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CARLEMAN INEQUALITIES FOR THE DIRAC AND LAPLACE OPERATORS AND STRONG UNIQUE CONTINUATION

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1. Introduction

Let U be a non-empty connected open subset of R^n, and u be a solution of the differential equation

$$(\Delta + \sum a_j \partial / \partial x_j + b)u = 0.$$

Here Δ is the Laplace operator, $a_j \in L^r(R^n), b \in L^s(R^n)$ for some suitable r, s. The main theorem says if u vanishes to infinite order at a point, then $u = 0$ identically. This is called a strong unique continuation theorem because it says that the behavior of a solution at a point determines the behavior in a neighborhood. The main step is to prove Carleman inequalities. We need two types of inequalities.

$$(1) \qquad \begin{aligned} &||e^{t\phi}\nabla f||_{L^{q_1}(U\setminus\{0\},dx)} \leq C||e^{t\phi}\Delta f||_{L^p(U\setminus\{0\},dx)} \\ &\qquad f \in C_0^\infty(U \setminus \{0\}) \ \frac{1}{p} - \frac{1}{q_1} = \frac{1}{r} \\ &||e^{t\phi} f||_{L^{q_2}(U\setminus\{0\},dx)} \leq C||e^{t\phi}\Delta f||_{L^p(U\setminus\{0\},dx)} \\ &\qquad f \in C_0^\infty(U \setminus \{0\}) \ \frac{1}{p} - \frac{1}{q_2} = \frac{1}{s} \end{aligned}$$

for C independent of t as $t \to \infty$, U an open neighborhood of the origin, where ϕ is a suitable weight function which is radial and

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decreasing. Once these two inequalities are proved a straightforward argument due to Carleman yields uniqueness. The key feature that distinguishes these inequalities from ordinary Sobolev inequalities is that the constant C is independent of the parameter t. In [8] dual version of (1) has been proved by finding a left parametrix and adopting appropriate pseudo-differential opperator theories as in Jerison[5], and Sogge[10,11]. Also a brief history on the unique continuation theory is given there. In this paper, we will see that actually a left inverse operator exists and that simplifies many steps. Borrowing from Alinhac- Baouendi [1] we use the weight function ϕ defined implicitly by $\phi(x) = \psi(y), \ \ y = -\psi(y) + e^{-\epsilon\psi(y)}$ when $y = log|x| < 0$. The same weight function was used in [8]. Then $e^{t\phi} \sim |x|^{-t}$. This is an algebraic blow up but still can be handled since u vanishes to infinite order at the origin. This is better than $|x|^{-t}$ which Jerison used in [5] because of convexity: $\partial^2\psi/\partial y^2 \geq e^{\epsilon y}$. In [9] inequality (2) has been proved for functions compactly supported in a shell using a weight function $\phi(x) = (log|x|)^2/2$. The idea was from Jerison [5]. We also give a proof of a strong unique continuation theorem for the Schödinger operator $D + V$, where D is the Dirac operator, and V is a potential function in some L^p space.

2. Statements of Results

The Dirac operator is a first-order constant coefficient operator on R^n of the form

$D = \sum_{j=1}^n \alpha_j \partial/\partial x_j$, where $\alpha_1, ...\alpha_n$ are skew hermitian matrices satisfying the Clifford relations:

$\alpha_j^* = -\alpha_j$ and $\alpha_j\alpha_k + \alpha_k\alpha_j = -2\delta_{jk};\ j, k = 1, ..., n.$

Also $D^2 = -\Delta$ and Carleman estimates for D imply estimates of type (1). Let $\phi(x) = \psi(y)$ be defined as above.

THEOREM 1. *Let $n \geq 3$, $p = (6n-4)/(3n+2)$, i.e., $1/p - 1/2 = 1/\gamma$. with $\gamma = (3n-2)/2$. There is a constant C depending only on n such that for all $t \in R$*

$$(1') \qquad ||e^{t\phi} f||_{L^2((-\infty,0)\times S, dydw)} \leq C||e^{t\phi} Df||_{L^p((-\infty,0)\times S, dydw)}$$

for all $f \in C_0^\infty((-\infty, 0) \times S, C^m)$.
Moreover,

$$(2) \quad ||e^{t\phi}\nabla f||_{L^2((-\infty,0)\times S, dydw)} \le C||e^{t\phi}\Delta f||_{L^p((-\infty,0)\times S, dydw)}$$

for $f \in C_0^\infty((-\infty, 0) \times S)$.

COROLLARY 1. *Let* Ω *be a connected, open subset of* R^n, $n \ge 3$. *If* $V \in L^\gamma(\Omega; M(m, C))$ *and* u *satisfies* $Du \in L^2(\Omega; C^m)$, $(D + V)u = 0$ *in* Ω.*and* $\int_{|x|<\epsilon} |u(x)|^2 dx = 0(\epsilon^N)$ *for any* N, *then* u *is identically zero in* Ω.

THEOREM 2. *Let* $n \ge 3$. *Then there is a constant* C *depending only on* n *such that*

$$||e^{t\phi}u||_{L^q((-\infty,0)\times S, dydw)} \le C||e^{t\phi}\Delta u||_{L^p((-\infty,0)\times S, dydw)}$$

for all $f \in C_0^\infty(R^n \setminus 0)$ *and for (1/q, 1/p) in the open triangle* ABC *with vertices* $A(1/2, 1/2)$, $B(n/(2n-2), 1/q_b)$, $C((n^2+2n-4)/(2n-2), 1/q_c)$ *where*

$$\frac{n}{2(n-1)} - \frac{1}{q_b} = \frac{n^2+2n+4}{2(n-1)} - \frac{1}{q_c} = \frac{2}{n}.$$

COROLLARY 2. *Let* $0 \in U \subset R^n$ *be an open set,* $u \in H^{2,p}_{loc}(U)$, $p = (6n-4)/(3n+2)$, u *satisfies the equation*

$$(3) \qquad \Delta u(x) + \sum a_j \partial/\partial x_j u(x) + bu(x) = 0$$

where $a_j \in L^r_{loc}(R^n)$, $b \in L^s_{loc}(R^n)$, *for r=(3n-2)/2,* $s > n/2$ *and*

$$\int_{\frac{r}{2}<|x|<r} |\nabla u(x)|^p dx = \int_{\frac{r}{2}<|x|<r} |u(x)|^p dx = 0(r^N)$$

for any N, *then* $u \equiv 0$ *in* U.

First, we want to set up some notations and elementary results, following Jerison [5].

2.1. Polar coordinates

Let S denote the unit sphere in R^n. For $y \in R$, and $w \in S. x = e^y w$ gives polar coordinates on R^n, i.e., $y = log|x|$ and $w = x/|x|$. The operator $L = \sum_{j<k} \alpha_j \alpha_k (x_j \partial/\partial x_k - x_k \partial/\partial x_j)$ acts only in the w -variables- $[L, \partial/\partial y] = 0$. We will view L as an operator on the sphere S. Let

$$\hat{\alpha} = \sum_{j=1}^{n} \alpha_j x_j/|x|, \quad then$$

$$\hat{\alpha} D = e^{-y}(\partial/\partial y - L);$$

and since $\hat{\alpha}^2 = -1$,

$$e^y D = \hat{\alpha}(\partial/\partial y - L) \tag{4}$$

Note that $\hat{\alpha}$ is unitary and $L^* = L$. If we recall that

$$\Delta = e^{-2y}(\partial^2/\partial y^2 + (n-2)\partial/\partial y + \Delta_S), \tag{5}$$

where Δ_S denotes the Laplace-Beltrami operator of the sphere. It follows from
$D^* = D, \quad D^2 = -\Delta$ that

$$L(L + n - 2) = -\Delta_S \tag{6}$$

In general if $\psi \in C^\infty(R)$, then (4) implies that in polar coordinates $x = e^y w$,

$$e^{t\psi(y)} e^y D e^{-t\psi(y)} h = \hat{\alpha} A_t h \tag{7}$$

where $A_t = \partial/\partial y - (t\psi'(y) + L)$.

Now we want to prove theorem 1.

Proof of Theorem 1. We will try to show the following equivalent inequality.

$$||e^{t\psi} f||_{L^2(R^- \times S, dx)} \leq C||A_t f||_{L^p(R^- \times S, dx)} \quad for\ f \in C_0^\infty(U)$$

Let π_k denote the projection of $L^2(S:C^m)$ onto $E_k = ker(L-k)$, $k \in Z$(See [5] for more details). We can rewrite

$$A_t f = \sum_k (\partial/\partial y - (t\psi'(y) + k))\pi_k f.$$

First, consider the following operator

$$\Omega = d/dy - y.$$

In [5] Jerison exhibited the following exact formula for the symbol of a left inverse of Ω: there is a unique operator B on R satisfying $B\Omega = I$ and $B(e^{-y^2/2}) = 0$ given by

$$Bf(y) = (1/2\pi)\int F_0(y,\eta)e^{iy\eta}\hat{f}(\eta)d\eta, where$$

(8)

$$F_0(y,\eta) = \sqrt{2}\int_0^\infty e^{-s^2-2sy}dse^{-iy\eta-(y^2+\eta^2)/2} - \int_0^\infty e^{-s^2-s(y-i\eta)}ds.$$

Also the following symbol estimate is true.

(9)

$$|(\partial/\partial y)^j(\partial/\partial)^l F_o(y,\eta)| \leq C_{j,l}(1+|y+i\eta|)^{-1-j-l} \quad j,l = 0,1,...$$

Inspired from the classical case, we can find a left inverse operator B_t satisfying $f(y) = B_t A_t f(y)$. If we assume B_t has kernel K_t, then the above is equivalent to

$$\begin{aligned}\pi_k f(y) &= \int K_t(y,s)(\partial/\partial s - t\psi'(s) + k)\pi_k f(s)ds \\ &= \int(-\partial/\partial s - t\psi'(y) + k)K_t(y,s)\pi_k f(s)ds.\end{aligned}$$

Let

$$K_{1,t}(y,s) = H(y-s)e^{t(\psi(y)-\psi(s))+k(y-s)}.$$

Then

$$B_{1,t}\pi_k f(y,w) = \sum_k \int_{-\infty}^y e^{t(\psi(y)-\psi(s))+k(y-s)}\pi_k f(s,.)(w)ds.$$

This integral converges only for $k < -t\psi'(y)$. On the other hand, the fact that $f \in C_0^\infty(R^- \times S)$ and

$$(-\partial/\partial s - t\psi'(s) - k)e^{t(\psi(y)-\psi(s))+k(y-s)} = 0$$

tells us

$$K_{2,t}(y,s) = (H(y-s)-1)e^{t(\psi(y)-\psi(s))+k(y-s)}$$

is another kernel. From this we have another expression which is

$$B_{2,t}\pi_k f(y,w) = -\sum_k \int_y^\infty g(s)e^{t(\psi(y)-\psi(s))+k(y-s)}\pi_k f(s,.)(w)ds,$$

where the integral converges for $k > -t\psi'(y)$. here g(s) is a cut off function we introduce for later use such that $g \in C^\infty$ which has value 1 for $s < 0$, and equals 0 for $s > 1$. Then g(s)f(s) = f(s) for $s < 0$. Now we can write

$$\psi(s) - \psi(y) = (s-y)\psi'(y) + \frac{(s-y)^2}{2}h(y,s),$$

where

$$h(y,s) = \int_0^1 (1-\xi)\psi''(y+\xi(s-y))d\xi \quad \geq 0.$$

After substituting $f(s) = \int \hat{f}(\eta)e^{is\eta}d\eta,\ \ and\ y - s = s'$, we get

$$B_{1,t}f(y,w) = \sum_k \int \sigma_{1,t}(y,\eta,k)\hat{f}(\eta,.)(w)e^{iy\eta}d\eta\pi_k$$

where

$$\sigma_{1,t}(y,\eta,k) = \int_{-\infty}^y e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds \quad (-t\psi'(y) > k).$$

Also

$$B_{2,t}f(y,w) = \sum_k \int \sigma_{2,t}(y,\eta,k)\hat{f}(\eta,.)(w)e^{iy\eta}d\eta\pi_k$$

where

$$\sigma_{2,t}(y,\eta,k) = -\int_y^\infty g(s)e^{t(\psi(y)-\psi(s))+k(y-s)+i(s-y)\eta}ds$$

$(k > -t\psi'(y))$. Since $\sigma_{1,t}(y,\eta,k)$ are defined only for $-t\psi'(y) > k$, we want to extend this for $k > -t\psi'(y)$. Also we want to extend $\sigma_{2,t}(y,\eta,k)$ for $k < -t\psi'(y)$. For that we will follow Stein's method ([12],p182).

LEMMA. *There exist a continuous function ϕ defined on $[1,\infty)$ which is rapidly decreasing at ∞, that is $\phi(\lambda) = 0(\lambda^{-N})$, as $\lambda \to \infty$, for every N, and which satisfies the following properties.*

$$\int_1^\infty \phi(\lambda)d\lambda = 0, \quad \int_1^\infty \lambda^l \phi(\lambda)d\lambda = 0, \quad for\ k = 1, 2, \ldots.$$

With this definition, we can extend our symbol by

$$\begin{aligned}\tilde{\sigma}_{1,t}(y,\eta,k) &= \sigma_{1,t}(y,\eta,k) \qquad (k < -t\psi'(y)) \\ &= \int_1^\infty \sigma_{1,t}(y,\eta,(1-2\lambda)(k+t\psi'(y)) - t\psi'(y))\phi(\lambda)d\lambda \\ &\qquad (k > -t\psi'(y)).\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_{2,t}(y,\eta,k) &= \sigma_{2,t}(y,\eta,k) \quad (k > -t\psi'(y)) \\ &= \int_1^\infty \sigma_{2,t}(y,\eta,(1-2\lambda)(k+t\psi'(y)) - t\psi'(y))\phi(\lambda)d\lambda \\ &\qquad (k < -t\psi'(y)).\end{aligned}$$

Now we want to find the size of symbols. In [8] we have estimated $\sigma_{1,t}(y,\eta,k)$ and $\sigma_{2,t}(y,\eta,k)$. Then using those estimates we can do the same for the extended symbols.

Claim

(10)

$$|(\frac{\partial}{\partial y})^N (\frac{\partial}{\partial \eta})^M d_k^m \tilde{\sigma}_{i,t}(y,\eta,k)| \leq \frac{C_{N,M}}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{M+m+1}}$$

$$\times (1 + \frac{t\psi''(y)}{\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|})^N \quad i = 1, 2.$$

This can be proved by integration by parts and the following properties of $h(y,s)$.

$$h(y, y-s) \sim e^{\epsilon y}\frac{e^{-s} - 1 + s}{s^2}$$

$$\frac{1}{4}e^{\epsilon y} \leq h(y, y-s) \leq \frac{1}{2}e^{\epsilon y} \quad for\ 0 \leq s \leq 2$$

$$e^{\epsilon y}\frac{1}{2s} \leq h(y, y-s) \leq e^{\epsilon y}\frac{1}{s} \quad for\ s \geq 2$$

With these estimates, we can write B_t as follows.

$$B_{1,t}f(y,w) = \sum_k \int \sigma_{1,t}(y,\eta,k)\hat{f}(\eta,.)(w)e^{iy\eta}d\eta\pi_k$$

$$B_{2,t}f(y,w) = \sum_k \int \sigma_{2,t}(y,\eta,k)\hat{f}(\eta,.)(w)e^{iy\eta}d\eta\pi_k$$

Now we want to show the following estimates.

$$||B_{i,t}f||_{L^2(R^- \times S, dx)} \leq C||f||_{L^p(R^- \times S, dx)} \quad i = 1, 2.$$

The main tool in the proof is the spherical restriction theorem of C.Sogge[10].

THEOREM. *Let ξ_k denote the projection operator from $L^2(S)$ to the space of spherical harmonics of degree k. Then there is a constant c such that*

$$||\xi_k g||_{L^{p'}(S)} \leq ck^{1-2/n}||g||_{L^p(S)}$$

where $p = 2n/(n+2)$, $p' = 2n/(n-2)$.

Formula (6) implies that $(L + (n-2)/2)^2 = -\Delta_s + (n-2)^2/4$. Hence

$$T = sgn(L + (n-2)/2) = (L + (n-2)/2)(-\Delta_s + (n-2)^2/4)^{-1/2}$$

is a classical pseudodifferential operator on S. Thus T is bounded from $L^q(S; C^m)$ to $L^q(S; C^m)$ for all q, $1 < q < \infty$. Moreover,

$$\pi_k = \frac{1}{2}(1+T)\xi_k, \quad k = 0, 1, 2...$$

$$\pi_k = \frac{1}{2}(1-T)\xi_k, \quad k = 1-n, -n, -n-1, ...$$

Therefore, Sogge's theorem implies that

$$||\pi_k g||_{L^{p'}(S;C^m)} \le Ck^{1-2/n}||g||_{L^p(S;C^m)}.$$

Define $\pi_{M,N}$ by

$$\pi_k\pi_{M,N}g = \{\pi_k g \ \ if\ M \le k \le N,\ \ 0\ otherwise.\}$$

The triangle inequality implies

$$||\pi_{M,N}g||_{L^{p'}(S;C^m)} \le CN^{1-2/n}(N-M+1)||g||_{L^p(S;C^m)}.$$

Next use a device due to P.Tomas[13]:

$$||\pi_{M,N}g||^2_{L^2} = \int_S <\pi_{M,N}g, g> \ \le ||\pi_{M,N}g||_{L^{p'}}||g||_{L^p}$$

$$\le CN^{1-2/n}(N-M+1)||g||^2_{L^p}.$$

We conclude that

$$||\pi_{M,N}g||_{L^2(S;C^m)} \le CN^{1/p'}(N-M+1)^{1/2}||g||_{L^p(S;C^m)}.$$

Also by duality

$$||\pi_{M,N}g||_{L^{p'}(S;C^m)} \le CN^{1/p'}(N-M+1)^{1/2}||g||_{L^2(S;C^m)}.$$

If we interpolate with the trivial estimate

$$||\pi_{M,N}g||_{L^2(S;C^m)} \le ||g||_{L^2(S;C^m)},$$

we find that

(11)

$$||\pi_{M,N}g||_{L^q(S;C^m)} \le C(N^{\frac{n-2}{2}}(N-M+1)^{n/2})^{1/2-1/q}||g||_{L^2(S;C^m)},$$

for $2 \le q \le p' = 2n/(n-2)$.

Let N be the integer satisfying $2^{N-1} \le 10e^{\epsilon j/2}t^{1/2} \le 2^N$. Consider a partition of unity $\{\phi_\beta\}_{\beta=0}^N$ of the positive real axis satisfying

$$\sum_{\beta=0}^{N}\phi_\beta(r) = 1 \qquad all\ r > 0,$$

$$supp\phi_\beta \subset \{r : 2^{\beta-2} \leq r \leq 2^\beta\}, \beta = 1, 2, ...N-1.$$

(12) $$supp\phi_0 \subset \{r : r \leq 1\}, \qquad supp\phi_N \subset \{r : r \geq s/400\},$$

$$|(\partial/\partial r)^l \phi_\beta(r)| \leq C_l 2^{-\beta l}, \quad l = 0, 1....$$

Now using duality at (11) we get the spherical restriction theorem we need;

(11′)

$$||\pi_{M,N} g||_{L^2(S;C^m)} \leq C(N^{\frac{n-2}{2}}(N-M+1)^{n/2})^{1/2-1/q}||g||_{L^{q'}(S;C^m)},$$

$$for\ q' = p = \frac{6n-4}{3n+2}, \quad q = \frac{6n-4}{3n-6}.$$

With this theorem we introduce partition of unity $\{\phi_\beta\}$ satisfying (12) and define

$$\sigma^\beta_{1,t}(y,\eta,k) = \phi_\beta(\frac{1}{\sqrt{a} + |t\psi'(y) + k - i\eta|})\sigma_{1,t}(y,\eta,k).$$

Then $\sigma_{1,t}(y,\eta,k)$ satisfies

$$|(\partial/\partial\eta)^j(\partial/\partial y)^l \sigma_{1,t}(y,\eta,k)| \leq C_{j,l}(\sqrt{a} + |t\psi'(y) + k - i\eta|)^{-1-j-l}(a + |t\psi'(y) + k|)^l.$$

(13) $$|(\partial/\partial\eta)^j(\sigma_{1,t}(y,\eta,k) - \sigma_{1,t}(y,\eta,k+1))| \leq C_j \cdot (\sqrt{a} + |t\psi'(y) + k - i\eta|)^{-2-j}.$$

From (13) and the property of $|(\partial/\partial\eta)^l\phi_\beta(r)| \leq 2^{-\beta l}$, we deduce that the following inequalities hold uniformly for $y \in I = I_l = (-l, -l+1)$

$$|(\partial/\partial\eta)^j F^\beta_t(y,\eta,k)| \leq C_j(2^\beta\sqrt{a})^{-1-j}.$$

(14) $$|(\partial/\partial\eta)^j(F^\beta_t(y,\eta,k) - F^\beta_t(y,\eta,k+1))| \leq C_j(2^\beta\sqrt{a})^{-2-j}.$$

Now define

$$F_t^\beta f(y,w) = \sum_k \frac{1}{2\pi} \int \sigma_{1,t}^\beta(y,\eta,k)\pi_k \tilde{f}(\eta,.)(w)e^{iy\eta}d\eta.$$

Then $B_{1,t} = \sum_{\beta=0}^N F_t^\beta$. Let $M = [-t\psi'(y) - 2^\beta\sqrt{a}]$, $M' = [M + 22^\beta\sqrt{a}] + 1$. Denote

$$T_t^\beta(y,\eta)g(w) = \sum_K F_t^\beta(y,\eta,k)\pi_k g(w).$$

Here $F_t^\beta(y,\eta,k) = 0$ unless $M \leq k \leq M'$. A summation by parts gives

$$T_t^\beta(y,\eta) = \sum_M^{M'} (F_t^\beta(y,\eta,k) - F_t^\beta(y,\eta,k+1))\pi_{M,k}, \quad for M \leq k \leq M'.$$

From (11′) and (14) we obtain

$$\begin{aligned} &||(\partial/\partial\eta)^j T_t^\beta(y,\eta)\pi_{M,k}g||_{L^2(S;C^m)} \\ &\leq C_j(2^\beta\sqrt{a})^{-1-j}(t^{\frac{n-2}{2}}(2^\beta\sqrt{a})^{n/2})^{1/2-1/q}||g||_{L^p(S;C^m)} \end{aligned}$$

uniformly for $y \in I$. Define

$$\begin{aligned} K_t^\beta(y,z) &= \frac{1}{2\pi}\int T_t^\beta(y,\eta)e^{iz\eta}d\eta \\ &= \frac{1}{2\pi}\int (\partial/\partial\eta)^j T_t^\beta(y,\eta)\frac{1}{(iz)^j}e^{iz\eta}d\eta. \end{aligned}$$

Since the length of the interval in η where T_t^β is non-zero is less than $2 \times 2^\beta\sqrt{a}$,

$$\begin{aligned} ||K_t^\beta(y,z)g||_{L^2(S;C^m)} \leq\ & C(1 + |2^\beta\sqrt{a}z|)^{-10} \\ & \cdot (t^{\frac{n-2}{2}}(2^\beta\sqrt{a})^{n/2})^{1/2-1/q}||g||_{L^p(S;C^m)}. \end{aligned}$$

Note that

$$F_t^\beta f(y,w) = \int K_t^\beta(y,y-y')f(y',.)(w)dy'.$$

LEMMA. *Let $H(y.y')$ be a bounded operator from $L^p(S)$ to $L^q(S)$ of operator norm $\leq h(y-y')$ for each $y, y' \in R$. Suppose that $h \in L^r(R)$ for $1/r + 1/p = 1 + 1/s$. Then*

$$Tf(y, w) = \int H(y, y')f(y', .)(w)dy'$$

satisfies

$$||Tf||_{L^s(R^- \times S, dx)} \leq ||h||_{L^r(R)}||f||_{L^p(R^- \times S, dx)}.$$

Applying the lemma when $s = 2$, we obtain for $\beta \leq N - 1$

$$||F_t^\beta||_{L^2(R^- \times S, dx)} \leq ||(1 + |2^\beta \sqrt{a} z|)^{-10}||_{L^r(dz)} \cdot (t^{\frac{n-2}{2}}(2^\beta \sqrt{a})^{n/2})^{1/2 - 1/q}||g||_{L^p(R^- \times S, dx)},$$

for $\frac{1}{r} + \frac{1}{p} = \frac{1}{2} + 1$.
Note that

$$||(1 + |2^\beta \sqrt{a} z|)^{-10}||_{L^r(dz)} \leq C(2^\beta \sqrt{a})^{-1/r}.$$

Then for $\frac{1}{\gamma} = \frac{1}{2} - \frac{1}{q} = \frac{2}{3n-2}$, and $\frac{1}{r} = \frac{3n-4}{3n-2}$ we obtain(after some calculation)

$$||F_t^\beta f||_{L^2(R^- \times S, dx)} \leq C 2^{-(n-2)\beta/\gamma}(e^{\epsilon y})^m ||f||_{L^p(R^- \times S, dx)},$$

$m > -1/3$. Since 2^β has negative power, if we sum the series in β we get exactly the same bound as in the $L^2 \to L^q$ estimate[8]i.e.,

$$||\sum_\beta^{N-1} F_t^\beta f||_{L^2(R^- \times S, e^{ny} dy dw)} \leq C||f||_{L^p(R^- \times S, e^{(n-\epsilon/3)y} dy dw)}.$$

Now the case $\beta = N$, we need a different estimate for F_t^N. We had

$$F_t^N f(y, w) = \sum_k \int \sigma_{1,t}^N(y, \eta, k)\tilde{f}(\eta, .)(w)e^{iy\eta} d\eta.$$

Since σ_t^N had support where

$$|t\psi'(y) + k - i\eta| \geq 2^N \sqrt{a},$$

we had
$|t\psi'(y) + k - i\eta| > c(1 + |\eta| + |k|)$ uniformly for $y < 0$. Hence
(15)

$$|(\partial/\partial\eta)^j (\partial/\partial y)^m d_k^l \sigma_t^N (y, \eta, k)| \leq C_{j,m}(1 + |\eta| + |k|)^{-1-j-l}$$
$$j = 0, 1, \ldots.$$

Here d_k^l is a difference operator of order l in the k variable. The above estimate means that $\sigma_{1,t}^N$ is a standard symbol in the (y, η, k) variables of order -1. So F_t^N is a standard pseudodifferential operator of order -1,(Taylor[13] p296) and we can write F_t^N as

$$F_t^N f(y, w) = \int_{S\times R} K_t((w, y), (w', y')) f(w', y') dw' dy'.$$

Then the corresponding kernel K_t has bounds

$$|K_t((w', y')(w, y))| \leq C(|w - w'| + |y - y'|)^{-n+1}.$$

Hence F_t^N is a bounded operator from $L^p \to L^{\bar{p}}$ for $\frac{1}{p} - \frac{1}{\bar{p}} = \frac{1}{n}$ (Stein,[12],p128) i.e.,

$$||F_t^N f||_{L^{\bar{p}}(R^- \times S, dx)} \leq C||f||_{L^p(R^- \times S, dx)}. \qquad (*)$$

On the otherhand, compactness of (w, w') and $n \geq 3$ gives us

$$\int |K_t((w', y'), (w, y))| dw' dy' \leq C$$

and

$$\int |K_t((w', y'), (w, y))| dw dy \leq C.$$

Now we can apply Young's inequality to get $L^p \to L^p$ boundedness, i.e.,

$$||F_t^N f||_{L^p(R^- \times S, dx)} \leq C||f||_{L^p(R^- \times S, dx)} \qquad (**)$$

Now if we intrerpolate the above two estimates (*) , and (**) we obtain

$$||F_t^N f||_{L^q(R^- \times S, dx)} \leq C||f||_{L^p(R^- \times S, dx)},$$

for all $p \leq q \leq \tilde{p}$. And in particular this holds for q=2. So if we combine this with $\beta \leq N-1$ cases, we obtain

$$||B_{1,t} f||_{L^2(R^- \times S, e^{(n+\frac{\epsilon}{2})y} dy)} \leq C||f||_{L^p(R^- \times S, e^{ny} dydw)}.$$

Exactly same process shows same inequality holds for $B_{2,t}$. Now define

$$P_1 f = \sum_{k < -t\psi'(y)} \pi_k f.$$

$$P_2 f = \sum_{k > -t\psi'(y)} \pi_k f.$$

Then $f = P_1 f + P_2 f$, $P_1 B_1 A_t f = P_1 f$, and $P_2 B_2 A_t f = P_2 f$. Now

$$\begin{aligned}
||f||_{L^2(R^- \times S, e^{(n+\frac{\epsilon}{2})y} dydw)} &= ||P_1 f + P_2 f||_{L^2(R^- \times S, e^{(n+\frac{\epsilon}{2})y} dydw)} \\
&\leq ||P_1 f||_{L^2(R^- \times S, e^{(n+\frac{\epsilon}{2})y} dydw)} \\
&\quad + ||P_2 f||_{L^2(R^- \times S, e^{(n+\frac{\epsilon}{2})y} dydw)} \\
&= ||P_1 B_{1,t} A_t f||_{L^2(R^- \times S, e^{(n+\frac{\epsilon}{2})y} dydw)} \\
&\quad + ||P_2 B_{2,t} A_t f||_{L^2(R^- \times S, e^{(n+\frac{\epsilon}{2})y} dydw)} \\
&\leq ||B_{1,t} A_t f||_{L^2(R^- \times S, e^{(n+\frac{\epsilon}{2})y} dydw)} \\
&\quad + ||B_{2,t} A_2 f||_{L^2(R^- \times S, e^{(n+\frac{\epsilon}{2})y} dydw)} \\
&\leq 2C||A_t f||_{L^p(R^- \times S, dx)}.
\end{aligned}$$

This is our desired inequality, and using (7), this is equivalent to the following estiamte.

$(1'')$

$$||e^{t\psi} f||_{L^2(R^- \times S, e^{(n+\epsilon)y} dydw)} \leq C||e^{t\psi} e^y Df||_{L^p(R^- \times S, e^{ny} dydw)}.$$

Now for the future use, we would like to note that the following corollary also holds.

Claim.

$$||e^{t\psi}f||_{L^p(R^-\times S,dx)} \le C||e^{t\psi}Df||_{L^p(R^-\times S,dx)}. \quad f \in C_0^\infty(R^- \times S).$$

We can apply same argument as the $L^p \to L^2$ estimate. The idea is to prove slightly stonger estimate, i.e., ($L^p(R^- \times S) \to L^p(L^2(S), dy)$ estimate.) In this context we want to show the following

$$||B_{i,t}g||_{L^p(L^2(S),dy)} \le C||g||_{L^p(R^-\times S,dx)} \quad i = 1, 2.$$

First we have

$$||K_t^\beta(y,z)g||_{L^2(S)} \le C(1+|2^\beta\sqrt{a}z|)^{-10} \cdot (t^{\frac{n-2}{2}}(2^\beta\sqrt{a})^{n/2})^{\frac{1}{2}-\frac{1}{q}}||g||_{L^p(S)}.$$

And

$$F_t^\beta f(y,w) = \int K_t^\beta(y, y-y')f(y',.)(w)dy'.$$

LEMMA. *Let $H(y,y')$ be a bounded operator from $L^p(S)$ to $L^p(S)$ of operator norm $\le h(y-y')$ for each $y, y' \in R$. Suppose that $h \in L^1(R)$. Then*

$$Tf(y,w) = \int H(y,y')f(y')(w)dy'$$

satisfies

$$||Tf||_{L^p(L^2(S),dy)} \le ||h||_{L^1(R)}||f||_{L^p(R^-\times S)}.$$

Note that

$$||(1+|2^\beta\sqrt{a}z|)^{-10}||_{L^1(dz)} \le C(2^\beta\sqrt{a})^{-1}.$$

Then the lemma implies

$$||\sum_\beta^{N-1} F_t^\beta f||_{L^p(L^2(S),dy)} \le C||f||_{L^p(R^-\times S,e^{(n-\epsilon/3)y}dydw)}.$$

On the other hand, for $\beta = N$, we need only

$$||F_t^N f||_{L^p(R^- \times S, dx)} \le C||f||_{L^p(R^- \times S, dx)}. \qquad (**)$$

After combining these two, we obtain

$$||B_{i,t} f||_{L^p(L^2(S), dy)} \le C||f||_{L^p(R^- \times S, e^{(n-\epsilon/3)y} dy dw)}, \quad i = 1, 2.$$

Now we will prove our inequality. First, from the definition of P_i, $i = 1, 2$, it is easy to check

$$||P_i f||_{L^p(L^2(S), dy)} \le ||f||_{L^p(L^2(S), dy)},$$

and

$$||f||_{L^p(R^- \times S, dx)} \le C||f||_{L^p(L^2(S), dy)}.$$

Then the above inequalities and projection method give the desired inequality.

(16) $$||f||_{L^p(R^- \times S, dx)} \le C||A_t f||_{L^p(R^- \times S, dx)}.$$

3. The Laplace operator

Proof of Theorem 3. We can decompose the right hand side of (3) i.e.,

$$(e^{t\psi(y)} e^{2y} \Delta e^{-t\psi(y)}) f(y, w) = \sum_k (\partial/\partial y - t\psi'(y) - k)(\partial/\partial y - t\psi'(y) + k + n - 2)\xi_k f(y, .)(w).$$

Let's denote this as $(A_t f)(y, w)$. The first component appeared in the Dirac operator case. On the other hand the second component has a nice property, i.e. whose symbol $i\eta - t\psi'(y) + k + n - 2$ never vanishes. If we denote $\sigma_t(y, \eta, k)$ be the symbol of the left inverse of $\partial/\partial y - t\psi'(y) - k$, then

$$\frac{\sigma_t(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2}$$

is a left parametrics. Now we want to inspect the property of the left parametrics. Since

$$\sigma_t(y,\eta,k) \sim \frac{1}{\sqrt{a}+|t\psi'(y)+k-i\eta|}, \quad a = t\psi''(y) \sim te^{\epsilon y}.$$

if we choose some large N_0, depending on t such that $te^{-\epsilon N_0} = \frac{1}{100}$, then

$$\sigma_t(y,\eta,k) \leq \frac{1}{\sqrt{a}}; \; for \; y > -N_0.$$

On the other hand, for $y \leq -N_0$, $\sqrt{a}$ is vanishing fast as y goes to $-\infty$. But we can choose t such that $|t-y| > 1/4$. Then in this range $t(\psi'(y)+1) \sim te^{\epsilon y} << 1/10$. So

$$|t\psi'(y)+k-i\eta| \geq |-t+k-i\eta| - |t\psi'(y)+1|.$$

And

$$\sigma_t(y,\eta,k) \leq \frac{1}{|t\psi'(y)+k-i\eta|} \leq c \;\; for \;\; y \leq -N_0.$$

In either case,

$$\sigma_t(y,\eta,k) \sim 0(1), \qquad \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2} \sim t^{-1}.$$

For $f \in C_0^\infty(R\times S)$, denote

$$F_tf(y,w) = \sum_k \int \frac{\sigma_t(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2} e^{iy\eta}\pi_k\tilde{f}(\eta,.)(w)d\eta.$$

First we want to show the following inequality.
(17)

$$||F_tf||_{L^q(I\times S,dx)} \leq Ce^{-\epsilon y/2}||f||_{L^p(I\times S,dx)} \quad for \; f\in C_0^\infty(I\times S).$$

Because of the behavior of $\sigma_t(y,\eta,k)$, we want to proceed separately depending on the size of y. As before we introduce partition of unity $\{\phi_\beta\}_{\beta=0}^N$ of the positive real axis satisfying (12). First, define

$$\sigma_t^\beta(y,\eta,k) = \phi_\beta(\frac{1}{\sqrt{a}}|t\psi'(y)+k-i\eta|)\sigma_t(y,\eta,k) \qquad y > -N_0.$$

$$\sigma_t^\beta(y,\eta,k) = \phi_\beta(|t\psi'(y)+k-i\eta|)\sigma_t(y,\eta,k) \qquad y < -N_0.$$

First, we will work on the case $y > -N_0$. Then from (10) it is easy to show
(10′)

$$|(\partial/\partial\eta)^j \frac{\sigma_t^\beta(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}| \le (2^\beta\sqrt{a})^{-1-j}t^{-1}, \quad \beta \le N-1.$$

In the case $\beta = N$, from (15) we have
(10″)

$$|(\partial/\partial\eta)^j d_y^l \frac{\sigma_t^N(y,\eta,k)}{i\eta - t\psi'(y)+k+n-2}| \le C_{jl}(1+|k|+|\eta|)^{-2-j-l}.$$

With these symbol estimates, we need Sogge's theorem for spherical harmonics[10].

THEOREM. *Let* $1 \le s \le 2(n-1)/n, \quad r = ns'/(n-2)$. *Then*

$$||\xi_k f||_{L^r(S^n)} \le Ck^{1/s'}||f||_{L^s(S^n)}.$$

Using the duality, this is equivalent to

$$||\xi_k f||_{L^{s'}(S^n)} \le Ck^{1/s'}||f||_{L^{r'}(S^n)}.$$

From the conditions of r, s we have the relation $\frac{1}{s} - \frac{1}{r} > \frac{2}{n}$. Also, $\frac{1}{r'} - \frac{1}{s'} > \frac{2}{n}$. Now if we interpolate above two inequalities, we obtain

$$(18) \qquad ||\xi_k f||_{L^q(S^n)} \le Ck^{1/s'}||f||_{L^p(S^n)},$$

for $1/q = t/r + (1-t)/s', \;\; 1/p = t/s + (1-t)/r'$ and $1/r' = 1 - 1/r > \frac{n^2+2n-4}{2n(n-1)}$, and $1/q - 1/p > 2/n$.
Let

$$F_t^\beta f(y,w) = \sum_k \frac{1}{2\pi}\int \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y)+k-i\eta+n-2}\xi_k\hat{f}(\eta,.)(w)e^{iy\eta}d\eta,$$
$$\beta = 0,1,\ldots.$$

Then $F_t^\beta f(y,w) = \sum_k \frac{1}{2\pi}\int K_t^\beta(y,y')f(y')(w)dy'$, where

$$K_t^\beta(y,z) = \sum_k \frac{1}{2\pi}\int \frac{\sigma_t^\beta(y,\eta,k)}{t\psi'(y)+k-i\eta+n-2}e^{iz\eta}d\eta\xi_k, \quad z = y-y'.$$

the integration in η is over an interval of length $2\times 2^\beta\sqrt{a}$. It turns out $K_t^\beta(z)$ is a bounded operator from $L^p(S^n)\to L^q(S^n)$ whose operator norm is bounded by

$$C(2^\beta\sqrt{a})t^{-1+1/s'}(1+|2^\beta\sqrt{a}z|)^{-10}.$$

Next, let $1/l+1/p = 1/q+1$, and apply Young's theorem and obtain

$$||F_t^\beta f||_{L^q(I\times S,dx)} \leq (2^\beta\sqrt{a})t^{-1+1/s'}(2^\beta\sqrt{a})^{-1/l}||f||_{L^p(I\times S,dx)}.$$

Since 2^β has power $1-1/l = 1/p-1/q = 2/n+\delta$, and $-1+1/s' < -n/(2n-2)$, we obtain

$$(19) \qquad ||\sum_\beta F_t^\beta f||_{L^q(I\times S,dx)} \leq Ct^\delta||f||_{L^p(I\times S,dx)}.$$

Now we want to estimate F_t^N.

It is obvious from (10″) that the operators $(\partial/\partial y)^2F_t^N$ and $\Delta_sF_t^N$(with symbols $-\eta^2\frac{\sigma_t^N}{i\eta t\psi'(y)+k+n-2}$ and $-k(k+n-2)\frac{\sigma_t^N(y,\eta,k)}{i\eta-t\psi'(y)+k+n-2})$, are standard zero order multipliers.

They are therefore bounded from L^p to L^p *for* $1<p<\infty$. Sovolev theorem [4] implies

$$||h||_{L^q(I\times S)} \leq C||(\partial/\partial y)^2h||_{L^p(I\times S)} + ||\Delta_sh||_{L^p(I\times S)},$$

for $1/p-1/q\leq 2/n$, for all $h\in C_0^\infty((-\infty,-N_0)\times S)$. So we find that

$$||F_t^Nf||_{L^q((-\infty,-N_0)\times S)} \leq C||f||_{L^p((-\infty,-N_0)\times S)}.$$

Now we want to show the claim. In (19) take δ small enough and use interpolation theorem with

$$||F_t^\beta f||_{L^2(I\times S)} \le t^{-1}(2^\beta\sqrt{a}z)^{-j}||f||_{L^2(I\times S)}$$

which implies $||\sum_\beta F_t^\beta f||_{L^2(I\times S)} \le Ct^{-2/3}e^{-\epsilon y/2}||f||_{L^2(I\times S)}$. We gain some power of t after interpolation and obtain the desired estimate (17) for $1/p - 1/q = 2/n - \epsilon$. The case $y < -N_0$ works in a similar way and we will leave it to the reader. Since we have the estimate only on $I \times S$ we want to extend this result to the whole $R^- \times S$. First we want to remind a result we had before.

$$(1'') \quad ||e^{t\psi}f||_{L^2(R^-\times S, e^{(n+\epsilon)y}dydw)} \le C'||e^{t\psi}e^yDf||_{L^p(R^-\times S,dx)}.$$

Now after replacing $f = Dg$ in $(1'')$, the above inequality implies
(20)

$$||e^{t\psi}e^yDf||_{L^2(R^-\times S, e^{(n+\epsilon)y}dydw)} \le C||e^{t\psi}e^{2y}\Delta f||_{L^p(R^-\times S, e^{ny}dydw)}.$$

Since

$$||Df||^2_{L^2(R^-\times S,dx)} = ||\sum_j \alpha_j\partial f/\partial x_j||^2_{L^2(R^-\times S,dx)}$$
$$= \sum_j ||\partial f/\partial x_j||^2_{L^2(R^-\times S,dx)} = ||\nabla f||^2_{L^2(R^-\times S,dx)}$$

we deduce from $(1'')$
(2′)

$$||e^{t\psi(y)}\nabla f||_{L^2(R^-\times S, e^{ny}dydw)} \le C||e^{t\psi(y)}\Delta f||_{L^p(R^-\times S, e^{ny}dydw)}.$$

On the other hand, By replacing $f = Dg$ in (16) it follows
(16′)

$$||e^{t\psi}e^yDf||_{L^p(R^-\times S, e^{(n+\epsilon y)}dydw)} \le C||e^{t\psi}e^{2y}\Delta f||_{L^p(R^-\times S, e^{ny}dydw)}.$$

If we combine (16) and (16′), we have
(21)

$$|||e^{t\psi}f||_{L^p(R^-\times S, e^{(n+2\epsilon)y}dydw)} \le C||e^{t\psi}e^{2y}\Delta f||_{L^p(R^-\times S, e^{ny}dydw)}.$$

With these estimates, we can continue to rewrite $f = \sum_j \phi_j f$, for $\{\phi_j\}$ partition of unities, having support in

$(-j-5/4, -j+5/4)$ and for each $x \in R^-$, there are only finitely many $'\phi_j's$ such that $\phi_j(x) \neq 0$. Then.

(22)

$$||f||_{L^q(R^-\times S, e^{(n+2\epsilon q)y}dydw)} \leq (\sum_j ||\phi_j f||^q_{L^q(I_j\times S, e^{(n+2\epsilon q)y})})^{1/q}$$
$$\leq (\sum_j ||A_t(\phi_j f)||^q_{L^p(I_j\times S, e^{ny}dydw)})^{1/q}.$$

Since

$$A_t(\phi_j f) = (\phi_j'')f + \sum_k \phi_j'(\partial/\partial y - t\psi'(y) - k)\xi_k f$$
$$+ \sum_k \phi_j'(\partial/\partial y - t\psi'(y) + k + n - 2)\xi_k f + \phi_j A_t f, .$$

We want to estimate each of them separately. First, we find that the first and the last term on the right hand side after summing on j, are bounded by

$$C||A_t f||_{L^p(R^-\times S, dx)}.$$

Claim We can get similar estimates for the intermediate terms, i.e.,

(*)
$$\sum_j ||\phi_j' \sum_k (\partial/\partial y - t\psi'(y) - k)\xi_k f||_{L^p(I_j\times S, e^{(n+2\epsilon)y})}$$
$$\leq C||A_t f||_{L^p(R^-\times S, e^{ny}dydw)}.$$

(**)
$$\sum_j ||\phi_j' \sum_k (\partial/\partial y - t\psi'(y) + k + n - 2)\xi_k f||_{L^p(I_j\times S, e^{(n+2\epsilon)y}dydw)}$$
$$\leq C||A_t f||_{L^p(R^-\times S, e^{ny}dydw)}.$$

Then after getting terms together, and from (22), (*), (**) we get

$$||f||_{L^q(R^-\times S, e^{(n+2\epsilon q+2\epsilon q/p)y}dydw)} \leq C||A_t f||_{L^p(R^-\times S, dx)}.$$

Now if we choose small ϵ, then $e^{2py} \leq e^{2\epsilon q + 2\epsilon q/p}$. (16′), (21) and the above inequality implies

$$(3) \qquad ||e^{t\psi} f||_{L^q(R^- \times S, dx)} \leq C||e^{t\psi}\Delta f||_{L^p(R^- \times S, dx)}.$$

Proof of the claim. Since

$$\begin{aligned}\sum_j ||\phi_j' \sum_k (\partial/\partial y - t\psi'(y) - k)\xi_k f||_{L^p(I_j \times S, e^{(n+2\epsilon)y} dydw)} \\ \leq C||\sum_k (\partial/\partial y - t\psi'(y) - k)\xi_k f||_{L^p(R^- \times S, e^{(n+2\epsilon)y} dydw)} \\ = C||e^{t\psi} e^y D e^{-t\psi} f||_{L^p(R^- \times S, e^{(n+2\epsilon)y} dydw)} \\ \leq C'||e^{t\psi} e^{2y} \Delta e^{-t\psi} f||_{L^p(R^- \times S, e^{ny} dydw)}.\end{aligned}$$

The last inequality follows from (16′). On the other hand, (**) is equivalent to

$$\begin{aligned}\sum_j ||\phi_j' f||_{L^p(R^- \times S, e^{(n+2\epsilon)y} dydw)} \\ \leq C||\sum_k (\partial/\partial y - t\psi'(y) - k)\xi_k f||_{L^p(R^- \times S, e^{ny} dydw)}.\end{aligned}$$

But the second inequality is (16) and we have proved the theorem.

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