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ON THE SPECTRUM OF THE RHALY OPERATORS ON bv

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ABSTRACT. In this paper, we determine the spectrum of the Rhaly matrix R_a as an operator on the space bv, when $\lim_n (n+1)a_n \neq 0$ and exists

Given a scalar sequence $a = (a_n)$ of scalars, the Rhaly matrix R_a is the lower triangular matrix with constant row-segments

$$R_a = \begin{pmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_1 & 0 & \dots \\ a_2 & a_2 & a_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

The Cesàro matrix is $R_{1/n}$ and more generally, if we take $a_n = n^{-z}$ we get the $z-$Cesàro matrix C_z.

bv is the space of sequences of bounded variation normed by

$$\|x\|_{bv} := |\lim x| + \sum_k |x_k - x_{k+1}| < \infty. \tag{2}$$

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From [5] any matrix A is in bv iff A satisfies

$$\|A\|_{bv} := \sup_m \sum_n | \sum_{k=0}^{m} (a_{nk} - a_{n-1,k})| \ < \infty \tag{3}$$

and

$$A\delta \in bv \quad \text{where } \delta := (1, 1, \dots). \tag{4}$$

We represent the set of eigenvalues of the Rhaly matrix R_a and the spectrum of R_a on the Banach space X by $\pi_0(R_a, X)$ and $\sigma(R_a, X)$, respectively. Under the above conditions, the purpose of this study is to determine the spectrum of Rhaly operator R_a as an operator on the Banach space bv_0.

The spectra of the Cesàro matrix on bv has been studied by Okutoyi[2].

In [4] taking $0 \leq \lim_n (n+1)a_n < \infty$ Rhaly showed that R_a is a bounded operator on the Hilbert space ℓ_2 of square summable sequences, and he also determined its spectrum as

$$(\ \sigma(R_a, \ell_2) = \{ \ \lambda \ : \ |\lambda - L| < L\} \ \cup \ \{ \ a_n \ : \ n = 0, 1, 2, \dots\}).$$

In this paper the author determines the spectrum of the Rhaly matrix R_a as an operator on bv with assumption $0 < L = \lim_n (n+1)a_n < \infty$. We assume that $L = \lim_n (n+1)a_n$ exists, is finite and that nonzero; that $a_n > 0$ for all n; that $S := \{ \ a_n \ : \ n = 0, 1, 2, \ \dots \ \}$.

According to the above assumption we can give following theorems for (a_n) sequence.

THEOREM 1. $\{(n+1)a_n\} \in bv$ *iff* $R_a \in B(bv)$.

PROOF. From [5], $R_a \in B(bv)$ if and only if (3) and (4) are satisfied. Since $\delta = (1, 1, \dots)$, we have $(y_n) = y = R_a\delta = (a_0, 2a_1, 3a_2, \dots, (n+1)a_n, \dots)$. Hence $(y_n) \in bv \Leftrightarrow \sum_n |(n+2)a_{n+1} - (n+1)a_n| < \infty \Leftrightarrow$

$\{(n+1)a_n\} \in bv$. Since $\{(n+1)a_n\} \in bv$, we have

$$\begin{aligned}
\|R_a\|_{bv} &:= \sup_m \sum_{n=0}^{\infty} |\sum_{k=0}^{m}(a_{nk} - a_{n-1,k})| \\
&= \sup_m \left[\sum_{n=0}^{m} |\sum_{k=0}^{n}(a_{nk} - a_{n-1,k})| + \sum_{n=m+1}^{\infty} |\sum_{k=0}^{m}(a_{nk} - a_{n-1,k})|\right] \\
&= \sup_m \left[a_0 + \sum_{n=1}^{m} |\left(\sum_{k=0}^{n-1}(a_{nk} - a_{n-1,k})\right) + a_n| \right.\\
&\left. + \sum_{n=m+1}^{\infty} |\sum_{k=0}^{m}(a_n - a_{n-1})|\right] \\
&\leq a_0 + \sup_m \left[\sum_{n=1}^{m} |(n+1)a_n - na_{n-1}| \right.\\
&\left. +(m+1) \sum_{n=m+1}^{\infty} |\frac{n}{n}a_{n-1} - \frac{(n+1)}{(n+1)}a_n + \frac{(n+1)}{n}a_n - \frac{(n+1)}{n}a_n|\right] \\
&\leq a_0 + \sum_{n=1}^{\infty} |(n+1)a_n - na_{n-1}| + \sup_m \sum_{n=m+1}^{\infty} |(n+1)a_n - na_{n-1}| \\
&+ \sup_m (m+1) \sum_{n=m+1}^{\infty} |\frac{1}{n} - \frac{1}{n+1}| a_n(n+1) \\
&\leq O(1) + O(1) + O(1) + \sup_m \left[(m+1) \lim_{k\to\infty} \left(\frac{1}{m+1} - \frac{1}{k+1}\right)\right] \\
&= O(1).
\end{aligned}$$

THEOREM 2. *If* $\{(n+1)a_n\} \in bv$ *and* $\lim_n (n+1)a_n = L < \infty$, *then*

$$S \cap (2L, \infty) \subseteq \pi_0(R_a, bv). \tag{5}$$

If $T : bv \to bv$ is a bounded matrix operator with matrix A, then

$T^* : bv^* \to bv^*$ acting on $C \oplus bs$ has matrix representation of the form

$$\begin{pmatrix} \overline{\chi} & \vartheta_0 - \overline{\chi} & \vartheta_1 - \overline{\chi} & \vartheta_2 - \overline{\chi} & \cdots \\ u_0 & a_{00} - u_0 & a_{10} - u_0 & a_{20} - u_0 & \cdots \\ u_1 & a_{01} - u_1 & a_{11} - u_1 & u_{21} - u_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{6}$$

where $\overline{\chi}$ is the limit of the sequence of row sums of A, u_k is the column vector whose kth entry is the limit of the kth column of A for each k and $\vartheta_k = \sum_{m=0}^{\infty} a_{km} = P_k(T(\delta^n))$ [2].

Lemma 3. *Let If $R_a : bv \to bv$ $\{(n+1)a_n\} \in bv$, then $R_a^* \in B(bv^*)$ and*

$$R_a^* = \begin{pmatrix} L & \{(k+1)a_k - L\}_{k=1}^{\infty} \\ 0 & R_a^t \end{pmatrix}. \tag{7}$$

Proof. From above, we have $\overline{\chi} = \lim_{n\to\infty} \sum_{v=0}^{\infty} a_{nv} = \lim_{n\to\infty} \sum_{v=0}^{n} a_n$ $= \lim_{n\to\infty}(n+1)a_n = L$, $u_k := \lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} a_n = 0$ for each $k \geq 0$ and $\vartheta_k := (P_k \circ T)(\delta) = P_k(T(\delta)) = P_k(a_0, 2a_1, \ldots, (k+1)a_k, \ldots) =$ $(k+1)a_k$.

Lemma 4. *Let $0 < L = \lim_n (n+1)a_n < \infty$ and $Z_n := \prod_{\vartheta=0}^{n} \left(1 - \frac{a_\vartheta}{\lambda}\right)$, $\lambda \neq 0$, $\lambda \in C$. Then the partial sums of $\sum_{\vartheta=1}^{\infty} Z_n$ are bounded if and only if $LRe\frac{1}{\lambda} \geq 1$, $\lambda \neq L$.*

Proof. We show the proof of the Lemma as proved in [2, Lemma 1.6]. The series $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ is uniformly convergent in every subinterval of $|u| < 1$. Hence $\ln(1-u) = -u + O(u^2)$, uniformly in $|u| < 1/2$,

$u \in C$. Since $a_\vartheta \to 0$, for a given $\lambda \neq 0$ there exists ϑ_0 such that $\frac{a_\vartheta}{\lambda} \leq \frac{1}{2}$ for $\vartheta > \vartheta_0$,

$$\begin{aligned}
\ln Z_n &= \ln \prod_{\vartheta=0}^{n} (1 - \frac{a_\vartheta}{\lambda}) = \sum_{\vartheta=0}^{n} \ln(1 - \frac{a_\vartheta}{\lambda}) \\
&= C + \sum_{\vartheta=\vartheta_0}^{n} \ln(1 - \frac{a_\vartheta}{\lambda}) \\
&= C + \sum_{\vartheta=\vartheta_0}^{n} \left(-\frac{a_\vartheta}{\lambda} + O\left(\frac{a_\vartheta^2}{|\lambda|^2} \right) \right) \\
&= C - \frac{L}{\lambda} \sum_{\vartheta=\vartheta_0}^{n} \frac{1}{\vartheta + 1} + \frac{L^2}{|\lambda|^2} \sum_{\vartheta=\vartheta_0}^{n} O\left(a_\vartheta^2\right)
\end{aligned}$$

where $t_\vartheta = O(\frac{1}{\vartheta^2})$. Now since $t_\vartheta = O(\frac{1}{\vartheta^2})$,

$$\sum_{\vartheta=\vartheta_0}^{n} t_\vartheta = \sum_{\vartheta=\vartheta_0}^{\infty} t_\vartheta - \sum_{\vartheta=n+1}^{\infty} t_\vartheta = C + O(\frac{1}{n}).$$

Since if $C_n = \sum_{\vartheta=0}^{n} \frac{1}{\vartheta + 1} - \log n$, then

$$\begin{aligned}
C_{n+1} - C_n &= \frac{1}{2+n} - \log \frac{n+1}{n} = \frac{1}{2+n} - \log(1 + \frac{1}{n}) \\
&= \frac{1}{2+n} - \frac{1}{n} + O(\frac{1}{n^2})
\end{aligned}$$

and we have

$$\sum_{\vartheta=\vartheta_0}^{n} \frac{1}{\vartheta + 1} = C + \log n + O(\frac{1}{n}).$$

So

$$\begin{aligned} C_{n+1} &= C_0 + \sum_{\vartheta=0}^{n} (C_{\vartheta+1} - C_\vartheta) \\ &= C_0 + \sum_{\vartheta=0}^{\infty} (C_{\vartheta+1} - C_\vartheta) \sum_{\vartheta=n+1}^{\infty} (C_{\vartheta+1} - C_\vartheta) \\ &= C + O(\frac{1}{n}). \end{aligned}$$

Hence as $n \to \infty$,

$$\log Z_n = C - \frac{L}{n} \log n + O(\frac{1}{n}),$$

that is

$$\begin{aligned} Z_n &= exp(C - \frac{L}{n} \log n + O(\frac{1}{n})) \\ &= exp(C) n^{-\frac{L}{\lambda}} (1 + O(\frac{1}{n})) \\ &= A n^{\frac{L}{\lambda}} O(n^{-LRe\frac{1}{\lambda}-1}). \end{aligned}$$

If $L\lambda \neq 1$, $LRe(\frac{1}{\lambda}) \geq 1$, then $s_n = \sum_{k=1}^{n} k^{-\frac{L}{\lambda}}$ is bounded and $\sum_{n=1}^{\infty} n^{-LRe\frac{1}{\lambda}-1} < \infty$, so that the partial sums of $\sum_n Z_n$ are bounded. If $0 < LRe(\frac{1}{\lambda}) < 1$ or $L\lambda = 1$, then the partial sums of $\sum_{n=1}^{\infty} n^{-LRe\frac{1}{\lambda}}$ are unbounded, but still we have $\sum_{n=1}^{\infty} n^{-LRe\frac{1}{\lambda}-1} < \infty$. If $0 < LRe\frac{1}{\lambda} \leq 0$ then

$$\sum_{n=1}^{N} n^{-\frac{L}{\lambda}} \asymp \frac{N^{1-\frac{L}{\lambda}}}{1 - \frac{L}{\lambda}} \tag{8}$$

where $a_n \asymp b_n$ means that there exist $m, M \in R^+$ such that $mb_n < a_n < Mb_n$.

Using (8), we see that the partial sums of $\sum_{n=1}^{\infty} n^{-\frac{L}{\lambda}}$ are unbounded although $\sum_{n=1}^{\infty} n^{-LRe\frac{1}{\lambda}-1} < \infty$ and hence we obtain that the partial sums of $\sum_n Z_n$ are bounded if and only if $LRe\frac{1}{\lambda} \geq 1$.

THEOREM 5. *If* $\{(n+1)a_n\} \in bv$ *and* $0 < L = \lim_n (n+1)a_n < \infty$, *then*

$$S \cup \{\lambda : \ |\lambda - \frac{L}{2}| \leq \frac{L}{2}\} \subset \pi_0(R_a^*, bv^* \cong C \oplus bs).$$

PROOF. If $R_a^* x = \lambda x$, then $Lx_0 + (L-a_0)x_1 + (L-a_1)x_2 + \cdots = \lambda x_0$ and

$$\lambda a_n^{-1} x_{n+1} = (\lambda a_n^{-1} - 1)x_n. \tag{9}$$

Hence $0 \in \pi_0(R_a^*, bv^*)$ (because if $\lambda = 0$, then $x = (x_0, \frac{-L}{L-a_0}x_0, 0, 0, \dots)$ $\in bv$). From (9), we have

$$x_{n+1} \ = \ (1 - \frac{a_n}{\lambda})x_n. \tag{10}$$

If $\lambda = a_m$, $\lambda \in \pi_0(R_a^*, bv^*)$ (because for $n \geq m+1, \ x_n = 0$). From (10), we have

$$x_n \ = \ \prod_{j=0}^{n-1} (1 - \frac{a_j}{\lambda})x_0. \tag{11}$$

From Lemma 4 the other λ's have the properties $\alpha L \geq 1$. Hence we obtain

$$S \cup \{\lambda : \ |\lambda - \frac{L}{2}| \leq \frac{L}{2}\} \subset \pi_0(R_a^*, bv^* \cong C \oplus bs).$$

If $T_\lambda x = y$, then we have

$$\begin{aligned}
x_0 &= \frac{1}{\lambda - a_0} y_0 \\
x_1 &= \frac{1}{\lambda - a_1} y_1 + \frac{a_1}{(\lambda - a_1)(\lambda - a_0)} y_0 \\
x_2 &= \frac{1}{\lambda - a_2} y_2 + \frac{a_1}{(\lambda - a_2)(\lambda - a_1)} y_1 + \frac{a_2 \lambda}{(\lambda - a_2)(\lambda - a_1)(\lambda - a_0)} y_1 \\
&\vdots \qquad \vdots \qquad \vdots \\
x_n &= \frac{1}{\lambda - a_n} y_n + \frac{a_n}{(\lambda - a_n)(\lambda - a_{n-1})} y_{n-1} \\
&+ \frac{a_n \lambda}{(\lambda - a_n)(\lambda - a_{n-1})(\lambda - a_{n-2})} y_{n-2} \\
&+ \ldots + \frac{a_n}{\lambda^2 \prod_{k=1}^{n} (1 - \frac{a_k}{\lambda})} y_1 + \frac{a_n}{\lambda^2 \prod_{k=0}^{n} (1 - \frac{a_k}{\lambda})} y_0 \\
&\vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

Therefore $T_\lambda^{-1} = (\lambda I - R_a)^{-1} = (b_{nk})$ is given by

$$T_\lambda^{-1} = (b_{nk}) = \begin{cases} \dfrac{1}{\lambda - a_n}, & k = n \\[2ex] \dfrac{a_n}{\lambda^2 \prod_{j=k}^{n} \left(1 - \dfrac{a_j}{\lambda}\right)}, & k < n \\[2ex] 0, & \textit{otherwise.} \end{cases} \tag{12}$$

LEMMA 6. *If* $Re\frac{1}{\lambda} = \alpha$, *then*

$$\prod_{k=0}^{N-1} |1 - \frac{a_k}{\lambda}| \simeq \frac{1}{N^{\alpha L}} \qquad (13)$$

as $N \longrightarrow \infty$. *We use the notation* $a_n \simeq b_n$ *in the sense that* $\left(\frac{a_n}{b_n}\right)$, $\left(\frac{b_n}{a_n}\right)$ *are both bounded.*

PROOF. See [6].

THEOREM 7. *If* $a_n = \frac{L}{n+1} + \frac{b_n}{n+1}$ *where* $(\frac{b_n}{n+1}) \in \ell_1$, *then*

$$\sigma(R_a, bv) = \left\{ \lambda \ : \ |\lambda - \frac{L}{2}| \le \frac{L}{2} \right\} \cup S.$$

PROOF By Theorem 5, we have

$$\left\{ \lambda \ : \ |\lambda - \frac{L}{2}| < \frac{L}{2} \right\} \cup S \subseteq \pi_0(R_a^*, bv^*) \subseteq \sigma(R_a^*, bv^*) = \sigma(R_a, bv).$$

To complete the proof let us show that,

$$\sigma(R_a, bv) \subseteq \left\{ \lambda \ : \ |\lambda - \frac{L}{2}| \le \frac{L}{2} \right\} \cup S.$$

Now, let $|\lambda - \frac{L}{2}| > \frac{L}{2}$, (*which means* $\alpha L < 1$) and $\lambda \neq a_m$ $(m = 0, 1, 2, ...)$. We prove that the matrix $T^{-1} = (b_{nk})$ given by (12) satisfies the properties in eqs. (3) and (4).

Since $\alpha L < 1 \Leftrightarrow |\lambda - \frac{L}{2}| > \frac{L}{2}$ and $\alpha L < 1$ and $\lambda \neq a_m$ $(m = 0, 1, 2, ...)$, then

$$\lim_{n\to\infty} b_{nk} = \lim_{n\to\infty} \frac{a_n}{|\lambda|^2 \prod_{j=k}^{n} |1 - \frac{a_j}{\lambda}|} = \lim_{n\to\infty} \frac{(n+1)^{\alpha L} a_n}{(k)^{\alpha L}} = 0 \qquad (14)$$

for every k. Hence (3) is satisfied.

Now let's show that the eq. (4) is satisfied. Let

$$\sum_{n=0}^{\infty}|\sum_{m=0}^{N}(b_{nm}-b_{n-1,m})|=\sum_1+\sum_2+\sum_3$$

where

$$\sum_1=\sum_{n=0}^{N}|\sum_{m=0}^{n}b_{nm}-\sum_{m=0}^{n-1}b_{n-1,m}|,\ 0\leq n\leq N$$

$$\sum_2=|\sum_{m=0}^{N+1}b_{N+1,m}-b_{N+1,N+1}+\sum_{m=0}^{N+1}b_{N,m}|,\ n=N+1$$

$$\sum_3=\sum_{n=N+2}^{\infty}|\sum_{m=0}^{N}(b_{nm}-b_{n-1,m})|,\ N+2\leq n\leq\infty.$$

Now lets show that $\sum_1=O(1)$. Taking $a_n^0=\dfrac{L}{n+1}$ and $M_\lambda=(\lambda I-R_{a^0})$ then $M_\lambda^{-1}\delta=\sum_{k=0}^{n}b_{nk}^0$ where $M_\lambda^{-1}M_\lambda\delta=\delta$. So that, we obtain

$$M_\lambda^{-1}(\lambda I-R_{a^0})\delta=M_\lambda^{-1}(\lambda I-LC_1)\delta M_\lambda^{-1}(\lambda-L)\delta=\delta.$$

By using $M_\lambda^{-1}\delta=\dfrac{1}{\lambda-L}\delta$ we derive $\sum_{k=0}^{n}b_{nk}^0=\dfrac{1}{\lambda-L}$. As a result

$$\sum_{n=0}^{N}|\sum_{k=0}^{n}(b_{nk}^0-b_{n-1,k}^0)|=|b_{00}^0|+\sum_{n=1}^{N}|\frac{1}{\lambda-L}-\frac{1}{\lambda-L}|=\frac{1}{|\lambda-L|},$$

i.e.;

$$\sum_{n=1}^{N}|\sum_{k=0}^{n}(b_{nk}^0-b_{n-1,k}^0)|=0. \tag{15}$$

We have

$$\sum_1 = \sum_{n=0}^{N} |\sum_{k=0}^{n} b_{nk} - \sum_{k=0}^{n-1} b_{n-1,k}|$$

$$= |b_{00}| + \sum_{n=1}^{N} |\sum_{k=0}^{n} b_{nk} - \sum_{k=0}^{n-1} b_{n-1,k}|$$

$$= |b_{00}| + \sum_{n=1}^{N} |b_{nn} - b_{n-1,n-1} + \sum_{k=0}^{n-1} b_{nk} - \sum_{k=0}^{n-2} b_{n-1,k}|$$

$$= \frac{1}{|\lambda - a_0|} + \sum_{n=1}^{N} | \frac{1}{\lambda - a_n} - \frac{1}{\lambda - a_{n-1}}$$

$$+ \sum_{k=0}^{n-1} \frac{a_n}{\lambda^2 \prod_{j=k}^{n} (1 - \frac{a_j}{\lambda})} - \sum_{k=0}^{n-2} \frac{a_{n-1}}{\lambda^2 \prod_{j=k}^{n-1} (1 - \frac{a_j}{\lambda})} |$$

$$= \frac{1}{|\lambda - a_0|} + \sum_{n=1}^{N} | \frac{a_n - a_{n-1}}{(\lambda - a_n)(\lambda - a_{n-1})} + a_n \sum_{k=0}^{n-1} w_k$$

$$- a_{n-1}(1 - \frac{a_n}{\lambda}) \sum_{k=0}^{n-2} w_k |$$

$$= \frac{1}{|\lambda - a_0|} + \sum_{n=1}^{N} | \frac{a_n - a_{n-1}}{(\lambda - a_n)(\lambda - a_{n-1})} + a_n w_{n-1}$$

$$+ a_n \sum_{k=0}^{n-2} w_k + (-a_{n-1} + \frac{a_n a_{n-1}}{\lambda}) \sum_{k=0}^{n-2} w_k |$$

$$= \frac{1}{|\lambda - a_0|} + \sum_{n=1}^{N} | \frac{2a_n - a_{n-1}}{(\lambda - a_n)(\lambda - a_{n-1})} + (a_n - a_{n-1}) \sum_{k=0}^{n-2} w_k$$

$$+ \frac{a_n a_{n-1}}{\lambda} \sum_{k=0}^{n-2} w_k - \sum_{k=0}^{n} (b^0_{nk} - b^0_{n-1,k}) |$$

$$
\begin{aligned}
&= \frac{1}{|\lambda - a_0|} + \sum_{n=1}^{N} \Big| \frac{2a_n - a_{n-1}}{(\lambda - a_n)(\lambda - a_{n-1})} - \frac{2a_n^0 - a_{n-1}^0}{(\lambda - a_n^0)(\lambda - a_{n-1}^0)} \\
&+ (a_n - a_{n-1}) \sum_{k=0}^{n-2} w_k - (a_n^0 - a_{n-1}^0) \sum_{k=0}^{n-2} w_k^0 \\
&+ \frac{a_n a_{n-1}}{\lambda} \sum_{k=0}^{n-2} w_k - \frac{a_n^0 a_{n-1}^0}{\lambda} \sum_{k=0}^{n-2} w_k^0 \Big| \\
&\le \frac{1}{|\lambda - a_0|} + \sum_{n=1}^{N} \Big| \frac{2a_n - a_{n-1}}{(\lambda - a_n)(\lambda - a_{n-1})} - \frac{2a_n^0 - a_{n-1}^0}{(\lambda - a_n^0)(\lambda - a_{n-1}^0)} \Big| \\
&+ \sum_{n=1}^{N} \Big|(a_n - a_{n-1}) \sum_{k=0}^{n-2} w_k - (a_n^0 - a_{n-1}^0) \sum_{k=0}^{n-2} w_k^0 \Big| \\
&+ \sum_{n=1}^{N} \Big| \frac{a_n a_{n-1}}{\lambda} \sum_{k=0}^{n-2} w_k - \frac{a_n^0 a_{n-1}^0}{\lambda} \sum_{k=0}^{n-2} w_k^0 \Big| \\
&= \frac{1}{|\lambda - a_0|} + I_1 + I_2 + I_3,
\end{aligned}
$$

where $a_n^0 = \dfrac{L}{n+1}$, $w_k = \dfrac{1}{\lambda^2 \prod_{j=k}^{n} (1 - \frac{a_j}{\lambda})}$ and $w_k^0 = \dfrac{1}{\lambda^2 \prod_{j=k}^{n} (1 - \frac{a_j^0}{\lambda})}$.

Since $\alpha L < 1$, we obtain

$$
\begin{aligned}
\sum_{k=0}^{n-2} |w_k| &= |w_0| \Big(1 + \sum_{k=1}^{n-2} \prod_{j=k}^{n} (1 - \frac{a_j}{\lambda}) \Big) \\
&= \frac{(n+1)^{\alpha L}}{|\lambda|^2} \Big(1 + \sum_{k=1}^{n-2} \frac{1}{k^{\alpha L}} \Big) \\
&\le O(1)(n+1)^{\alpha L} \Big(1 + \int_0^{n-2} \frac{dx}{x^{\alpha L}} \Big)
\end{aligned}
$$

$$\begin{aligned}&\leq O(1)(n+1)^{\alpha L}\left(1+\frac{1}{1-\alpha L}(n-2)^{1-\alpha L}\right)\\&=O(1)(n+1)^{\alpha L}+O(1)(n+1)=O(n+1).\end{aligned} \tag{16}$$

Since

$$\begin{aligned}\ln(1-\frac{a_j^0}{\lambda})-\ln(1-\frac{a_j}{\lambda})&=-\frac{L}{\lambda(j+1)}+O(\frac{1}{(j+1)^2})+\frac{a_j}{\lambda}-O(\frac{1}{a_j^2})\\&=-\frac{1}{\lambda}\left(a_j-\frac{L}{j+1}\right)+O(\frac{1}{(j+1)^2}),\end{aligned} \tag{17}$$

we have

$$\begin{aligned}\left|\prod_{j=k}^{n}\left(\frac{1-\frac{a_j^0}{\lambda}}{1-\frac{a_j}{\lambda}}\right)-1\right|&=\left|\exp\left[\sum_{j=k}^{n}\ln(1-\frac{a_j^0}{\lambda})-\ln(1-\frac{a_j}{\lambda})\right]-1\right|\\&=O(1)\sum_{j=k}^{n}|\ln(1-\frac{a_j^0}{\lambda})-\ln(1-\frac{a_j}{\lambda})|\\&=O(1)\sum_{j=k}^{n}\left[\left(a_j-\frac{L}{j+1}\right)+O\left(\frac{1}{(j+1)^2}\right)\right].\end{aligned} \tag{18}$$

Then since $a_j=\dfrac{L}{j+1}+\dfrac{b_j}{j+1}$ and $(\frac{b_j}{j+1})\in\ell_1$, it follows that

$$|\sum_{k=0}^{n-2}(w_k-w_k^0)|\leq|w_0-w_0^0|+|\sum_{k=1}^{n-2}(w_k-w_k^0)|$$

$$=\left|\frac{1}{\lambda^2\prod_{j=0}^{n}(1-\frac{a_j}{\lambda})}-\frac{1}{\lambda^2\prod_{j=0}^{n}(1-\frac{a_j^0}{\lambda})}\right|$$

$$+\left|\sum_{k=1}^{n-2}\left(\frac{1}{\lambda^2\prod_{j=k}^{n}(1-\frac{a_j}{\lambda})}-\frac{1}{\lambda^2\prod_{j=k}^{n}(1-\frac{a_j^0}{\lambda})}\right)\right|$$

$$\leq O(1)\frac{1}{\prod_{j=k}^{n}|1-\frac{a_j}{\lambda}|}\left\{|\prod_{j=0}^{n}\left(\frac{1-\frac{a_j^0}{\lambda}}{1-\frac{a_j}{\lambda}}\right)-1|+O(1)\sum_{k=1}^{n-2k-1}\prod_{j=0}|1-\frac{a_j^0}{\lambda}|\right.$$

$$\left.\cdot|\prod_{j=k}^{n}\left(\frac{1-\frac{a_j^0}{\lambda}}{1-\frac{a_j}{\lambda}}\right)-1|\right\}$$

$$=O(1)(n+1)^{\alpha L}\left\{O(1)\sum_{j=0}^{n}\left[\left(a_j-\frac{L}{j+1}\right)+O(\frac{1}{(j+1)^2})\right]\right.$$

$$\left.+O(1)\sum_{k=1}^{n-2}\frac{1}{k^{\alpha L}}\sum_{j=k}^{n}\left[\left(a_j-\frac{L}{j+1}\right)+O(\frac{1}{(j+1)^2})\right]\right\}$$

$$=O(1)(n+1)^{\alpha L}\left\{O(1)\sum_{j=0}^{n}\left[\frac{b_j}{j+1}+O(\frac{1}{(j+1)^2})\right]\right.$$

$$\left.+\sum_{k=1}^{n-2}\frac{1}{k^{\alpha L}}\sum_{j=k}^{n}\left[\frac{b_j}{j+1}+O(\frac{1}{(j+1)^2})\right]\right\}$$

$$=O(1)(n+1)^{\alpha L}\left\{O(1)+\sum_{j=1}^{n}\left[\frac{b_j}{j+1}+O(\frac{1}{(j+1)^2})\right]\sum_{k=1}^{j}\frac{1}{k^{\alpha L}}\right\}$$

$$=O(1)(n+1)^{\alpha L}\left\{O(1)+\sum_{j=1}^{n}\left[\frac{b_j}{j+1}+O(\frac{1}{(j+1)^2})\right]\frac{j^{1-\alpha L}}{1-\alpha L}\right\}$$

$$=O(1)(n+1)^{\alpha L}\left\{O(1)+O(1)\sum_{j=1}^{n}\left[\frac{b_j}{(j+1)^{\alpha L}}+O(\frac{1}{(j+1)^{1+\alpha L}})\right]\right\}. \tag{19}$$

Hence

$$I_1 = \sum_{n=1}^{N} \left| \frac{2a_n - a_{n-1}}{(\lambda - a_n)(\lambda - a_{n-1})} - \frac{2a_n^0 - a_{n-1}^0}{(\lambda - a_n^0)(\lambda - a_{n-1}^0)} \right|$$

$$\begin{aligned}
&= \sum_{n=1}^{N} \left| \frac{2(a_n - a_n^0) - (a_{n-1} - a_{n-1}^0)}{(\lambda - a_n)(\lambda - a_{n-1})} \right. \\
&+ \left. \left[\frac{1}{(\lambda - a_n)(\lambda - a_{n-1})} - \frac{1}{(\lambda - a_n^0)(\lambda - a_{n-1}^0)} \right] (2a_n^0 - a_{n-1}^0) \right| \\
&\leq \sum_{n=1}^{N} \frac{2|a_n - a_n^0| + |a_{n-1} - a_{n-1}^0|}{|\lambda - a_n|\, |\lambda - a_{n-1}|} \\
&+ \sum_{n=1}^{N} \frac{|\lambda(a_n + a_{n-1} - a_n^0 - a_{n-1}^0) + (a_n^0 a_{n-1}^0 - a_n a_{n-1})|}{|\lambda - a_n^0|\, |\lambda - a_{n-1}^0|\, |\lambda - a_n|\, |\lambda - a_{n-1}|} \left| \frac{2L}{n+1} - \frac{L}{n} \right| \\
&\leq \frac{1}{\delta^2} \sum_{n=1}^{N} \left(\frac{2|b_n|}{n+1} + \frac{|b_{n-1}|}{n} \right) + \frac{1}{\delta^4} \sum_{n=1}^{N} \left\{ |\lambda| \left(\frac{|b_n|}{n+1} + \frac{|b_{n-1}|}{n} \right) \right. \\
&+ \left. \frac{L}{n(n+1)} (|b_n| + |b_{n-1}| + |b_n|\, |b_{n-1}|) \frac{n-1}{n(n+1)} \right\} \\
&= O(1)
\end{aligned} \tag{20}$$

is shown, where $\delta = \inf_{j=1}^{N} |\lambda - a_j|$. By given (a_n) we know that $(b_n) \in bv$ from (16) and (19), we have

$$\begin{aligned}
I_2 &= \sum_{n=1}^{N} \left| (a_n - a_{n-1}) \sum_{k=0}^{n-2} w_k - (a_n^0 - a_{n-1}^0) \sum_{k=0}^{n-2} w_k^0 \right| \\
&= \sum_{n=1}^{N} \left| (a_n - a_n^0 - a_{n-1} + a_{n-1}^0) \sum_{k=0}^{n-2} w_k - (a_n^0 - a_{n-1}^0) \sum_{k=0}^{n-2} (w_k - w_k^0) \right| \\
&= \sum_{n=1}^{N} \left| \frac{L}{n+1} + \frac{b_n}{n+1} - \frac{L}{n+1} - \frac{L}{n} - \frac{b_{n-1}}{n} + \frac{L}{n} \right| O(n+1)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{N}\frac{1}{n(n+1)}\sum_{k=0}^{n-2}|w_k - w_k^0|\\
&=\sum_{n=1}^{N}|\frac{b_n}{n+1}-\frac{b_{n-1}}{n+1}+\frac{b_{n-1}}{n+1}-\frac{b_{n-1}}{n}|\cdot O(n+1)\\
&+\sum_{n=1}^{N}\frac{1}{n(n+1)}O(1)(n+1)^{\alpha L}\\
&\cdot\left\{O(1)+O(1)\sum_{j=1}^{n}\left[\frac{b_j}{(j+1)^{\alpha L}}+O(\frac{1}{(j+1)^{1+\alpha L}})\right]\right\}\\
&=\sum_{n=1}^{N}|\frac{b_n}{n+1}-\frac{b_{n-1}}{n+1}|O(n+1)\sum_{n=1}^{N}\frac{|b_{n-1}|}{n(n+1)}O(n+1)\\
&+O(1)\sum_{j=1}^{N}\left(\sum_{n=j}^{N}\frac{1}{(n+1)^{2-\alpha L}}\right)\left[\frac{b_j}{(j+1)^{\alpha L}}+O(\frac{1}{(j+1)^{1+\alpha L}})\right]\\
&=O(1)+O(1)+O(1)\sum_{j=1}^{N}\sum_{n=j}^{N}(n+1)^{\alpha L-2}\\
&\cdot\left[\frac{b_j}{(j+1)^{\alpha L}}+O(\frac{1}{(j+1)^{1+\alpha L}})\right]\\
&=O(1)+O(1)\sum_{j=1}^{N}\left(\int_j^{N+1}x^{\alpha L-2}dx\right)\left[\frac{b_j}{(j+1)^{\alpha L}}+O(\frac{1}{(j+1)^{1+\alpha L}})\right]\\
&=O(1)+O(1)\sum_{j=1}^{N}\left(\int_j^{\infty}x^{\alpha L-2}dx\right)\left[\frac{b_j}{(j+1)^{\alpha L}}+O(\frac{1}{(j+1)^{1+\alpha L}})\right]\\
&=O(1)+O(1)\sum_{j=1}^{N}\left[\frac{b_j}{j+1}+O(\frac{1}{(j+1)^2})\right]=O(1)
\end{aligned}
\tag{21}
$$

and finally

$$\begin{aligned}
I_3 &= \sum_{n=1}^{N} \Big| \frac{a_n a_{n-1}}{\lambda} \sum_{k=0}^{n-2} w_k - \frac{a_n^0 a_{n-1}^0}{\lambda} \sum_{k=0}^{n-2} w_k^0 \Big| \\
&= \sum_{n=1}^{N} \Big| \frac{a_n a_{n-1} - a_n^0 a_{n-1}^0}{\lambda} \sum_{k=0}^{n-2} w_k + \frac{a_n^0 a_{n-1}^0}{\lambda} \sum_{k=0}^{n-2} (w_k - w_k^0) \Big| \\
&\le O(1) \sum_{n=1}^{N} \frac{|b_n| + |b_{n-1}| + |b_n||b_{n-1}|}{n(n+1)} \sum_{k=0}^{n-2} |w_k| \\
&+ O(1) \sum_{n=1}^{N} \frac{1}{n(n+1)} \Big| \sum_{k=0}^{n-2} (w_k - w_k^0) \Big| \\
&\le O(1) \sum_{n=1}^{N} \frac{|b_n| + |b_{n-1}| + |b_n||b_{n-1}|}{n} \\
&+ O(1) \sum_{n=1}^{N} \frac{1}{n(n+1)} \\
&\cdot \left\{ O(1)(n+1)^{\alpha L} + O(1) \sum_{j=1}^{n} \left[\frac{b_j}{(j+1)^{\alpha L}} + O\Big(\frac{1}{(j+1)^{1+\alpha L}}\Big) \right] \right\} \\
&= O(1) + O(1) \sum_{n=1}^{N} \frac{1}{(n+1)^{2-\alpha L}} \\
&+ O(1) \sum_{n=1}^{N} \frac{1}{(n+1)^2} \sum_{j=1}^{n} \left[\frac{b_j}{(j+1)^{\alpha L}} + O\Big(\frac{1}{(j+1)^{1+\alpha L}}\Big) \right] \\
&= O(1) + O(1) + O(1) \sum_{n=1}^{N} \frac{1}{(n+1)^2} = O(1).
\end{aligned} \tag{22}$$

Hence we have $\sum_1 = O(1)$.

$$\sum_2 = \Big| \sum_{m=0}^{N} b_{N+1,m} - b_{N+1,N+1} + \sum_{m=0}^{N+1} b_{N,m} \Big|$$

$$= \left|\sum_{m=0}^{N} \frac{a_{N+1}}{\lambda^2 \prod_{j=m}^{N+1} (1-\frac{a_j}{\lambda})} - \frac{1}{\lambda - a_{N+1}} - \frac{1}{\lambda - a_N} - \sum_{m=0}^{N-1} \frac{a_N}{\lambda^2 \prod_{j=m}^{N} (1-\frac{a_j}{\lambda})}\right|$$

$$\leq \sum_{m=0}^{N} \frac{a_{N+1}}{|\lambda|^2 \prod_{j=m}^{N+1} |1-\frac{a_j}{\lambda}|} + \frac{1}{|\lambda - a_{N+1}|} + \frac{1}{|\lambda - a_N|}$$

$$+ \sum_{m=0}^{N-1} \frac{a_N}{|\lambda|^2 \prod_{j=m}^{N} |1-\frac{a_j}{\lambda}|}$$

$$= \frac{1}{|\lambda - a_{N+1}|} + \frac{1}{|\lambda - a_N|}$$

$$+ \frac{a_{N+1}}{|\lambda|^2 \prod_{j=0}^{N+1} |1-\frac{a_j}{\lambda}|} \left(1 + \sum_{m=0}^{N} \prod_{j=0}^{m-1} |1-\frac{a_j}{\lambda}|\right)$$

$$+ \frac{a_N}{|\lambda|^2 \prod_{j=0}^{N} |1-\frac{a_j}{\lambda}|} \left(1 + \sum_{m=0}^{N-1} \prod_{j=0}^{m-1} |1-\frac{a_j}{\lambda}|\right)$$

$$= O(1) + \frac{(N+2)^{\alpha L} a_{N+1}}{|\lambda|^2} \left\{1 + \sum_{m=1}^{N} \frac{1}{m^{\alpha L}}\right\}$$

$$+ \frac{(N+1)^{\alpha L} a_N}{|\lambda|^2} \left\{1 + \sum_{m=1}^{N-1} \frac{1}{m^{\alpha L}}\right\}$$

$$= O(1) + K(N+2)^{\alpha L - 1} \left\{1 + \int_0^N \frac{1}{x^{\alpha L}} dx\right\}$$

$$+M(N+1)^{\alpha L-1}\left\{1+\int_0^{N-1}\frac{1}{x^{\alpha L}}dx,\right\}$$

since $\alpha L<1$, then $\sum_2=O(1)$.

$$\sum_3=\sum_{n=N+2}^{\infty}\left|\sum_{m=0}^{N}(b_{nm}-b_{n-1,m})\right|$$

$$=\sum_{n=N+2}^{\infty}\left|\sum_{m=0}^{N}\left(\frac{a_n}{\lambda^2\prod_{j=m}^{n}(1-\frac{a_j}{\lambda})}-\frac{a_{n-1}}{\lambda^2\prod_{j=m}^{n-1}(1-\frac{a_j}{\lambda})}\right)\right|$$

$$=\sum_{n=N+2}^{\infty}\left|\sum_{m=0}^{N}\frac{1}{\lambda^2\prod_{j=m}^{n}(1-\frac{a_j}{\lambda})}\left(a_n-a_{n-1}(1-\frac{a_n}{\lambda}\right)\right|$$

$$=\frac{1}{|\lambda|^2}\sum_{n=N+2}^{\infty}\frac{1}{\prod_{j=0}^{n}|1-\frac{a_j}{\lambda}|}\left\{1+\sum_{m=1}^{N}\prod_{j=0}^{m-1}|1-\frac{a_j}{\lambda}|\right\}$$

$$\cdot\left|a_na_{n-1}(\frac{1}{a_{n-1}}-\frac{1}{a_n}+\frac{1}{\lambda})\right|$$

$$=O(1)\sum_{n=N+2}^{\infty}(n+1)^{\alpha L}\left\{1+\sum_{m=1}^{N}\frac{1}{m^{\alpha L}}\right\}$$

$$\cdot\left|\frac{1}{n(n+1)a_{n-1}}-\frac{1}{n(n+1)a_n}+\frac{1}{n(n+1)\lambda}\right|$$

$$=O(1)\sum_{n=N+2}^{\infty}(n+1)^{\alpha L}\left\{1+\int_0^{N}\frac{dx}{x^{\alpha L}}\right\}$$

$$\cdot\left|\frac{K}{n+1}-\frac{M}{n}+\frac{1}{n(n+1)\lambda}\right|$$

$$
\begin{aligned}
&= O(1) \sum_{n=N+2}^{\infty} (n+1)^{\alpha L} \left\{ 1 + \frac{N^{1-\alpha L}}{1-\alpha L} \right\} \frac{1}{n(n+1)} (1 + \frac{1}{|\lambda|}) \\
&= O(N^{1-\alpha L}) \sum_{n=N+2}^{\infty} (n+1)^{\alpha L - 2} \\
&= O(N^{1-\alpha L}) \int_{n=N+1}^{\infty} (x+1)^{\alpha L - 2} dx \\
&= O(1).
\end{aligned}
$$

This agrees, with the result obtained by Okutoyi [2], for the special case $R_a = C_1$.

For the other special cases of spectrums of Rhaly matrices R_a we give the following examples.

EXAMPLE 1. If $a = (\frac{n+3}{n^2+1})$ then

$$\pi_0(R_a^*, bv^*) = \{ \lambda : | \lambda - \frac{1}{2} | \leq \frac{1}{2} \} \cup \{ 1, 2, 3 \},$$

and

$$\sigma(R_a, bv) = \{ \lambda : | \lambda - \frac{1}{2} | \leq \frac{1}{2} \} \cup \{ 2, 3 \}.$$

EXAMPLE 2. If $a = (\sin \frac{1}{n+1})$ then

$$\pi_0(R_a^*, bv^*) = \{ \lambda : | \lambda - \frac{1}{2} | < \frac{1}{2} \} \cup \{ 1 \},$$

and

$$\sigma(R_a, bv) = \{ \lambda : | \lambda - \frac{1}{2} | \leq \frac{1}{2} \}.$$

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