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RINGS WHOSE ADDITIVE MAPPINGS ARE GENERATED BY MULTIPLICATIVE AND RELATED RINGS

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1. Introduction

In this paper, we will investigate the rings in which all the additive endomorphisms or only the left multiplication endomorphisms are generated by ring endomorphisms. This is motivated by the work on the Sullivan Problem, that is, characterize those rings in which every additive endomorphism is a ring endomorphism, these rings are called *AE rings* [12], [1], [5], [6], [7], [9], and the current investigation of LSD-generated algebras [3] and SD-generated algebras [2]

Throughout this paper, R is an associative ring not necessarily with unity, $End(R,+)$ the ring of additive endomorphisms of R, and $End(R,+,\ )$ the monoid of ring endomorphisms of R. For $X \subseteq R$, we use $gp(X)$ for the subgroup of $(R,+)$ generated by X. For each $x \in R$, ${}_x\tau$ denotes the left multiplicative mapping (i.e., $a \longmapsto xa$, for all $a \in R$). Observe that ${}_x\tau \in End(R,+)$. $\mathcal{LGE}(R)$ is the set $\{x \in R \mid {}_x\tau \in gp(End(R,+,\cdot))\}$. Note that $\mathcal{LGE}(R)$ is a subring of R. Sometimes, we will use the notations: $\mathcal{I}(R)$ is the set of all idempotents, $\mathcal{N}(R)$ is the set of all nilpotents of R, and $End_{\mathbb{Z}}(R)$ instead of $End(R,+)$, $End(R)$ instead of $End(R,+,\cdot)$ and $GE(R)$ instead of $gp < End(R,+,\cdot) >$. Clearly, GE(R) is a subring of $End_{\mathbb{Z}}(R)$.

$\mathcal{L}(R)$ is the set $\{x \in R \mid xab = xaxb\}$. $(\mathcal{L}(R),\cdot)$ is a subsemigroup of $(R,\cdot)$, and $x \in \mathcal{L}(R)$ if and only if ${}_x\tau \in End(R,+,\cdot)$. Also

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$\mathcal{L}(R) \subseteq \mathcal{LGE}(R)$ and $\mathcal{L}(R)$ contains all one-sided unities of R, the left annihilators of R and all central idempotents.

We use $\mathcal{RGE}(R)$ and $\mathcal{R}(R)$ for the right sided analogs of $\mathcal{LGE}(R)$ and $\mathcal{L}(R)$, respectively. We call that a ring R is an *AGE ring* (*LGE ring*) if

$$End_{\mathbb{Z}}(R) = GE(R) \ (R = \mathcal{LGE}(R)).$$

Similarly we can define the *RGE ring* (i.e.,$R = \mathcal{RGE}(R)$).

Clearly, we see that every AE ring is AGE, LGE and RGE, but not conversely from the following examples. Note if the left regular representation of R into $End(R, +)$ is surjective, then R is an AGE ring.

R is called *LSD* (*LSD-generated*) if $R = \mathcal{L}(R)$ ($R = gp(\mathcal{L}(R))$), and also R is called *RSD* (*RSD-generated*) if $R = \mathcal{R}(R)$ ($R = gp(\mathcal{R}(R))$) [4] and [3]. R is called *SD* (*SD-generated*) if $R = \mathcal{L}(R) \cap \mathcal{R}(R)$ ($R = gp(\mathcal{L}(R) \cap \mathcal{R}(R))$ [2]. The classes of LSD, LSD-generated, SD and SD-generated rings are closed with respect to homomorphisms and direct sums. Observe that the class of LGE rings contains both the class of AGE rings and the class of LSD-generated rings. The class of AGE rings is contained in the class of RGE rings.

2. A study on rings whose additive mappings are generated by multiplicative and related rings

In the sequel, examples are provided to show that the classes of LGE, AGE and LSD-generated rings are distinct. Although the class of AE rings is a proper subclass of the class of SD rings, the class of AGE rings is not contained in the class of SD-generated in the following example.

EXAMPLE 2.1.

(1) *Since the rings $\mathbb{Z}$ and $\mathbb{Z}_n$ are additively generated by* 1, $End_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}$, $End_{\mathbb{Z}_n}(\mathbb{Z}) \cong \mathbb{Z}_n$, *thus we see that $\mathbb{Z}$ and $\mathbb{Z}_n$ are both AGE, LSD-generated and SD-generated rings. However, $\mathbb{Z}$ and $\mathbb{Z}_n$ are all not AE rings except the cases $\mathbb{Z}_1$ and $\mathbb{Z}_2$, because any*

nontrivial on $\mathbb{Z}$ or $\mathbb{Z}_n$ is an additive endomorphism but which is not a ring endomorphism.

(2) *On the other hand, $x \in \mathcal{L}(R)$ implies $x^3 = x^n$ for $n > 3$, then $\mathcal{L}(S) = \{0\}$ for any nonzero proper subring S of $\mathbb{Z}$. Hence any nonzero proper subring of $\mathbb{Z}$ is an AGE ring which is not LSD-generated and SD-generated.*

PROPOSITION 2.2. *For every AGE ring R, and for any positive integer n, we get that $\bigoplus_{i=1}^{n} R_i$ is an AGE ring, where $R_i \cong R$.*

PROOF. We prove the case for $n = 2$, that is, $R \bigoplus R$. Similarly, we can prove for the case $n > 2$. We must show that

$$End_{\mathbb{Z}}(R \bigoplus R) = GE(R \bigoplus R).$$

Since $End_{\mathbb{Z}}(R \bigoplus R) \cong Mat_2(End_{\mathbb{Z}}(R))$, we obtain that

$$End_{\mathbb{Z}}(R \bigoplus R) \cong \begin{pmatrix} End_{\mathbb{Z}}(R) & End_{\mathbb{Z}}(R) \\ End_{\mathbb{Z}}(R) & End_{\mathbb{Z}}(R) \end{pmatrix} = \begin{pmatrix} GE(R) & GE(R) \\ GE(R) & GE(R) \end{pmatrix}.$$

Let $f \in End_{\mathbb{Z}}(R \bigoplus R)$ such that

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, \; f_{ij} \in GE(R).$$

Then

$$f_{11} = \sum_i \lambda_i h_i, \; f_{12} = \sum_j \lambda_j h_j, \; f_{21} = \sum_k \lambda_k h_k, \; f_{22} = \sum_t \lambda_t h_t,$$

where, $\lambda's \in \mathbb{Z}$ and $h's \in End(R)$. Thus f is expressed of the form

$$f = \sum_i \lambda_i \begin{pmatrix} h_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_j \lambda_j \begin{pmatrix} 0 & h_j \\ 0 & 0 \end{pmatrix} + \sum_k \lambda_k \begin{pmatrix} 0 & 0 \\ h_k & 0 \end{pmatrix} + \sum_t \lambda_t \begin{pmatrix} 0 & 0 \\ 0 & h_t \end{pmatrix}.$$

Since all $\begin{pmatrix} h_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & h_j \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ h_k & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & h_t \end{pmatrix}$ are ring endomorphisms of $R \bigoplus R$. Hence $R \bigoplus R$ is an AGE ring. □

From Example 2.1 and Proposition 2.2, there exist numerously many examples of AGE rings and LSD-generated rings.

LEMMA 2.3. *For any surjective ring endomorphism h, $\mathcal{L}(R)$ and $\mathcal{R}(R)$ are all invariant subsemigroups under h.*

PROPOSITION 2.4. *Let R be a ring with unity. If R is an AGE ring with $S \subseteq End(R)$ such that $End_{\mathbb{Z}}(R) = gp < S >$, and each element of S is onto, then R is an LSD-generated, moreover SD-generated.*

PROOF. Let $x \in R$. Consider a left translation mapping $\phi_x : R \longrightarrow R$ by $\phi_x(a) = xa$ for all $a \in R$, which is a group endomorphism. Since R is an AGE ring,

$$\phi_x = \sum_i^n \lambda_i h_i,$$

where $\lambda_i \in \mathbb{Z}$ and $h_i \in End(R)$ such that h_i is onto, $i = 1, 2, \cdots, n$. Since $1 \in R$, $\phi_x(1) = \sum_i^n \lambda_i h_i(1)$, that is, $x = \sum_i^n \lambda_i h_i(1)$ and since $1 \in \mathcal{L}(R) \cap \mathcal{R}(R)$ by Lemma 2.3, $h_i(1) \in \mathcal{L}(R) \cap \mathcal{R}(R)$. Hence R is LSD–generated and RSD–generated, so SD–generated. □

EXAMPLE 2.5.

(1) *If S is an LSD-generated ring, then*

$$R = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$$

is also LSD-generated by the set

$$\{\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 0 & y \\ 0 & y \end{pmatrix} \mid v, x, y \in \mathcal{L}(S)\}.$$

(2) *If S is an RSD-generated ring, then*

$$R = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$$

is also RSD-generated by the set

$$\{\begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} y & y \\ 0 & 0 \end{pmatrix} \mid v, x, y \in \mathcal{R}(S)\}.$$

In particular,

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$$

is an LSD-generated ring with the generators: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, *and also an RSD-generated ring with the generators:* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, *but which is not an SD-generated ring. Clearly,* $\begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$ *is an SD-generated ring.*

Similarly,

$$R = \begin{pmatrix} \mathbb{Z}_n & \mathbb{Z}_n \\ 0 & \mathbb{Z}_n \end{pmatrix}$$

is both LSD-generated and RSD-generated, but which is not SD-generated.

EXAMPLE 2.6. *Let S be an LSD semigroup (i.e., $xab = xaxb$, for all $x, a, b \in S$). Then the semigroup ring $K[S]$, where K is $\mathbb{Z}$ or $\mathbb{Z}_n$, is an LSD-generated ring. In particular, let S be a nonempty set and define multiplication on S by $st = t$, for each $s, t \in S$. Then $\mathbb{Z}[S]$ and $\mathbb{Z}_n[S]$ are LSD-generated rings. Furthermore if $|S| = 2$, then $\mathbb{Z}_2[S]$ is an LSD ring which is not an AGE ring.*

EXAMPLE 2.7. *Let R be a ring and $X \subseteq R$ such that $R = gp(X)$.*

(1) *If I is the ideal generated by $\{axay - axy \mid a, x, y \in X\}$, then R/I is an LSD-generated ring.*
(2) *If J is the ideal generated by $\{xbyb - xyb \mid b, x, y \in X\}$, then R/J is an RSD-generated ring.*

PROPOSITION 2.8. *Let $Y \subseteq End(R, +, \cdot)$ and $S \subseteq R$ such that $f(S) \subseteq gp(S)$, for each $f \in Y$.*

(1) *If R is an LGE ring and for each $x \in R$, ${}_x\tau = \sum_{i\in I} \pm f_i$, where each $f_i \in Y$, then $gp(S)$ is a left ideal of R.*
(2) *If R is an AGE ring and $Y = End(R, +, \cdot)$, then $h(S) \subseteq gp(S)$, for each $h \in End(R, +)$.*

PROOF. (1) Let $x \in R$ and $w \in gp(S)$. Then $w = \sum_{j \in J} k_j s_j$, where each $k_j \in \mathbb{Z}$ and each $s_j \in S$. Also there exist $f_i \in Y$ such that

$$_x\tau = \sum_{i \in I} \pm f_i.$$

Hence

$$xw = {}_x\tau(w) = \sum_{i \in I} \pm f_i(w) = \sum_{i \in I} \pm f_i(\sum_{j \in J} k_j s_j) = \sum_{i \in I} \sum_{j \in J} \pm k_j f_i(s_j) \in gp(S$$

Thus $gp(S)$ is a left ideal of R.
(2) The proof of this part is similar to that of part (1). □

Proposition 2.8 can be used to show that $R = F[x]$ is not an LGE ring, where F is a field. Assume that $0 \neq f \in End(R, +, \cdot)$. Then $f(1) = 1$ since R is an integral domain. By Proposition 2.8 (1), if R is an LGE ring, then $gp(\mathcal{U}(R)) = R$, where $\mathcal{U}(R)$ is the unit group of R. This is a contradiction.

COROLLARY 2.9. *Let $S = \mathcal{I}(R)$, $\mathcal{N}(R)$. Then we have the following statements:*

(1) *If R and Y are as in Proposition 2.8 (1), then $gp(S)$ is a left ideal of R.*
(2) *If R and Y are as in Proposition 2.8 (2), then $h(S) \subsetneq gp(S)$, for each $h \in End(R, +)$.*

Observe that Example 2.5 is an LSD-generated ring which is not an AGE ring. To see this, let $h : R \longrightarrow R$ be defined by

$$h(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

Then $h \in End(R, +)$, but $h(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \notin \mathcal{N}(R) = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$. By Corollary 2.9 (2), R is not an AGE ring.

COROLLARY 2.10.

(1) *If* R *is an LGE ring with a right unity, then* $R = gp(\mathcal{I}(R))$.

(2) *If* R *is a simple AGE ring, then* $R = gp(\mathcal{N}(R))$.

It is immediate that the classes of LGE rings and LSD-generated rings are closed with respect to direct sums.

Let $Y \subseteq End(R, +, \cdot)$ and let R^Y denote

$$\{x \in R | f(x) = x, \text{ for each } f \in Y\}.$$

Observe that R^Y is a subring of R (when Y is a group acting as automorphisms on R, then R^Y is called the *fixed ring* under Y).

PROPOSITION 2. 11. *Let* $X \subseteq \mathcal{LGE}(R)$. *For each* $x \in X$, *pick a representation of* ${}_x\tau = \sum_{i \in I} k_i f_i$ *such that* $k_i \in \mathbb{Z}$ *and* $f_i \in End(R, +, \cdot)$. *Let* Y_x *be the set of* f_i *in this representation. Let* $Y = \cup_{x \in X} Y_x$. *If* $< X >$ *is the subring generated by* X, *then* R^Y *is a left* $< X >$-*module. If* $< X >= R$, *then* R^Y *is a left ideal of* R *and* R *is an LGE ring.*

PROOF. Let $x \in X$ and $s \in R^Y$. Then there exists a representation of ${}_x\tau = \sum_{i \in I} k_i f_i$ such that $k_i \in \mathbb{Z}$ and $f_i \in Y$. Hence

$$xs = {}_x\tau(s) = \sum_{i \in I} k_i f_i(s) = (\sum_{i \in I} k_i)s \in R^Y.$$

Since X generates R, then R^Y is a left ideal of R. □

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