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GENERALIZATION OF A FIRST ORDER NON–LINEAR COMPLEX ELLIPTIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS IN SOBOLEV SPACE

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Abstract In this paper we discuss on the existence of general solution of Partial Differential Equations

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w})$$

in the Sololev Space $W_{1,p}(D)$, that is generalization of a first order Non-linear Elliptic System of Partial Differential Equations

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \frac{\partial w}{\partial z}).$$

1. Introduction

Suppose that D is a domain with finite area in the complex plane and $F = F(z, w, \frac{\partial w}{\partial z})$, $G = G(z, w, \bar{w}) \in L_p(D)$, $1 < p < \infty$, and define the weakly singular and strongly singular operators T_D and $\prod_D$:

$$T_D f(z) = -\frac{1}{\pi} \iint_D \frac{1}{\xi - z} f(\xi) d\zeta d\eta$$

$$\prod_D f(z) = -\frac{1}{\pi} \iint_D \frac{1}{(\xi - z)^2} f(\xi) d\zeta d\eta$$

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that $\xi = \zeta + i\eta, z = x + iy, \frac{\partial T_D f(z)}{\partial \bar{z}} = f(z), \frac{\partial T_D f(z)}{\partial z} = \prod_D f(z)$ and if $f \in L_p(D)$ then $T_D f$ is bounded and Holder continuous,[1]. T_D maps the Banach space $L_p(D), 1 < p < \infty$, into the Sobolev space $W_{1,p}(D)$ [1].

Furthermore, we assume that $w \in W_{1,p}(D), 1 < p < \infty$, is an arbitrary solution of :

$$\frac{\partial w}{\partial \bar{z}} = F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w}). \tag{1}$$

We define a function ϕ as follow:

$$\phi(z) = w(z) - T_D[F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w})]. \tag{2}$$

on differentiating ϕ partially with respect to $\bar{z}$ and z respectively, we obtain the following:

$$\begin{cases} \frac{\partial \phi}{\partial \bar{z}} = \frac{\partial w}{\partial \bar{z}} - [F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w})] = 0 \\ \frac{\partial \phi}{\partial z} = \frac{\partial w}{\partial z} - \prod_D[F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w})] \end{cases} \tag{3}$$

at least in the sobolev sense. Furthermore, since $F(z, w, \frac{\partial w}{\partial z})$, $G(z,w, \bar{w}) \in L_p(D), 1 < p < \infty$, the following estimates hold:

$$||\phi||_{p,D} \leq ||w||_{p,D} + ||T_D(F + G)||_{p,D}$$

$$||\frac{\partial \phi}{\partial z}||_{p,D} \leq ||\frac{\partial w}{\partial z}||_{p,D} + ||\prod_D(F + G)||_{p,D}.$$

It follows from the first equation in (3) and Weyls Lemma[1] that ϕ is a holomorphic function in D, it belongs to the Sobolev Space $W_{1,p}(D), 1 < p < \infty$. Moreover, we deduce that, if w is a solution of (1), then w necessarily is of the form:

$$w(z) = \phi(z) + T_D[F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w})$$

where ϕ is holomorphic in D. Furthermore

$$\frac{\partial w}{\partial z} = \phi'(z) + \prod_D[F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w})].$$

We now suppose that (w, h) is a solution of the following system:

$$(4)\qquad \begin{cases} w(z) &= \phi(z) + T_D[F(z, w, h) + G(z, w, \bar{w})] \\ h(z) &= \phi'(z) + \prod_D[F(z, w, h) + G(z, w, \bar{w})] \end{cases}$$

for a function $\phi \in W_{1,p}(D), 1 < p < \infty, S_D < \infty (S_D$ is the area of $D)$ and ϕ holomorphic in D. On differentiating the first equation in (4) partially with respect to $\bar{z}$ and z we obtain:

$$\begin{cases} \frac{\partial w}{\partial \bar{z}} = 0 + F(z, w, h) + G(z, w, \bar{w}) \\ \frac{\partial w}{\partial z} = \phi' + \prod_D[F(z, w, h) + G(z, w, \bar{w})] = h \end{cases}$$

this shows that w is a solution to the given differential equation (1). On substitution $h = \frac{\partial w}{\partial z}$ in (3), we obtain the following result:

THEOREM 1.1. *A function $w \in W_{1,p}(D), 1 < p < \infty, S_D < \infty$, is a solution to the partial differential equation (1), if and only if for a function $\phi \in W_{1,p}(D)$ and holomorphic in D, (w, h) satisfies the system (4).*

2. Existence of a General Solution in $W_{1,p}(D)$

We make the following assumptions:
I. The domain D has a finite area.
II. As a function of the variables $z \in D, w, \bar{w}, h; F(z, w, h) + G(z, w, \bar{w})$ is a continuous function of its variables.
III. The functions $F(z, w, h)$ and $G(z, w, \bar{w})$ satisfy a Lipschitz condition of the form:

$$|F(z, w, h) - F(z, \tilde{w}, \tilde{h})| \leq L_1|w - \tilde{w}| + L_2|h - \tilde{h}|$$
$$|G(z, w, \bar{w}) - G(z, w, \tilde{\bar{w}})| \leq L_3|w - \tilde{w}|$$

almost everywhere in D; whereas the constant L_2 is strictly less than 1, L_1 and L_3 are arbitrary positive numbers.
IV. There exist $w, h \in L_p(D), 1 < p < \infty$, such that $F(z, w, h)$, $G(z, w, \bar{w}) \in L_p(D)$.

REMARK:.

The assumption (III) and (IV) guarantee $F(z,w,h)+G(z,w,\bar{w}) \in L_p(D)$, whenever $w, h \in L_p(D)$. In fact we then have

$$\begin{aligned}|F(z,w,h)+G(z,w,\bar{w})| &\le |F(z,w,h)-F(z,w_0,h_0)| \\ &+ |G(z,w,\bar{w})-G(z,w_0,\bar{w}_0)| \\ &+ |F(z,w_0,h_0)+G(z,w_0,\bar{w}_0)| \\ &\le (L_1+L_3)|w-w_0| + L_2|h-h_0| \\ &+ |F(z,w_0,h_0)+G(z,w_0,\bar{w}_0)|.\end{aligned}$$

The function $F(z,w,h)+G(z,w,\bar{w})$ is thus measurable and it belongs to $L_p(D)$, since

$$\begin{aligned}||F(z,w,h)+G(z,w,\bar{w})||_{p,D} &\le (L_1+L_3)||w-w_0||_{p,D} \\ &+ L_2||h-h_0||_{p,D} \\ &+ ||F(z,w_0,h_0)+G(z,w_0,\bar{w}_0)||_{p,D}.\end{aligned}$$

We shall denote by $\mathfrak{J}_p(D)$ the set of pairs (w,h) for which $w, h \in L_p(D), 1 < p < \infty$, and define the norm by the relation

$$||(w,h)|| = ||(w,h)||_{p,\lambda} = max(\lambda||w||_p, ||h||_p) \quad \lambda > 0.$$

The set $\mathfrak{J}_p(D)$ is then a Banach Space.
We shall now tackle the system (4) in $\mathfrak{J}_p(D), 1 < p < \infty$. For a pair $(w,h) \in \mathfrak{J}_p(D)$ we define an operator L as follows:

$$L(w,h) = (W,H);$$

$$\begin{cases} W(z) = \phi(z) + T_D[F(z,w,h)+G(z,w,\bar{w})] \\ H(z) = \phi'(z) + \prod_D[F(z,w,h)+G(z,w,\bar{w})] \end{cases}$$

where ϕ is a fixed holomorphic function in D and it belongs to $W_{1,p}(D), 1 < p < \infty$. On the strength of:

THEOREM 2.1. *If D is a domain of finite area S_D and $f \in L_p(D), 1 < p < \infty$, then $T_D f \in L_p(D)$ as well. The following estimate holds:*

$$||T_D f||_{p,D} \leq B(D)||f||_{p,D}.$$

[1].
And the Calderon-Zygmund Theorem (see Basic Integral Operators) it follows that $(W, H) \in \mathfrak{J}_p(D)$; i.e. the operator L maps the Banach space $\mathfrak{J}_p(D)$ into itself.
Suppose that $(W, H), (\tilde{W}, \tilde{H})$ are the images of two arbitrarily chosen elements $(W, H), (\tilde{W}, \tilde{H}) \in \mathfrak{J}_p(D)$ respectively:

$$\begin{cases} W(z) = \phi(z) + T_D[F(z, w, h) + G(z, w, \bar{w})] \\ H(z) = \phi'(z) + \prod_D[F(z, w, h) + G(z, w, \bar{w})] \end{cases}$$

$$\begin{cases} \tilde{W}(z) = \phi(z) + T_D[F(z, \tilde{w}, \tilde{h}) + G(z, \tilde{w}, \tilde{\bar{w}})] \\ \tilde{H}(z) = \phi'(z) + \prod_D[F(z, \tilde{w}, \tilde{h}) + G(z, \tilde{w}, \tilde{\bar{w}})] \end{cases}$$

It then follows that

$$\begin{aligned}
\lambda||W - \tilde{W}||_p &\leq \lambda||T_D||_p||F(z, w, h) + G(z, w, \bar{w}) \\
&\qquad - [F(z, \tilde{w}, \tilde{h}) + G(z, \tilde{w}, \tilde{\bar{w}})]||_p \\
&\leq \lambda B(D)||F(z, w, h) + G(z, w. \bar{w}) \\
&\qquad - [F(z, \tilde{w}, \tilde{h}) + G(z, \tilde{w}, \tilde{\bar{w}})]||_p \\
&\leq \lambda B(D)[||F(z, w, h) - F(z, \tilde{w}, \tilde{h})|| \\
&\qquad + ||G(z, w. \bar{w}) - G(z, \tilde{w}, \tilde{\bar{w}})||] \\
&\leq \lambda B(D)[L_1||w - \tilde{w}|| + L_2||h - \tilde{h}|| + L_3||w - \tilde{w}||] \\
&= \lambda B(D)[(L_1 + L_3)||w - \tilde{w}|| + L_2||h - \tilde{h}||] \\
&\leq B(D)[(L_1 + L_3) + \lambda L_2]||(w, h) - (\tilde{w}, \tilde{h})||_{p,\lambda}
\end{aligned}$$

because

$$\begin{aligned}
||(w, h) - (\tilde{w}, \tilde{h})||_{p,\lambda} &= ||(w - \tilde{w}, h - \tilde{h})|| \\
&= max(\lambda||w - \tilde{w}||, ||h - \tilde{h}||)
\end{aligned}$$

if

$$\lambda||w-\tilde{w}|| \geq ||h-\tilde{h}||$$

then

$$\begin{aligned}\lambda||W-\tilde{W}||_{p,\lambda} &\leq \lambda B(D)[(L_1+L_3)||w-\tilde{w}||+L_2||h-\tilde{h}||] \\ &\leq \lambda B(D)[(L_1+L_3)||w-\tilde{w}||+\lambda L_2||w-\tilde{w}||] \\ &= B(D)[\lambda(L_1+L_3)+\lambda^2 L_2]||w-\tilde{w}|| \qquad (5)\end{aligned}$$

on the other hand

$$\begin{aligned}&B(D)[(L_1+L_3)+\lambda L_2]||(w,h)-(\tilde{w},\tilde{h})||_{p,\lambda} = \\ &\qquad B(D)[\lambda(L_1+L_3)+\lambda^2 L_2]||w-\tilde{w}||. \qquad (6)\end{aligned}$$

Or, suppose that

$$\lambda||w-\tilde{w}|| \leq ||h-\tilde{h}||$$

then

$$\begin{aligned}\lambda||W-\tilde{W}||_{p,\lambda} &\leq \lambda B(D)[(L_1+L_3)\frac{1}{\lambda}||h-\tilde{h}||+L_2||h-\tilde{h}||] \\ &= B(D)[(L_1+L_3)+\lambda L_2]||h-\tilde{h}||. \qquad (7)\end{aligned}$$

On the other hand

$$\begin{aligned}&B(D)[(L_1+L_3)+\lambda L_2]||(w,h)-(\tilde{w},\tilde{h})||_{p,\lambda} = \\ &\qquad B(D)[(L_1+L_3)+\lambda L_2]||h-\tilde{h}||. \qquad (8)\end{aligned}$$

consequently [from (5),(6),(7),(8)]:

$$\lambda||W-\tilde{W}||_p \leq B(D)[(L_1+L_3)+\lambda L_2]||(w,h)-(\tilde{w},\tilde{h})||_{p,\lambda}$$

similarly

$$||H-\tilde{H}||_p \leq A(D)[\frac{1}{\lambda}(L_1+L_3)+L_2]||(w,h)-(\tilde{w},\tilde{h})||_{p,\lambda}.$$

This means that

$$||(W, H) - (\tilde{W}, \tilde{H})|| \leq [(L_1 + L_3) + \lambda L_2] max(B(D) + \frac{1}{\lambda}A(D))||(w, h) - (\tilde{w}, \tilde{h})||.$$

Thus if

$$[(L_1 + L_3) + \lambda L_2] max(B(D) + \frac{1}{\lambda}A(D)) < 1$$

then the operator L is contractive in $\mathfrak{J}_p(D)$ and, as such, there exists a unique fixed element (w, h) of the operator L, which is also a solution to (4):

$$\begin{cases} w = \phi + T_D[F(z, w, h) + G(z, w, \bar{w})] \\ h = \phi' + \prod_D[F(z, w, h) + G(z, w, \bar{w})]. \end{cases}$$

The corresponding w is then, by theorem(1.1) a general solution to the complex differential equation (1).

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