for all $t > 0$. Thus (1.2) becomes equivalent to

$$\dot{x} = Lx$$

which has the solution $x(t) = e^{Lt}x_0$, and uniqueness of solutions is obvious.

Conversely if $ker E \cap \mathcal{V}^* \neq 0$ let $g(t)$ be any function such that $g(t) \in ker E \cap \mathcal{V}^*$ for all $t \geq 0$. Then

$$x(t) = e^{Lt}x_0 + \int_0^t e^{L(t-\tau)} g(\tau) d\tau$$

satisfies, (1.2), whatever the function $g(t)$.

This completes the proof of theorem 2.1.

THEOREM 2.2. *Let $\mathcal{V}^* = sup\mathcal{I}$. Suppose $x_0 \in \mathcal{V}^*$ and $f(t) \in E\mathcal{V}^*$ for all $t \geq 0$. Then*

$$x(t) = e^{Lt}x_0 + \int_0^t e^{L(t-\tau)}g(\tau)d\tau \tag{2.4}$$

where $f(t) = Eg(t); g(t) \in \mathcal{V}^*$, and L is defined in theorem 2.1.

Proof. If we define $x(t)$ by (2.4) we have $x(0) = x_0$ and

$$E\dot{x} - Ax - f(t) = (EL - A)x + Eg(t) - f(t) = 0$$

since $g(t) \in \mathcal{V}^*$ and so $x \in \mathcal{V}^*$, therefore $(EL - A)x = 0$.

3. An algorithm to find $\mathcal{V}^*$

We are seeking an algorithm to find

$$\mathcal{V}^* = sup\{\mathcal{V} : A\mathcal{V} \subseteq E\mathcal{V}; \mathcal{V} \subseteq \mathcal{X}\}.$$

THEOREM 3.1. *Let matrices A, E, be $n \times n$. Define sequence $\mathcal{V}^\mu$ according to*

$$\mathcal{V}^0 = \mathcal{X},$$
$$\mathcal{V}^\mu = A^{-1}(E\mathcal{V}^{\mu-1}); \quad \mu = 1, 2, \ldots$$

where $A^{-1}U = \{x \in \mathcal{X} : Ax \in U\}$.

Then $\mathcal{V}^{\mu} \subseteq \mathcal{V}^{\mu-1}$, *and for some* $k \leq n$

$$\mathcal{V}^{\mu} = \mathcal{V}^{*}$$

for $\forall \mu \geq k$.

Proof. Clearly $\mathcal{V}^{1} \subseteq \mathcal{V}^{0}$, and if $\mathcal{V}^{\mu} \subseteq \mathcal{V}^{\mu-1}$, then

$$\begin{aligned}\mathcal{V}^{\mu+1} &= A^{-1}(E\mathcal{V}^{\mu}) \\ &\subseteq A^{-1}(E\mathcal{V}^{\mu-1}) \\ &= \mathcal{V}^{\mu}.\end{aligned}$$

So the sequence $\mathcal{V}^{\mu}$ is nonincreasing, thus for some $k \leq n$, $\mathcal{V}^{\mu} = \mathcal{V}^{k}(\mu \geq k)$. Now, $\mathcal{V} \subseteq \mathcal{V}^{*}$ if and only if

$$\mathcal{V} \subseteq A^{-1}(E\mathcal{V}) \tag{3.1}$$

From (3.1), $\mathcal{V} \subseteq \mathcal{V}^{0}$, and if $\mathcal{V} \subseteq \mathcal{V}^{\mu-1}$,

$$\mathcal{V} \subseteq A^{-1}(E\mathcal{V}) \subseteq A^{-1}(E\mathcal{V}^{\mu-1}) = \mathcal{V}^{\mu}.$$

Therefore, $\mathcal{V} \subseteq \mathcal{V}^{k} \equiv \mathcal{V}^{*}$, and as $\mathcal{V}$ was arbitrary the result follows.

The following method by using theorem 3.1 tells us how can we find $\mathcal{V}^{*}$.

We need the following terminology. If M, X, Y are matrices, with M given, a **maximal solution** of the equation $MX = 0$ (resp. $YM = 0$)means a solution X (resp. Y) of maximal rank, having linearly independent columns(resp. row). With reference to theorem 3.1, let $\mathcal{V}^{\mu} = ImV_{\mu}$, with $V_0 = I$(the identity matrix). Let W_{μ} be a maximal solution of

$$W_{\mu}(EV_{\mu-1}) = 0, \qquad \mu = 1, 2, \ldots$$

and obtain V_{μ} as a maximal solution of

$$(W_{\mu}A)V_{\mu} = 0, \qquad \mu = 1, 2, \ldots$$

At each stage we have $\mathcal{V}^{\mu} \subseteq \mathcal{V}^{\mu-1}$, i.e.(as a check),

$$rank(V_{\mu-1}, V_\mu) = rank V_{\mu-1};$$

and the stoping rule is $\mathcal{V}^\mu = \mathcal{V}^{\mu-1}$, i.e.

$$rank V_\mu = rank V_{\mu-1}.$$

Then $\mathcal{V}^* = \mathcal{V}^\mu$.

EXAMPLE.

$$E = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Then we get

$$W_1 = (1 \quad 0 \quad -1), \qquad V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

In second step we get

$$W_2 = (1 \quad 0 \quad -1), \qquad V_2 = V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Hence

$$\mathcal{V}^* = Im\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Now following proof of the first part of theorem 2.1 we get

$$L = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and then

$$e^{Lt} = \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since

$$\mathcal{V}^* = \left\{ \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \forall \alpha, \beta \in \Re \right\}$$

we get

$$E\mathcal{V}^* = \left\{ \begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix}, \forall \alpha, \beta \in \Re \right\}$$

Let $f(t)$ as

$$f(t) = \begin{pmatrix} e^t \\ 2e^t \\ e^t \end{pmatrix}$$

therefore $f(t) \in E\mathcal{V}^*$, then we have $g(t) = \begin{pmatrix} e^t \\ 2e^t \\ 0 \end{pmatrix} \in \mathcal{V}^*$

By assuming $x_0 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, then by (2.4) we get

$$x(t) = \begin{pmatrix} (1 + 3t + t^2)e^t \\ (2 + 2t)e^t \\ 0 \end{pmatrix}$$

We note that $kerE = Im \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$. So $kerE \cap \mathcal{V}^* = 0$, then the solution is unique.

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References

[1]. L. R. FLETCHER and A. AASARAAI, *On disturbance decoupling in descriptor systems*, SIAM J. CONTROL AND OPTIMIZATION. **27, No. 6.** (1989), 1319-1332.

[2]. W. M. WONHAM, *Linear Multivariable Control a Geometric Approach. Third edition*, Springer Verlag, New York. 1985.