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RINGS IN WHICH NILPOTENT ELEMENTS FORM AN IDEAL

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ABSTRACT We study the relationships between strongly prime ideals and completely prime ideals, concentrating on the connections among various radicals (prime radical, upper nilradical and generalized nilradical). Given a ring R, consider the condition (*) nilpotent elements of R form an ideal in R. We show that a ring R satisfies (*) if and only if every minimal strongly prime ideal of R is completely prime if and only if the upper nilradical coincides with the generalized nilradical in R

1. Introduction

This paper was motivated by the results in [1] and [4] which are related to nilradicals. Throughout this paper, all rings are associative with identity. Given a ring R we use $\mathbf{P}(R)$, $\mathbf{N}(R)$, and $\mathbf{Spec}_S(R)$ to represent the prime radical, the set of all nilpotent elements, and the set of all strongly prime ideals of R, respectively.

Recall that a ring R is called *strongly prime* if R is prime with no nonzero nil ideals and an ideal P of R is called *strongly prime* if R/P is strongly prime, and that the upper nilradical of a ring R is the unique

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maximal nil ideal of R (see [3, Proposition 2.6.2]); we denote it by $\mathbf{N}_r(R)$. Notice that

$$\begin{aligned}\mathbf{N}_r(R) &= \{a \in R \mid RaR \text{ is a nil ideal of } R\}\\ &= \bigcap\{P \mid P \text{ is a strongly prime ideal of } R\}\\ &= \bigcap\{P \mid P \text{ is a minimal strongly prime ideal of } R\}.\end{aligned}$$

It is straightforward to check that a ring R satisfies (*) if and only if $\mathbf{N}_r(R) = \mathbf{N}(R)$ if and only if $R/\mathbf{N}_r(R)$ is a reduced ring (i.e., a ring without nonzero nilpotent elements). It is clear that the Köthe's conjecture (i.e., the upper nilradical contains every nil left ideal) holds if given a ring satisfies (*); but the converse is not true in general considering 2-by-2 full matrix rings over reduced rings (see [3, Theorem 2.6.35]). A ring R is called *2-primal* if $\mathbf{P}(R) = \mathbf{N}(R)$. 2-primal rings satisfy (*) obviously; however the converse does not hold in general by [2, Example 3.3]. Commutative rings and reduced rings are 2-primal and so they satisfy (*).

2. Rings which satisfy (*)

An ideal P of a ring R is called a *minimal strongly prime* ideal of R if P is minimal in $\mathbf{Spec}_S(R)$ To observe the properties of minimal strongly prime ideals of rings which satisfy (*), we introduce the following concepts:

$$N(P) = \{a \in R \mid aRb \subseteq \mathbf{N}_r(R) \text{ for some } b \in R\backslash P\},$$

$$N_P = \{a \in R \mid ab \in \mathbf{N}_r(R) \text{ for some } b \in R\backslash P\},$$

$$\overline{N}_P = \{a \in R \mid a^m b \in \mathbf{N}_r(R) \text{ for some integer } m \text{ and some } b \in R\backslash P\},$$

where P is a strongly prime ideal of a ring R. It may be easily checked that for each prime ideal P of a ring R, $N(P) \subseteq P$ and $N(P) \subseteq N_P \subseteq \overline{N}_P$.

To obtain the following results, from Lemma 1 to Theorem 5, we use the methods that Shin used in [4].

A right (or left) ideal I of a ring R is said to have the IFP (*insertion of factors property*) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. Notice that the zero ideal of a reduced ring has the IFP. Given a ring R, recall that a subset S of $R\backslash\{0\}$ is called an m-system if $s_1, s_2 \in S$ implies $s_1 r s_2 \in S$ for some $r \in R$. Obviously the complement of any prime ideal is an m-system.

LEMMA 1. *For a ring R, the following statements are equivalent:*
(1) R satisfies (*).
(2) $\mathbf{N}_r(R)$ has the IFP.

PROOF. (1)⇒(2): Since $R/\mathbf{N}_r(R)$ is reduced by hypothesis, $ab \in \mathbf{N}_r(R)$ implies $aRb \subseteq \mathbf{N}_r(R)$ for $a, b \in R$.

(2)⇒(1): Let $a \in \mathbf{N}(R)$. Then $a^n = 0$ for some positive integer n We claim $a \in \mathbf{N}_r(R)$. Assume to the contrary that $a \notin \mathbf{N}_r(R)$. Then there exists a strongly prime ideal P such that $a \notin P$. Since $R\backslash P$ is an m-system, there exist $r_1, \ldots, r_{n-1} \in R$ such that $ar_1 a \cdots ar_{n-1}a \in R\backslash P$. But $ar_1 a \cdots ar_{n-1}a \in \mathbf{N}_r(R)$ since $\mathbf{N}_r(R)$ has the IFP. Consequently $ar_1 a \cdots ar_{n-1}a \in P$, a contradiction; and therefore R satisfies (*).

LEMMA 2 *If a ring R satisfies (*), then $N(P) = N_P = \overline{N}_P$ for each strongly prime ideal P of R.*

PROOF. It is trivial that $N(P) \subseteq N_P \subseteq \overline{N}_P$. Take $a \in \overline{N}_P$ and let $m \geq 1$ be minimal with $a^m b \in \mathbf{N}_r(R)$ for some $b \in R\backslash P$. By Lemma 1 $aRa^{m-1}b \in \mathbf{N}_r(R)$ and $a^{m-1}b \notin P$ so $a \in N(P)$.

THEOREM 3. *Suppose that a ring R satisfies (*). Then $N(P) = \bigcap\{Q \in \mathbf{Spec}_S(R) \mid N(P) \subseteq Q \subseteq P\}$ for each $P \in \mathbf{Spec}_S(R)$.*

PROOF. If $Q \subseteq P$ for $P, Q \in \mathbf{Spec}_S(R)$, then $N(P) \subseteq N(Q) \subseteq Q \subseteq P$; hence we have $N(P) \subseteq \bigcap\{Q \in \mathbf{Spec}_S(R) \mid N(P) \subseteq Q \subseteq P\}$. Conversely, suppose that $a \notin N(P)$. We claim that there exists a strongly prime ideal Q such that $a \notin Q$ and $Q \subseteq P$. The set $S = \{a, a^2, a^3, \ldots\}$ is closed under multiplication that does not contain 0 by Lemma 2, and note that $L \overset{\text{let}}{=} R\backslash P$ is a m-system. Let $T = \{a^{t_0}b_1a^{t_1}b_2\cdots b_na^{t_n} \neq 0 \mid b_i \in L, t_i \in \{0\} \cup \mathbb{Z}^+\}$, where $\mathbb{Z}^+$ is the set of positive integers . Let $M = S \cup T$. Note that $L \subseteq T$. Now we will show that M is closed under multiplication. If $x, y \in S$, then $xy \in S \subseteq M$. If $x \in S$ and $y \in T$ with $x = a^s, y = a^{t_0}b_1a^{t_1}b_2\cdots b_na^{t_n}$, then $xy \neq 0$. For, if $xy = 0$ then

$$xy = a^{s+t_0}b_1a^{t_1}b_2\cdots b_na^{t_n} = 0 \in \mathbf{N}_r(R).$$

By Lemma 1, we have that

$$(a^{s+t_0}a^{t_1}\cdots a^{t_n})(b_1\cdots b_n)\cdots(a^{s+t_0}a^{t_1}\cdots a^{t_n})(b_1\cdots b_n) \in \mathbf{N}_r(R),$$

and so

$$[(a^{s+t_0+\cdots+t_n})(b_1\cdots b_n)]^{n+1} \in \mathbf{N}_r(R).$$

Thus $(a^{s+t_0+\cdots+t_n})(b_1\cdots b_n) \in \mathbf{N}_r(R)$ because $\mathbf{N}_r(R) = \mathbf{N}(R)$. Since L is an m-system, there exist $r_1, \ldots, r_{n-1} \in R$ such that

$$b_1r_1\cdots b_{n-1}r_{n-1}b_n \in L.$$

Let $s+t_0+\cdots+t_n = w$ and $b_1r_1\cdots b_{n-1}r_{n-1}b_n = b$. Then $a^wb \in \mathbf{N}_r(R)$ and hence $a \in \overline{N}_P = N(P)$ by Lemma 2, which is a contradiction. Consequently $xy \in T \subseteq M$. Similarly, if $x, y \in T$ then $xy \neq 0$ and so $xy \in T \subseteq M$. Thus M is a multiplicatively closed system which is disjoint from (0); hence there exists a prime ideal Q that is disjoint from M. Therefore $a \notin Q$ and $Q \subseteq P$. To complete the proof, we have to show that Q is strongly prime. $(M + Q)/Q$ has no nilpotent elements but intersects every nonzero ideal in R/Q by the maximality of Q with respect to the property $M \cap Q = 0$, so Q is strongly prime.

COROLLARY 4. *Suppose that a ring R satisfies (*). Then for each strongly prime ideal P of R the following statements are equivalent:*

(1) P is a minimal strongly prime ideal of R.

(2) $N(P) = P$.

(3) For any $a \in P$, $ab \in \mathbf{N}_r(R)$ for some $b \in R\backslash P$.

PROOF. (1)$\Leftrightarrow$(2) follows from Theorem 3.

(2)$\Rightarrow$(3): For each $a \in P = N(P)$, $abab \in aRb \subseteq \mathbf{N}_r(R)$ for some $b \in R \backslash P$, hence ab is nilpotent and so $ab \in \mathbf{N}_r(R)$.

(3)$\Rightarrow$(2): If $a \in P$ and $ab \in \mathbf{N}_r(R)$ for some $b \in R \backslash P$, then $aRb \subseteq \mathbf{N}_r(R)$ because $R/\mathbf{N}_r(R)$ is reduced. Hence $a \in N(P)$ and so $N(P) = P$ since $N(P) \subseteq P$ always.

Recall that an ideal I of a ring R is called *completely prime* if R/I is a domain. We use $\mathbf{P}_C(R)$ for the intersection of all completely prime ideals of a ring R. Birkenmeier-Heatherly-Lee [1, Proposition 2.1] proved that a ring R is 2-primal if and only if $\mathbf{P}(R) = \mathbf{P}_C(R)$, and Shin [4, Proposition 1.11] proved that R is 2-primal if and only if every minimal prime ideal of R is completely prime. The following theorem, which contains similar connections to the preceding results among $\mathbf{N}(R), \mathbf{N}_r(R)$ and $\mathbf{P}_C(R)$, is our main result in this note.

THEOREM 5. *For a ring R, the following statements are equivalent:*
(1) R satisfies (*).
(2) Every minimal strongly prime ideal of R is completely prime.
(3) $\mathbf{N}_r(R) = \mathbf{P}_C(R)$.

PROOF. (1)$\Rightarrow$(2): Let P be a minimal strongly prime ideal of R such that $ab \in P$ and $b \notin P$. Then by Corollary 4, $(ab)c \in \mathbf{N}_r(R)$ for some $c \in R \backslash P$. Since $R \backslash P$ is a m-system and $b, c \in R \backslash P$, there exists $z \in R$ such that $bzc \in R \backslash P$. Also by Lemma 1, $\mathbf{N}_r(R)$ has the IFP. So we have $aR(bzc) \subseteq \mathbf{N}_r(R)$ and $a \in N(P) = P$. Therefore P is a completely prime ideal of R.

(2)$\Rightarrow$(3): $\mathbf{N}_r(R)$ is the intersection of minimal strongly prime ideals in R, so an intersection of completely prime ideals by the condition, and this contains $\mathbf{P}_C(R)$. Next since $R/\mathbf{P}_C(R)$ is reduced, also $\mathbf{N}_r(R) \subseteq \mathbf{P}_C(R)$.

(3) $\Rightarrow$ (1) : By hypothesis $R/\mathbf{N}_r(R)$ is a subdirect product of domains and so it is reduced; hence R satisfies (*).

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