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THE REPEATED ENVELOPING SEMIGROUP COMPACTIFICATIONS

A FATTAHI AND P MILNES

Faculty of Mathematics,
Ferdowsi University of Mashhad, P.O.Box 91775-1159, Mashhad, Iran.
Dept. of Mathematics,
University of Western Ontario, London, Ontario, Canada, N6A 5B7,
E-mail: abfattahi@yahoo.ca and fattahi@math.um.ac.ir ,
E-mail: milnes@uwo.ca.

Abstract This note consists of some efficient examples to support the notion of enveloping semigroup compactification and also employ this notion to obtain the universal reductive compactification.

1. Introduction

A semigroup S is called *right reductive* if for each $a, b \in S$, from $at{=}bt$ for every $t \in S$, it follows that $a = b$. For example, all right cancellative semigroups and semigroups with a right identity, are right reductive. Throughout this article S will be a semitopological semigroup.

By a *semigroup compactification* of S we mean a pair (ψ, X), where X is a compact Hausdorff right topological semigroup, and $\psi : S \longrightarrow X$ is a continuous homomorphism with dense image such that, for each $s \in S$, the mapping $x \longrightarrow \psi(s)x : X \longrightarrow X$ is continuous. The reader is referred to sections 3.1 and 3.3 of [1] for the one-to-one correspondence between compactifications of S and m-admissible subalgebras of $C(S)$ (= the C^*-algebra of all bounded complex-valued continuous functions on S), and also for a discussion of *universal P-compactifications.*

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Let (ψ, X) be a compactification of S, then the mapping $\sigma : S \times X \longrightarrow X$, defined by $\sigma(s,x) = \psi(s)x$, is separately continuous and so (S, X, σ) is a flow. If Σ_1 denotes the enveloping semigroup of the flow (S, X, σ) (i.e., the pointwise closure of semigroup $\{\sigma(s,.) : s \in S\}$ in X^X) and the mapping $\sigma_1 : S \longrightarrow \Sigma_1$ is defined by $\sigma_1(s) = \sigma(s,.)$ for all $s \in S$, then (by [1; 1.6.5]) (σ_1, Σ_1) is a compactification of S. It is easy to show that $\Sigma_1 = \{\lambda_x : x \in X\}$, where $\lambda_x(y) = xy$ for each $y \in X$. If we define the mapping $\theta_0 : X \longrightarrow \Sigma_1$, $\theta_0(x) = \lambda_x$, then θ_0 is a continuous homomorphism with the property that $\theta_0 \circ \psi = \sigma_1$. So (σ_1, Σ_1) is a factor of (ψ, X). By definition, θ_0 is one-to-one, if and only if X is right reductive. So we get the next, which is an extension of the Lawson's result [2; 2.4(ii)] :

PROPOSITION 1.1. *Let (ψ, X) be a compactification of S. Then $(\sigma_1, \Sigma_1) \cong (\psi, X)$, if and only if X is right reductive.*

2. Main Results

Let (ψ, X) be a compactification of S. If X is not right reductive, (σ_1, Σ_1) is a proper factor of (ψ, X). So we can continue this process by induction. Let (σ_n, Σ_n) be constructed . If Σ_n is not right reductive, we define Σ_{n+1} as the enveloping semigroup of the flow (S, Σ_n, σ), where $\sigma : S \times \Sigma_n \longrightarrow \Sigma_n$ is defined by $\sigma(s,x) = \sigma_n(s)x$. Trivially, $\Sigma_{n+1} = \{\lambda_x : x \in \Sigma_n\}$ and $(\sigma_{n+1}, \Sigma_{n+1})$ is a new compactification of S, where $\sigma_{n+1} : S \longrightarrow \Sigma_{n+1}$ is defined by $\sigma_{n+1}(s) = \sigma(s,.)$.

The method of arriving at Σ_n shows that Σ_n is not right reductive if and only if there exist $x, y \in X$ and $t_1, ... t_n \in X$ such that $xt_1t_2...t_n \neq yt_1t_2...t_n$ and $xu_1u_2...u_nu_{n+1} = yu_1u_2...u_nu_{n+1}$ for all $u_1, ... u_{n+1} \in X$. So this process stops at the nth stage, if $X^n = X^{n+1}$.

In the following we present an example (example 2.2) of a semigroup compactification X of a semigroup S for which, for each n, Σ_n is not right reductive. So we get an infinite lattice $\{(\sigma_n, \Sigma_n)\}$ of compactifications ordered by $(\sigma_n, \Sigma_n) > (\sigma_{n+1}, \Sigma_{n+1})$.

EXAMPLE 2.1. For a positive integer n, suppose α_n is a real number belonging to $[1 - \frac{1}{2^{1/(n+2)}}, 1 - \frac{1}{2^{1/(n+1)}})$ and let $S_n =$

$[1/2, 1-\alpha_n] \subset \mathbb{R}$ with multiplication $st = max\{1/2, s\cdot t\}$, where $s.t$ denotes the product of s and t in $\mathbb{R}$. S_n is a compact Hausdorff abelian topological semigroup. Set $X_n = S_n$ and let $i : S_n \longrightarrow X_n$ be the identity map, so that (i, X_n) is the identity compactification of S_n. We show that for X_n, Σ_n is not right reductive, but Σ_{n+1} is right reductive. Since $\alpha_n < 1 - \frac{1}{2^{1/n+1}}$ we can choose $a, b \in X_n$ such that $\frac{1}{2(1-\alpha_n)^n} < a, b < 1 - \alpha_n$ and $a \neq b$. Set $t_1 = t_2 = ... = t_n = 1 - \alpha_n$. Hence $a.t_1 .....t_n = a.(1-\alpha_n)^n > 1/2$ and $b \cdot t_1 \cdot ... \cdot t_n = b \cdot (1-\alpha_n)^n > 1/2$. Therefore

$$at_1...t_n = a \cdot t_1 \cdot ... \cdot t_n \neq b \cdot t_1 \cdot ... \cdot t_n = bt_1...t_n.$$

But for every $a, b \in X_n$ and $t_1, ..., t_n, t_{n+1} \in X_n$, we have $a.t_1.... t_{n+1} < (1-\alpha_n)^{n+2} < 1/2$ and $b.t_1.....t_{n+1} < (1-\alpha_n)^{n+2} < 1/2$. Therefore

$$at_1...t_{n+1} = 1/2 = bt_1...t_{n+1},$$

which means that Σ_n is not right reductive but Σ_{n+1} is right reductive.

EXAMPLE 2.2. Let $S = \Pi_{m\in\mathbb{N}} S_m$ with the product topology and product multiplication, where for each positive integer m, S_m is the semigroup in example 2.1. S is a compact topological semigroup. Much as in example 2.1., let (i, X) be the identity compactification of S. We show that for X, for each $n \in \mathbb{N}$, Σ_n is not right reductive. Let $n \in \mathbb{N}$ and $A, B \in X$ be such that the nth coordinate of A and B is a_n and b_n respectively and other coordinates of A and B are equal to $1/2$, and a_n, b_n belongs to $(\frac{1}{2(1-\alpha_n)^n}, 1-\alpha_n)$ and $a_n \neq b_n$.

Set $T_1 = T_2 = ... = T_n = (1/2, 1/2, ..., 1/2, 1-\alpha_n, 1/2, 1/2...) \in X$, i.e., the nth coordinate is $1-\alpha_n$ and the others are $1/2$. Since $a_n \cdot (1-\alpha_n)^n > 1/2$ and $b_n \cdot (1-\alpha_n)^n > 1/2$, we have:

$$AT_1T_2...T_n \neq BT_1T_2...T_n$$

But since $\alpha_n > 1 - \frac{1}{2^{1/(n+2)}}$, for every $U_1, U_2, ..., U_{n+1} \in X$ we have:

$$AU_1U_2...U_{n+1} = (1/2) = BU_1U_2...U_{n+1}.$$

Set $\mathcal{F} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n = \bigcap_{n\in\mathbb{N}} \sigma_n^*(\mathcal{C}(\Sigma_n))$ (where * denotes the conjugate mapping). We show that $\mathcal{F}$ contains only constant functions. Suppose $f \in \mathcal{F}$. For each positive integer n, $f = \sigma_n^*(g_n)$, for some $g_n \in \mathcal{C}(\Sigma_n)$. So for each $s \in S$:

$$\begin{aligned} f(s) = g_n(\sigma_n(s)) &= g_n(\theta_{n-1} \circ \sigma_{n-1}(s)) = \dots \\ &= g_n(\theta_{n-1} \circ \theta_{n-2} \circ \dots \circ \theta_0 \circ i(s)), \end{aligned}$$

where $\theta_i : \Sigma_i \longrightarrow \Sigma_{i+1}$ is defined by $\theta_i(x) = \lambda_x$.

If $s = (a_1, a_2, \dots, a_n, a_{n+1}, \dots)$, it is easy to show that

$$\theta_{n-1} \circ \theta_{n-2} \circ \dots \circ \theta_0 \circ i(s) = \theta_{n-1} \circ \theta_{n-2} \circ \dots \circ \theta_0 \circ i(s_n),$$

where $s_n = (1/2, 1/2, \dots, 1/2, a_n, a_{n+1}, \dots)$ is in S. So $f(s) = f(s_n)$.

But $s_n \longrightarrow (1/2)$ and therefore $f(s_n) \longrightarrow f((1/2))$. Hence $f(s) = f(s_n) = f((1/2))$, i.e., f is a constant function. Thus $S^{\mathcal{F}}$ is the trivial compactification, containing only the identity map.

REMARK 2.3. In general, we can use the notion of enveloping compactification to present the universal reductive compactification, whose existence is guaranteed by subdirect product metods (see [1; 3.3.4]). Suppose (ψ, X) is the universal compactification of S. If, for some $n \in \mathbb{N}$, Σ_n is right reductive, then (σ_n, Σ_n) is the universal reductive compactification of S. Otherwise, let $(\psi_1, X_1) = \wedge_{n\in\mathbb{N}}(\sigma_n, \Sigma_n)$ be the infimum of the lattice $\{(\sigma_n, \Sigma_n)\}$ (see [1; 3.3.22]). If X_1 is not right reductive, we continue this process with (ψ_1, X_1). By transfinite induction, we obtain a well ordered set $\{(\sigma_\alpha, \Sigma_\alpha)\}$ of compactifications of S, so that, for each β, if α is an immediate predecessor of β, i.e. $\beta = \alpha^+$, define a campactification $(\sigma_\beta, \Sigma_\beta)$ of S, where $\Sigma_\beta = \{\lambda_x : x \in \Sigma_\alpha\}$, and $\sigma_\beta : S \longrightarrow \Sigma_\beta$ is defined by $\sigma_\beta(s) = \sigma(s, .)$. If β is a limit ordinal, we define $(\sigma_\beta, \Sigma_\beta)$ by $\wedge_{\alpha\in\beta}(\sigma_\alpha, \Sigma_\alpha)$. The well ordered set $\{(\sigma_\alpha, \Sigma_\alpha)\}$ is similar to a unique ordinal number α_0. Therefore $\wedge_{\alpha\in\alpha_0}(\sigma_\alpha, \Sigma_\alpha)$ must be the universal reductive compactification of S.

References

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