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SOME PROPERTIES AROUND $1\frac{1}{2}$ STARCOMPACT SPACES

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Abstract A $1\frac{1}{2}$-starcompact space has one of the most curious properties among the spaces of starcompactness. It is not too far away from countably compact spaces and may be considered as the first candidate for extending theorems about countably compact spaces. Unfortunately, $1\frac{1}{2}$-starcompactness is not so easy to be recognized as 2-starcompactness which will follow from countable pracompactness. We investigate some properties around $1\frac{1}{2}$-starcompact spaces.

1. Introduction

The study of star-covering properties of a topological space could be started around 1970s by W. Fleischman [5] or even earlier. A systematic study on them was done by van Douwen et al in 1991 [2]. One of recent and systematic investigation in this area is Matveev's survey [8].

In this paper, we investigate some properties and their interrelationships around $1\frac{1}{2}$-starcompact spaces. More precisely, we construct an example of a first-countable normal $\mathcal{P}$-starcompact that is neither $\mathcal{K}$- starcompact nor $n\frac{1}{2}$-starcompact. In Section 3, we give a proof of a very useful theorem that every countably pracompact space is 1-cl-starcompact (and hence 2-starcompact). We

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study countably pracompact spaces and prove theorems related to right separated spaces and theorems on spaces with property wD. We give an example of a countably pracompact space which is not countably compact. We also introduce an example of countably pracompact space which is not $1\frac{1}{2}$-starcompact.

We shall start with some basic notations and terminology which will be used throughout this paper. As far as topological concepts are concerned, we follow [4]. Let X be a space and let $\mathcal{U}$ be a collection of subsets of X. For any non-empty $A \subset X$, let $\mathrm{St}(A,\mathcal{U}) = \mathrm{St}^1(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$ and define $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^n(A,\mathcal{U}),\mathcal{U})$ for $n \in \mathbb{N}$. $\mathrm{St}^n(\{x\},\mathcal{U})$ is simply written as $\mathrm{St}^n(x,\mathcal{U})$. For each $n \in \mathbb{N}$, a space X is called *n-starcompact* (resp. *$n\frac{1}{2}$-starcompact*) [8] provided that for every open cover $\mathcal{U}$ of X there is a finite subset F of X (resp. a finite subcollection $\mathcal{V}$ of $\mathcal{U}$) such that $\mathrm{St}^n(F,\mathcal{U}) = X$ (resp. $\mathrm{St}^n(\cup\mathcal{V},\mathcal{U}) = X$).

Let $\mathbb{N}_{1/2} = \{n - 1/2 : n \in \mathbb{N}\}$ and $\widetilde{\mathbb{N}} = \mathbb{N} \cup \mathbb{N}_{1/2}$. Then for each $n \in \widetilde{\mathbb{N}}$, every n-starcompact space is $n\frac{1}{2}$-starcompact.

For a family $\mathcal{U}$ of sets in a space X and $k \in \widetilde{\mathbb{N}}$ we denote

$$\mathcal{U}^k = \begin{cases} \{St^k(x,\mathcal{U}) : x \in X\} \text{ if } k \in \mathbb{N} \\ \{St^{k-1/2}(U,\mathcal{U}) : U \in \mathcal{U}\} \text{ if } k \in \mathbb{N}_{1/2}. \end{cases}$$

DEFINITION 1. A space X is k-starcompact ($k \in \widetilde{\mathbb{N}}$) if the following condition holds
(St^k) for every open cover $\mathcal{U}$ of X the cover $\mathcal{U}^k$ has a finite subcover.

A space X is ω-starcompact if the following condition holds (St^{ω}) for every open cover $\mathcal{U}$ of X the cover $\mathcal{U}^k$ has a finite subcover for some $k \in \widetilde{\mathbb{N}}$.

We use the terminology in [8] which is different from the one used in [2] where what we call $k\frac{1}{2}$-starcompactness was called k-starcompactness and what we call k-starcompactness was called strong k-starcompactness. The reason for this is to reserve the words "weakly" and "strongly" for introducing other properties.

Many examples and properties of k-starcompact spaces were discovered in [2] and [7]. It is known that 1-starcompactness is equivalent to countable compactness for Hausdorff spaces. In the class of regular spaces, every k-starcompact space is $2\frac{1}{2}$-starcompact if $k \geq 3$ and $k \in \widetilde{\mathbb{N}}$. Furthermore, $2\frac{1}{2}$-starcompactness is equivalent to pseudocompactness for Tychonoff spaces. See Diagram 2 below for their implications.

DEFINITION 2. A space X is k-cl-starcompact ($k \in \widetilde{\mathbb{N}}$) if the following condition holds
(St^k_{cl}) For every open cover $\mathcal{U}$ of X the cover $\mathcal{U}^k$ has a finite subfamily the union of which is dense in X.

The next two conditions can be defined only for $k \in \mathbb{N}$.

DEFINITION 3. A space X is weakly k-starcompact if the following condition holds
(wSt^k) for every open cover $\mathcal{U}$ of X there exists a finite subset $A \subset X$ such that for every open neighborhood O of A, $St^k(O, \mathcal{U}) = X$.

A space X is weakly-k-cl-starcompact if the following condition holds
(wSt^k_{cl}) for every open cover $\mathcal{U}$ of X there exists a finite subset $A \subset X$ such that for any open neighborhood O of A $\overline{St^k(O, \mathcal{U})} = X$.

PROPOSITION 1.1. *[8] The following implications hold for any X and $k \in \mathbf{N}$ without assumption of any axiom of separation.*

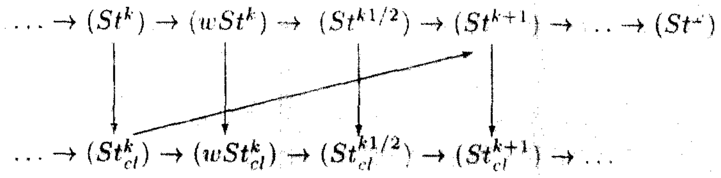

Diagram 1

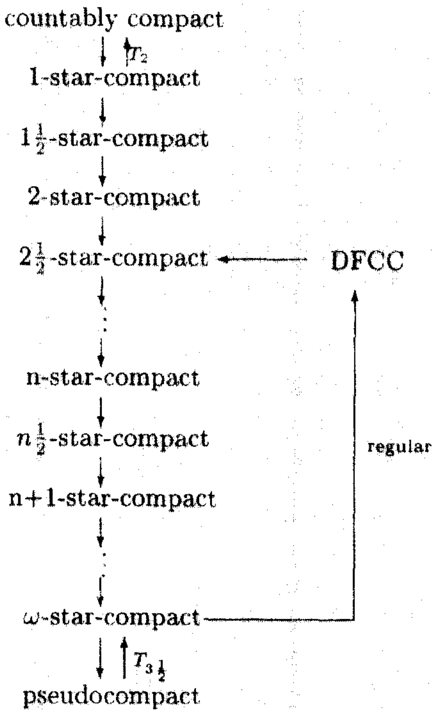

Diagram 2

A space X has the *discrete finite chain condition* (henceforth abbreviated DFCC) provided every discrete family of nonempty open sets is finite.

THEOREM 1.1. *[8] The following conditions are equivalent in* *the class of regular spaces:*

1. *(DFCC),*
2. *($St^{2\frac{1}{2}}$),*
3. *(St^k) for any $k \geq 2\frac{1}{2}$.*
4. *(St^ω),*
5. *$St_{cl}^{1\frac{1}{2}}$),*
6. *(St_{cl}^k) for any $k \geq 1\frac{1}{2}$,*
7. *(wSt^k) for any $k \in \mathbb{N}$, $k \geq 3$,*
8. *(wSt_{cl}^k) for any $k \in \mathbb{N}$, $k \geq 2$.*

In the class of Tychonoff spaces all these conditions are equivalent to pseudocompactness.

2. $1\frac{1}{2}$-starcompact spaces

A $1\frac{1}{2}$-starcompact space has one of the most curious properties among the spaces of starcompactness. It is not too far away from countably compact spaces and may be considered as the first candidate for extending theorems about countably compact spaces. While, it has some features of pseudocompactness. Unfortunately, $1\frac{1}{2}$-starcompactness is not so easy to be recognized as 2-starcompactness which will follow from countable pracompactness (see below). We introduce a couple of well- known theorems providing that a given space is not $1\frac{1}{2}$-starcompact.

THEOREM 2.1. *[2] If a regular space X contains a closed discrete subspace Y such that $|Y| = w(X) \geq \omega$, then X is not $1\frac{1}{2}$-starcompact.*

For example, the usual space $\mathbb{R}$ is not $1\frac{1}{2}$-starcompact since it is clearly a regular space which has $\mathbb{N}$ as a closed discrete subspace. It is, in fact, not $n\frac{1}{2}$-starcompact for any $n \in \omega$ since it is not pseudocompact.

For any space $\langle X, \mathcal{T}\rangle$, $w(X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a base for $X\}$ is called the *weight* of X. The *extent* $e(X)$ of X is defined as follows: $e(X) = sup\{\,|D| : D \subset X,\ D$ is closed and discrete $\}$. Clearly, the extent is a generalization of Lindelöf degree. For any space X, $e(X) \leq w(X)$. The above theorem shows us that $e(X) < w(X)$ for every regular $1\frac{1}{2}$-starcompact space X. In particular, a "very good" space $D^\kappa \setminus \{p\}$, where p is arbitrary point of D^κ and κ is an infinite cardinal number, is 2-starcompact but not $1\frac{1}{2}$-starcompact. Indeed, $e(D^\kappa \setminus \{p\}) = w(D^\kappa \setminus \{p\}) = \kappa$.

Since both inequalities $w(X) > |X|$ and $w(X) < |X|$ are possible, the next theorem is independent of the previous one.

THEOREM 2.2. *[8] If a regular space X contains a closed discrete subspace Y such that $|Y| = |X| \geq \omega$, then X is not $1\frac{1}{2}$-starcompact.*

However, for every cardinal κ there exists a Tychonoff $1\frac{1}{2}$-starcompact space X such that $e(X) = \kappa$ (see Theorem 36 in [8]). Maybe "discrete" in the last two theorems can be replaced by a weaker condition.

Big extent is not a necessary condition for not being $1\frac{1}{2}$-starcompact: there is a 2-starcompact space of countable extent which is not $1\frac{1}{2}$-starcompact constructed by means of the "Noble plank" (see Example 3 in [8]). Other examples of "good" pseudocompact spaces of countable extent which are not $1\frac{1}{2}$-starcompact can appear if the answer to the following question will be negative.

QUESTION 1[QUESTION 2 IN [8]]. *Suppose X is the union of countably many dense, countably compact (in itself) subspaces. Must X be $1\frac{1}{2}$-starcompact?*

Let $X = \prod_{s\in S} X_s$ be the product space and p be a fixed point of X. The subspace

$$\sum(p) = \{x \in X : |\{s \in S : x_s \neq p_s\}| \leq \omega\}$$

of X is called a *$\sum$-product* of spaces $\{X_s : s \in S\}$ (about p).

Observe that a $\sum$-product of spaces $\{X_s\}_{s\in S}$ is a proper subspace of the product space $\prod_{s\in S} X_s$ if and only if uncountably many spaces X_s contain at least two elements. Such $\sum$-products are called *proper*.

In particular, let X be the union of countably many distinct Σ-products in D^{ω_1}. Is X $1\frac{1}{2}$-starcompact?

Ikenaga and Tani [6] call a space X *K-starcompact* provided that for every open cover $\mathcal{U}$, there is a compact subspace $K \subset X$

such that $St(K,\mathcal{U}) = X$. It is clear that

$$(\text{starcompact}) \Rightarrow (\mathcal{K}\text{-starcompact}) \Rightarrow (1\frac{1}{2}\text{-starcompact}).$$

Also, it is clear that $\mathcal{K}$-starcompactness is a $1\frac{1}{2}$-starcompactness-type property.

DEFINITION 4. A space X is called $\mathcal{K}$-starcompact (resp. $\mathcal{L}$-starcompact, $\mathcal{P}$-starcompact, $\mathcal{M}$-starcompact) if for every open cover $\mathcal{U}$ of X there is a compact (resp. Lindelöf, paracompact, metacompact) subspace A of X such that $St(A,\mathcal{U}) = X$.

The following diagram gives us obvious implications between the concepts above.

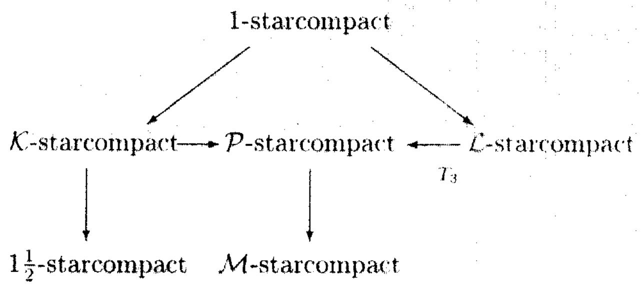

Diagram 3

EXAMPLE 2.1. A first-countable normal $\mathcal{P}$-starcompact that is neither $\mathcal{K}$- starcompact nor $n\frac{1}{2}$-starcompact.

Proof. Let $\mathbb{R}$ be the set of real numbers and A a countable set disjoint from $\mathbb{R}$. Write $A = \{a_m : m \in \omega\}$. Let $X = A \cup \mathbb{R}$ and topologize X as follows;

For each $m \in \omega$, let $\mathcal{B}_m = \{\{a_m\} \cup (p,q) : m-1 \leq p < m < q \leq m+1 \text{ and } p,q \in \mathbb{Q}\}$, and let $\mathcal{B}_{\mathbb{R}}$ be the usual base for $\mathbb{R}$. Let $\mathcal{J}_X$ be the topology on X obtained from $\{\mathcal{B}_m : m \in \omega\}$ and $\mathcal{B}_{\mathbb{R}}$.

Observe that $\mathbb{R}$ is a paracompact subspace of $\langle X, \mathcal{J}_X\rangle$, and for every open cover $\mathcal{U}$ of X, $St(\mathbb{R},\mathcal{U}) = X$. Thus X is $\mathcal{P}$-starcompact. Moreover, X is first-countable and normal.

Claim. X is not $\mathcal{K}$-starcompact. Indeed, for each $m \in \omega$, let $U_m = \{a_m\} \cup (m-1, -m+1)$ and $V_m = (-m-1, -m+1)$. Then clearly $\mathcal{U} = \{U_m : m \in \omega\} \cup \{V_m : m \in \omega\}$ is an open cover of X. Let K be an arbitrary compact subspace of X. The open cover $\mathcal{U}$ defined witnesses that $St(K, \mathcal{U}) \neq X$, so X is not $\mathcal{K}$-starcompact. Furthermore, with this open cover, we can see that X is not $n\frac{1}{2}$-starcompact for every $n \in \omega\backslash 1$, and thus is not ω- starcompact.

3. Countably pracompact spaces

Recall that a space X is *countably compact* if every infinite subset of X has a limit point. A subspace Y of a space X is *relatively countably compact* (or weakly countably compact or conditionally compact) if every infinite subset of Y has a limit point in X. Recall that a space X has the *discrete finite chain condition* (henceforth abbreviated DFCC) provided that every discrete family of nonempty open sets is finite.

DEFINITION 5. A space X is countably pracompact if it has a dense subspace Y such that Y is relatively countably compact.

PROPOSITION 3.1. *(1) Every countably compact space is countably pracompact.*

(2) Every countably pracompact space is DFCC.

Proof. (1) Let$X$ be a countably compact space. Choose $Y = X$. Then Y is dense in X and Y is relatively countably compact being Y countably compact. (2) Suppose X is not DFCC. Then there exists a countably infinite discrete open collection $\mathcal{D} = \{D_n : n \in \omega\}$. Let $D \subset X$ be dense. We claim that D is not relatively countably compact, i.e., there exists a countably infinite subset A of D such that A has no limit point. Since D is dense and each D_n is open, $D \cap D_n \neq \emptyset$. Choose one $x_n \in D \cap D_n$ for each $n \in \omega$. Let $A = \{x_n : n \in \omega\}$. Then A is a countably infinite subset of D. It is easy to check that A has no limit point.

The following theorem is a very useful one which is given by Theorem 15 in [8] without proof. We give here a proof of this theorem.

THEOREM 3.1. *Every countably pracompact space is 1-cl-star-*

compact (and hence 2-starcompact).

Proof. Suppose X is countably pracompact. Let $Y \subset X$ be a dense subspace which is relatively countably compact in X. By way of contradiction, we assume that X is not 1-cl-starcompact. Then there exists an open cover $\mathcal{U}$ of X such that the cover $\mathcal{U}' = \{St(x,\mathcal{U}) : x \in X\}$ does not have a finite subfamily whose union is dense in X, i.e., for any finite set $A = \{x_0, x_1, \ldots, x_n\} \subset X$,

$$\overline{St(A,\mathcal{U})} = \overline{St(x_0,\mathcal{U}) \cup St(x_1,\mathcal{U}) \cup \ldots \cup St(x_n,\mathcal{U})} \neq X \qquad (*)$$

Since Y is dense in X, $Y \setminus St(A,\mathcal{U}) \neq \emptyset$ for any finite $A \subset Y \subset X$ since if $Y = St(A,\mathcal{U})$ for some finite $A \subset Y$, then $X = \overline{Y} = \overline{St(A,\mathcal{U})}$, contradicting $(*)$. Then, by induction, we can construct an infinite sequence $S = \{p_n : n \in \omega\}$ with $p_n \in Y$ and $p_n \notin St(\{p_m : m < n\},\mathcal{U})$ for each $n \in \omega$. It is easy to see that S is discrete. Therefore S is an infinite subset of Y which does not have a limit point in X, i.e., Y is not relatively compact in X. This is a contradiction.

The following is an example of a countably pracompact space which is not countably compact.

EXAMPLE 3.1. Consider a maximal almost disjoint family $\mathcal{C} = \{A_s : s \in S\}$ of infinite subsets of ω. We may assume that $S \cap \omega = \emptyset$. Define a topology on the set $X = S \cup \omega$ as follows:

All points of ω are isolated in X, and an arbitrary basic neighborhood of any point $s \in S$ consists of the point s and all but finitely many points of A_s.

Then the space X is Tychonoff since X has a clopen base, the set ω is both countably compact and dense in X. Therefore X is countably pracompact, but X is not countably compact since S is an infinite closed discrete subspace of X.

In addition, the space X above is locally compact locally metrizable Moore space.

The following example is known in [8] as a mini-2- starcompact space. We introduce it here as an example of countably pracompact space which is not $1\frac{1}{2}$-starcompact.

EXAMPLE 3.2. Let $\mathcal{C}$ be a maximal almost disjoint family of infinite subsets of ω. Denote $\mathcal{A}$ the family of all infinite subsets of the elements of $\mathcal{C}$. For every $A \in \mathcal{A}$ choose a point $x_A \in \beta\omega\backslash\omega$ so that $x_A \in \overline{A}$ and $x_A \neq x_{A'}$ whenever $A \neq A'$. Denote $Y = \{x_A : A \in \mathcal{A}\}$ and $X = Y \cup \omega$. Let $\mathcal{T}_0$ be the topology on X inherited from $\beta\omega$. Now we define a finer topology $\mathcal{T}$ on X declaring the basic open neighborhood of x_A to take form $U\backslash(Y\backslash\{x_A\})$, where U is a neighborhood of x_A in $\mathcal{T}_0$.

The space $\langle X, \mathcal{T}\rangle$ is countably pracompact because ω is dense and relatively countably compact in $\langle X, \mathcal{T}\rangle$. Indeed, let B be an infinite subset of B. Then by maximality of $\mathcal{C}$, there exists $C \in \mathcal{C}$ such that $|B \cap C| = \omega$. So $B \cap C \in \mathcal{A}$. Then $x_{B\cap C}$ is a limit point of B in $\langle X, \mathcal{T}\rangle$. It is shown in [8] that for any infinite subspace Z of $\langle X, \mathcal{T}\rangle$, Z is not $1\frac{1}{2}$-starcompact. In particular, $\langle X, \mathcal{T}\rangle$ is not $1\frac{1}{2}$.

PROPOSITION 3.2. *The closure of the set of isolated points in a DFCC space is countably pracompact.*

Proof. Let X be a DFCC space and $A = \{x \in X : x$ is an isolated points in $X\}$. Then $\overline{A}$ is countably pracompact; clearly, A is a dense subspace of $\overline{A}$.

We claim that A is relatively countably compact in $\overline{A}$. Let $B \subset A$ be any subset. Then $\mathcal{B} = \{\{x\} : x \in B\}$ is a discrete collection of non-empty open subsets of X. Since X is DFCC, $\mathcal{B}$ must be finite and so B is finite. Therefore B has a limit point in $\overline{A}$.

A space X is *right(left)-separated* if and only if there exists a well-ordering $\prec$ on X such that every initial segment $(\infty, x]$ is open (closed), i.e., $\{x_{\alpha'} : \alpha' \leq \alpha\}$ is open (closed) for each α.

THEOREM 3.2. *If X is right-separated and A is a set of all isolated points in X, then A is dense in X, i.e., $\overline{A} = X$.*

Proof. Let $U \subset X$ be a non-empty open subset. We want to show that $A \cap U \neq \emptyset$, i.e., U has an isolated point. Let $\alpha = \min\{\beta : x_\beta \in U\}$. Then $\{x_\beta : \beta < \alpha + 1\} \cap U = \{x_\alpha\}$ is open. Thus x_α is an isolated point of U.

THEOREM 3.3. *A DFCC, right-separated space is countably pracompact.*

Proof. It follows from Proposition 3.2 and Theorem 3.2

In general, DFCC does not imply countable pracompactness ([1]).

A space is called *pseudo-normal* if every countable closed subset has arbitrary small closed neighborhoods. A space is said to have *property D* if every countable closed discrete set has arbitrary small closed neighborhoods, and is said to have *property wD* (for weak D) if every infinite closed discrete set has an infinite subset which has arbitrary small closed neighborhoods. Clearly,

normal $\Longrightarrow$ pseudo-normal $\Longrightarrow$ property D $\Longrightarrow$ property wD.

The following proposition ([3]) is frequently useful when one works with property D or wD.

PROPOSITION 3.3. *The following conditions on a countable closed discrete set D in a space X are equivalent:*

(1) *D has arbitrary small closed neighborhoods*
(2) *there is an indexed discrete open family $\{U_x : x \in D\}$ in X satisfying $x \in U_x$ for every $x \in D$.*

Recall that countably compact $\Longrightarrow$ countably pracompact $\Longrightarrow$ DFCC., and that every right separated DFCC space is countably pracompact.

PROPOSITION 3.4. *Every DFCC space with property wD is countably compact.*

Proof. Suppose X is not countably compact and has property wD. Then there exists an infinite closed discrete subset $A \subset X$. Also, there exists an infinite subset $B \subset A$ which has arbitrary small closed neighborhoods. By Proposition 3.3, there exists an indexed discrete open family $\mathcal{U} = \{U_x : x \in B\}$ in X satisfying for all $x \in B$ $x \in U_x$. Since $\mathcal{U}$ is infinite, X is not DFCC.

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