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ASYMPTOTIC STABILITY OF SOME SEQUENCES RELATED TO INTEGRAL CLOSURE

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Abstract In this paper we will show that if E is an injective module over a commutative ring A, then the sequence of sets

$$Ass_A(A/(I^n)^{*(E)}),\ \ n \in N,$$

is increasing and ultimately constant. Also we will obtain some results concerning the integral closure of ideals related to some modules.

1. Introduction

Throughout this paper A denotes a commutative Noetherian ring (with a non-zero identity).

We recall that if L is an A-module, a prime ideal P of A is said to be associated prime of L if there exists an element $x \in L$ such that $(0 :_A Ax) = P$ (see [5]). The set of associated primes of L is denoted by $Ass_A(L)$.

Now we recall some of concepts from [11] and [3]. Let H be either an Artinian or an injective A- module, and Let I be an ideal of A. Then I is said to be a reduction of the ideal J of A relative to H if $I \subseteq J$ and there exists a positive integer n such that

$$(0 :_H IJ^n) = (0 :_H J^{n+1}).$$

An element x of A is said to be integrally dependent on I relative to H if there exists a positive integer n such that

$$(0 :_H \sum_{i=1}^{n} x^{n-i} I^i) \subseteq (0 :_H x^n).$$

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Furtheremore, the set of elements of A which are integrally dependent on I relative to H is an ideal of A, called the integral closure of I relative to H and it is denoted by $I^{\star(H)}$. In fact it is the largest ideal of A which has I as a reduction relative to H.

In this paper we will consider the asymptotic stability of certain sequences of sets connected with integral closures of ideals relative to some modules over commutative Noetherian rings.

Throughout the remainder of this paper I will denote an ideal of A.

2. Auxiliary results

REMARK 2.1. We have
(a) (See [3, 2.6]) and [6, Lemma 4].)

$$I^{\star(E)} = \bigcap_{P \in Ass_A(E)} (\bar{I}A_P)^c,$$

where $(\bar{I}A_P)^c$ denotes the contraction of $\bar{I}A_P$ with respect to $\phi_P : A \to A_P$. (Here $\bar{I}A_P$ denotes the integral closure of IA_P in the sense of [7].)

(b) (See [12, (1.19)].) Let F a flat A-module. Then for any arbitrary A-module H we have

$$Ass_A(H \otimes_A F) = \{P \in Ass_A(H) : P \subseteq P' \text{ for some } P' \in Coass_A(F)\}.$$

(Here $Coass_A(F)$ denotes (see [12]) the set of coassociated primes of F.)

LEMMA 2.2. *Let E be an injective A-module and let $S = A \setminus \sum_{P \in Ass_A(E)} P$. Then*

$$S^{-1}(I^{\star(E)}) = (S^{-1}I)^{\star(E)}.$$

(Note that E has an structure as $S^{-1}A$-module (see [2, (2.8)]).)

Proof. The injective module E has an structure as an $S^{-1}A$-module and that

$$(0 :_E I) = (0 :_E S^{-1}I).$$

(See [2, (2.8)].) Now let $\lambda \in (S^{-1}I)^{\star(E)}$. Then $\lambda = x/s$ for some $x \in A$. It follows that $x/1 \in (S^{-1}I)^{\star(E)}$. So there exists $n \in N$ such that

$$(0 :_E \sum_{i=1}^{n}(x/1)^{n-i}(S^{-1}I)^i) \subseteq (0 :_E (x/1)^n).$$

Then we have

$$(0 :_E \sum_{i=1}^{n}(x/1)^{n-i}(S^{-1}I)^i) = \cap_{i=1}^{n}(0 :_E S^{-1}(x^{n-i}I^i)) =$$

$$\cap_{i=1}^{n}(0 :_E x^{n-i}I^i) = (0 :_E \sum_{i=1}^{n} x^{n-i}I^i).$$

On the other hand we have

$$(0 :_E (x/1)^n) = (0 :_E x^n/1) = (0 :_E x^n).$$

Hence

$$(0 :_E \sum_{i=1}^{n} x^{n-i}I^i) \subseteq (0 :_E x^n).$$

It implies that $x \in I^{\star(E)}$ so that $\lambda = x/s \in S^{-1}(I^{\star(E)})$. To see the reverse inclusion let $\lambda \in S^{-1}(I^{\star(E)})$. It implies that $\lambda = x/s$ for some $x \in I^{\star(E)}$. Then by similar arguments as above we have $x/1 \in (S^{-1}I)^{\star(E)}$ so that $\lambda \in (S^{-1}I)^{\star(E)}$. This completes the proof.

LEMMA 2.3. *Let H be an Artinian A-module. Then we have the following.*

(a) If J is another ideal of A with $I \subseteq J$, then $I^{\star(H)} \subseteq J^{\star(H)}$. (This is also true when our module is an injective A-module)

(b) If M is a maximal ideal of A and $H = \cup_{n=1}^{\infty}(0 :_H M^n)$, then

$$(IA_M)^{\star(H)} = I^{\star(H)}A_M.$$

(Note that the Artinian module H has an structure as an A_M-module (see [9].)

(c) Let $H = \oplus_{i=1}^{m} H_i$ for some $m \in N$. (Note that this is always possible because H is an Artinian A-module (see [9, (1.4)]). Then we have

$$I^{\star(H)} = \bigcap_{i=1}^{m} I^{\star(H_i)}$$

Proof. (a) Let $x \in I^{\star(H)}$. then there exists a positive integer n such that

$$(0 :_H \sum_{i=1}^{n} x^{n-i} I^i) \subseteq (0 :_H x^n).$$

Now we have

$$(0 :_H \sum_{i=1}^{n} x^{n-i} J^i) \subseteq (0 :_H \sum_{i=1}^{n} x^{n-i} I^i) \subseteq (0 :_H x^n).$$

Hence $x \in J^{\star(H)}$.

(b) H has an structure as A_M- module and that

$$(0 :_H I) = (0 :_H IA_M).$$

(See [9].) Now we have similar arguments as in the proof of Lemma 2.2.

(c) It is clear that I is a reduction of $I^{\star(H)}$ relative to H (see [11]). Hence there exists $t \in N$ such that

$$(0 :_H I(I^{\star(H)})^t) = (0 :_H (I^{\star(H)})^{t+1}).$$

Then we have

$$\oplus_{i=1}^{m}(0 :_{H_i} I(I^{\star(H)})^t) = \oplus_{i=1}^{m}(0 :_{H_i} (I^{\star(H)})^{t+1}).$$

Hence for $i = 1, ..., m$,

$$(0 :_{H_i} I(I^{\star(H)})^t) = (0 :_{H_i} (I^{\star(H)})^{t+1}),$$

so that I is a reduction of $I^{\star(H)}$ relative to H_i. It follows that $I^{\star(H)} \subseteq I^{\star(H_i)}$ for $i = 1, ..., m$. Hence we have

$$I^{\star(H)} \subseteq \cap_{i=1}^{m} I^{\star(H_i)}.$$

To see the reverse inclusion, we note that for $i = 1, 2, \ldots, m$, I is a reduction of $I^{\star(H_i)}$ relative to H_i. Hence there exists $t_i \in N$ $(1 \leq i \leq m)$ such that

$$(0 :_{H_i} I(I^{\star(H_i)})^{t_i}) = (0 :_{H_i} (I^{\star(H_i)})^{t_i+1}).$$

Let $t = \max_{1 \leq i \leq m} t_i$, and set $t - t_i = k_i$ $(1 \leq i \leq m)$. Now we have

$$(0 :_H I(I^{\star(H_i)})^{t}) = \oplus_{i=1}^{m} (0 :_{H_i} I(I^{\star(H_i)})^{t_i+k_i}) =$$

$$\oplus_{i=1}^{m} (0 :_{H_i} (I^{\star(H_i)})^{t_i+k_i+1}) = (0 :_H (I^{\star(H_i)})^{t+1}).$$

Hence for each i $(1 \leq i \leq m)$, I is a reduction of $I^{\star(H_i)}$ relative to H. It implies that for $i = 1, \ldots, m$, $I^{\star(H_i)} \subseteq I^{\star(H)}$. Therefore we have

$$\cap_{i=1}^{m} I^{\star(H_i)} \subseteq I^{\star(H)}.$$

This completes the proof.

3. Asymptotic stability

Throughout this sectin, N will denote the set of positive integers.

In [8, (2.4)] Ratliff showed that the sequence of

$$Ass_A(A/(I^n)^-), \ n \in N,$$

is increasing and ultimately constant. (Here $(I^n)^-$ denotes the integral closure of I^n in the sense of [7].) We denote the ultimate constant value of this sequence by $\bar{A}s^\star(I, A)$.

LEMMA 3.1. *Let L, H, and E be respectively a finitely generated, an Artinian, and an injective A-module. Then the sequences of sets*

(a)

$$Ass_A((L \otimes_A H)/(I^n)^{\star(H)}(L \otimes_A H)), \ n \in N,$$

and

$$Ass_A((I^n)^{\star(H)}(L \otimes_A H)/(I^{n+1})^{\star(H)}(L \otimes_A H), \ n \in N,$$

(b)

$$Ass_A((L \otimes_A E)/(I^n)^{\star(E)}(L \otimes_A E)),\ n \in N$$

and

$$Ass_A((I^n)^{\star(E)}(L \otimes_A E)/(I^{n+1})^{\star(E)}(L \otimes_A E),\ n \in N.$$

are ultimately constant.

Proof. (a) Let

$$V_n = (L \otimes_A H)/(I^n)^{\star(H)}(L \otimes_A H) \text{ and}$$
$$T_n = (I^n)^{\star(H)}(L \otimes_A H)/(I^{n+1})^{\star(H)}(L \otimes_A H).$$

Now the sequence V_n, $n \in N$ is ultimately constant because $L \otimes_A H$ is an Artinian A-module and the sequence $(I^n)^{\star(H)}(L \otimes_A H)$, $n \in N$ is a decreasing sequence by Lemma 2.3 (a). In this case, $T_n = 0$ for sufficientely large n so that $Ass_A(T_n) = \emptyset$.

(b) Let

$$V_n = (L \otimes_A E)/(I^n)^{\star(E)}(L \otimes_A E) \text{ and}$$
$$T_n = ((I^n)^{\star(E)}(L \otimes_A E)/(I^{n+1})^{\star(E)}(L \otimes_A E).$$

Then we have

$$V_n \cong E/(I^n)^{\star(E)}E \otimes_A L.$$

Now for an arbitrary ideal J of A,

$$JE = (0 :_E (0 :_A J)$$

by [1, (3.2)]. Hence

$$(I^n)^{\star(E)}E = (0 :_E (0 :_A (I^n)^{\star(E)})).$$

Now the sequence V_n, $n \in N$ is ultimately constat because A is Noetherian and the sequence $(0 :_A (I^n)^{\star(E)})$, $n \in N$ is an increasing sequence by Lemma 2.3 (a). To see the second assertion, we consider the exact sequence of A-modules and A-homomorphisms

$$0 \to T_n \to V_{n+1} \to V_n \to 0.$$

Now By using the first assertion we deduce that $T_n = 0$ so that $Ass_A T_n = \emptyset$. This completes the proof.

THEOREM 3.2. *Let E be an injective A-module. Then the sequence of sets*

$$Ass_A(A/(I^n)^{\star(E)}),\ n \in N.$$

is increasing and ultimately constant. Further if we denote the ultimate costant value of the sequence of $Ass_A(A/(I^n)^{\star(E)})$, $n \in N$, by $\bar{As}^{\star}(I,E)$, then

$$\bar{As}^{\star}(I,E) = \{P \in \bar{As}^{\star}(I,A) : P \subseteq P' \text{ for some } P' \in Ass_A(E)\}.$$

Proof. By Remark 2.1 (a), we have

$$I^{\star(E)} = \bigcap_{P \in Ass_A(E)} (\bar{I}A_P)^c,$$

where $(\bar{I}A_P)^c$ denotes the contraction of $\bar{I}A_P$ with respect to $\phi_P : A \to A_P$. Now let

$$\bar{I} = Q_1 \cap Q_2 \cap \ldots \cap Q_m$$

be a minimal primary decomposition of $\bar{I}$, where Q_j $(1 \leq j \leq m)$ is a P_j-primary ideal of A. Then by [10, 5.41],

$$Ass_A(A/(\bar{I}A_P)^c) = \{P_j : P_j \in Ass_A(A/\bar{I}) \text{ and } P_j \subseteq P\}$$

Hence by above arguments we have

$$I^{\star(E)} = \bigcap_{P_j \subseteq P \text{ for some } P \in Ass_A(E)} Q_j,$$

$1 \leq j \leq m$. In fact it is a minimal primary decomposition of $I^{\star(E)}$. Hence

$$Ass_A(A/I^{\star(E)}) = \{P_j \in Ass_A(A/\bar{I}) : \\ P_j \subseteq P \text{ for some } P \in Ass_A(E)\}.$$

Therefore

$$Ass_A(A/(I^n)^{\star(E)}) = \{P \in Ass_A(A/(I^n)^-) : \\ P \subseteq P' \text{ for some } P' \in Ass_A(E)\}.$$

Now as we mentioned above,

$$Ass_A(A/(I^n)^-),\ n \in N,$$

is increasing and ultimately constant. So the result follows from this.

CROLLARY 3.3. *Let F and E be respectively a flat and an injective A-module. Then the sequence of sets*

$$Ass_A(F/(I^n)^{\star(E)}F),\ n \in N.$$

is increasing and ultimately constant. If we denote the ultimate constant value of this sequence by $\bar{As}^\star(I, F)$ then we have

$$\bar{As}^\star(I, F) = \{P \in \bar{As}^\star(I, E) : P \subseteq P' \text{ for some } P' \in Coas_A(F)\}.$$

Proof.

$$F/(I^n)^{\star(E)}F \cong A/(I^n)^{\star(E)} \otimes_A F.$$

Now the result follows from 3.2 and 2.1 (b).

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