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CHARACTERIZATIONS OF REGULAR *po*–SEMIGROUPS

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Abstract Lajos([1-3]) gave the ideal-theoretical characterizations of some classes of semigroups without "order" The first author([4]) gave the ideal-theoretical characterization of some classes of *po*-semigroup with order "$\leq$". In this paper we give the other characterizations.

1. Introduction

S. Lajos([1-3]) gave some characterizations of a regular semigroup without "order". Recently S. K. Lee([4]) gave the characterizations of the regularity in a *po*-semigroup (:ordered semigroup) by a different criteria.

The aim of this paper is to obtain other characterizations of the regularity in a *po*-semigroup S.

Kehayopulu([5-9]) considered the ordered semigroups. A *po*-semigroup (:ordered semigroup) is an ordered set $(S, \leq)$ at the same time a semigroup such that:

$$a \leq b \Longrightarrow ca \leq cb \text{ and } ac \leq bc$$

for all $c \in S$.

The following definitions are well known.

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DEFINITION 1. Let A be a non-empty subset of a *po*-semigroup S. A is called a *left*(resp. *right*) *ideal of* S if

i) $SA \subseteq A$(resp. $AS \subseteq A$).

ii) $a \in A$ and $b \leq a$ for $b \in S \Longrightarrow b \in A$.

A is called an *ideal of* S if A is both a left and a right ideal of S.

DEFINITION 2. Let Q be a non-empty subset of a *po*-semigroup S. Q is called a *quasi-ideal of* S if

i) $QS \cap SQ \subseteq Q$

ii) $a \in Q$ and $b \leq a$ for $b \in S \Longrightarrow b \in Q$.

Every left(resp. right) ideal is a quasi-ideal. Also every ideal is a quasi-ideal.

DEFINITIONS 3. Let B be a non-empty subset of a *po*-semigroup S. B is called a *bi-ideal of* S if

i) $BSB \subseteq B$

ii) $a \in B$ and $b \leq a$ for $b \in S \Longrightarrow b \in B$.

Every quasi-ideal is a bi-ideal.

DEFINITION 4. A subsemigroup T of a *po*-semigroup S is called *regular* if, for all $a \in T$ there exists $x \in T$ such that $a \leq axa$.

NOTATION. For a subset H of a *po*-semigroup S,

$$(H] = \{t \in S : t \leq h \text{ for some } h \in H\}.$$

We denote by $L(a)$(resp. $R(a), B(a), Q(a)$) the left(resp. right, bi-, quasi-) ideal generated by $a \in S$. And we denote by $I(a)$ the ideal of S generated by $a \in S$.

One can easily prove that

$$L(a) = (a \cup Sa],\ \ R(a) = (a \cup aS],\ \ B(a) = (a \cup aSa],$$

$$I(a) = (a \cup Sa \cup aS \cup SaS] \text{ and } Q(a) = (a \cup ((aS] \cap (Sa])].$$

(cf. also [8]).

REMARK 1. Let T be a subsemigroup of S. Then T is regular if and only if $a \in (aTa]$ for all $a \in T$.

2. Main Results

LEMMA([7, 8], ALSO SEE [4]). *Let S be a po-semigroup. We have the following.*

i) $A \subseteq (A]$ *for any* $A \subseteq S$.
ii) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$.
iii) *If A is some types of ideal, then* $A = (A]$.
iv) $(A](B] \subseteq (AB]$ *for all A and* $B \subseteq S$.

Recently S. K. Lee([4]) proved the following Theorem A.

THEOREM A. *A po-semigroup S is regular if and only if for every bi-ideal B, any three left ideals L_1, L_2, L and every ideal I of S, we have*

$$B \cap L_1 \cap L_2 \subseteq (BL_1L_2], \quad \textit{equivalently}, \quad B \cap I \cap L \subseteq (BIL].$$

Now we give new characterizations of regular *po*-semigroup S.

PROPOSITION 1. *Let S be a po-semigroup. Then the followings are true.*

(1) *If S is regular, then for each bi-ideal, each ideal I and each subset X of S, we have* $B \cap I \cap X \subseteq (BIX]$.
(2) *If* $B \cap Q \subseteq (BSQ]$ *for each bi-ideal B and each quasi-ideal Q of S, then S is regular.*

PROOF. (1) Let $a \in B \cap I \cap X$. Since S is regular, there exists $x \in S$ such that

$$a \leq axa \leq (axa)x(axa) = (axa)(xax)a \in (BSB)(SIS)X \subseteq BIX.$$

Thus $a \in (BIX]$, and so $B \cap I \cap X \subseteq (BIX]$.

(2) Let $a \in S$. We consider the bi-ideal $B(a)$ and the quasi-ideal $Q(a)$ of S generated by a, respectively. By hypothesis and Lemma, we

have

$$\begin{aligned} a \in B(a) \cap Q(a) &\subseteq (B(a)SQ(a)] = ((a \cup aSa]S(a \cup ((aS] \cap (Sa])] \\ &\subseteq ((a \cup aSa](S](a \cup (Sa])] \subseteq ((a \cup aSa](S]((a \cup Sa]]] \\ &= ((a \cup aSa](S](a \cup Sa]] \subseteq ((aSa \cup aS^2 \cup aSaSa \cup aSaS^2a]] \\ &\subseteq ((aSa]] = (aSa]. \end{aligned}$$

Then S is regular.

COROLLARY 1. *A po-semigroup S is regular if and only if for each bi-ideal B, each ideal I and each quasi-ideal Q of S, we have $B \cap I \cap Q \subseteq (BIQ]$.*

PROPOSITION 2. *Let S be a po-semigroup. Then the followings are true.*

(1) *If S is regular then, for each bi-ideal B, each ideal I and each subset X of S, $B \cap I \cap X \subseteq (XIB]$.*
(2) *If $B \cap Q \subseteq (QSB]$ for each bi-ideal B and each quasi-ideal Q of S then, S is regular.*

PROOF. (1) Let $a \in B \cap I \cap X$. Since S is regular, there exists $x \in S$ such that

$$a \leq axa \leq (axa)x(axa) = a(xax)(axa) \in X(SIS)(BSB) \subseteq XIB.$$

Thus $a \in (XIB]$, and so $X \cap I \cap B \subseteq (XIB]$.

(2) Let $a \in S$. We consider the bi-ideal $B(a)$ and the quasi-ideal $Q(a)$ of S generated by a, respectively. By hypothesis and Lemma, we have

$$\begin{aligned} a \in B(a) \cap Q(a) &\subseteq (Q(a)SB(a)] = ((a \cup ((aS] \cap (Sa])]S(a \cup aSa]] \\ &\subseteq ((a \cup (aS]](S](a \cup aSa]] \subseteq (((a \cup aS]](S](a \cup aSa]] \\ &= ((a \cup aS](S](a \cup aSa]] \subseteq ((aSa \cup aSaSa \cup aS^2a \cup aS^2aSa]] \\ &\subseteq ((aSa]] = (aSa]. \end{aligned}$$

Then S is regular.

COROLLARY 2. *A po-semigroup S is regular if and only if for each quasi-ideal Q, each ideal I and each bi-ideal B of S, we have $B \cap I \cap Q \subseteq (QIB]$.*

PROPOSITION 3. *Let S be a po-semigroup. Then the followings are true.*

(1) *If S is regular then, for each subset X, each ideal I and each left ideal L of S, $X \cap I \cap L \subseteq (XIL]$.*
(2) *If $B \cap L \subseteq (BSL]$ for each bi-ideal B and each left ideal L of S then, S is regular.*

PROOF. (1) Since every left ideal is a bi-ideal, we can obtain $X \cap I \cap L \subseteq (XIL]$ from (1) of Proposition 2.

(2) Let $a \in S$. We consider the bi-ideal $B(a)$ and the left ideal $L(a)$ of S generated by a, respectively. By hypothesis and Lemma, we have

$$\begin{aligned} a \in B(a) \cap L(a) &\subseteq (Q(a)SL(a)] = ((a \cup ((aS] \cap (Sa]))]S(a \cup Sa]] \\ &\subseteq ((a \cup (aS]](S](a \cup Sa]] \subseteq (((a \cup aS]](S](a \cup Sa]] \\ &= ((a \cup aS](S](a \cup Sa]] \subseteq ((aSa \cup aS^2a \cup aS^3a]] \\ &\subseteq ((aSa]] = (aSa]. \end{aligned}$$

Then S is regular.

COROLLARY 3 (THEOREM A). *A po-semigroup S is regular if and only if for each bi-ideal B, each ideal I and each left ideal L of S, we have $B \cap I \cap L \subseteq (BIL]$.*

PROPOSITION 4. *Let S be a po-semigroup. Then the following are true.*

(1) *If S is regular then, for each right ideal R, each ideal I and each subset X of S, we have $R \cap I \cap X \subseteq (RIX]$.*
(2) *If $R \cap B \subseteq (RSB]$ for each bi-ideal B and each right ideal R of S then, S is regular.*

PROOF. (1) Since every right ideal is a bi-ideal, we can obtain $R \cap I \cap X \subseteq (RIX]$ from (1) of Proposition 1.

(2) Let $a \in S$. We consider the the right ideal $R(a)$ and the bi-ideal $B(a)$ of S generated by a, respectively. By hypothesis and Lemma, we have

$$\begin{aligned} a \in R(a) \cap B(a) &\subseteq (R(a)SB(a)] = ((a \cup aS]S(a \cup aSa]] \\ &\subseteq ((a \cup aS](S](a \cup aSa]] = ((aSa \cup aSaSa \cup aS^2a \cup aS^2aSa]] \\ &\subseteq ((aSa]] = (aSa]. \end{aligned}$$

Then S is regular.

COROLLARY 4. *A po-semigroup S is regular if and only if each right ideal R, each ideal I and each bi-ideal B of S, we have $R \cap I \cap B \subseteq (RIB]$.*

REMARK 2. If we change the bi-ideal to a quasi-ideal and change the quasi-ideal to a bi-ideal in Proposition 3, 4 and Corollary 3, 4, then we have the similar results.

PROPOSITION 5. *Let S be a po-semigroup. Then the followings are true.*

(1) *If S is regular then, for each right ideal R, each subset X and each ideal I of S, we have $R \cap X \cap L \subseteq (RXL]$.*
(2) *If $R \cap L \subseteq (RSL]$ for each right ideal B and each left ideal R of S then, S is regular.*

PROOF. (1) Let $a \in R \cap X \cap L$. Since S is regular, there exists $x \in S$ such that

$$a \leq axa \leq (axa)x(axa) = (axax)a(xa) \in (RSRS)X(SL) \subseteq RXL.$$

Thus $a \in (RXL]$, and so $R \cap I \cap X \subseteq (RXL]$.

(2) Let $a \in S$. We consider the the right ideal $R(a)$ and the left ideal $L(a)$ of S generated by a, respectively. By hypothesis and Lemma, we

have

$$\begin{aligned} a \in R(a) \cap L(a) &\subseteq (R(a)SL(a)] = ((a \cup aS]S(a \cup Sa]] \\ &\subseteq ((a \cup aS](S](a \cup Sa]] = ((aSa \cup aS^2a \cup aS^2a \cup aS^3a]] \\ &\subseteq ((aSa]] = (aSa]. \end{aligned}$$

Then S is regular.

COROLLARY 5. *A po-semigroup S is regular if and only if each right ideal R, each subset X and each left ideal L of S, we have $R \cap X \cap L \subseteq (RXL]$.*

Hence we have characterizations of the regularity of *po*-semigroup from Corollary 1, 2, 3, 4, 5 and Remark 2.

THEOREM. *Let S be a po-semigroup. Then the followings are equivalent.*

(1) *S is regular.*
(2) *$B \cap I \cap Q \subseteq (BIQ]$ for each bi-ideal B, each ideal I and each quasi-ideal Q of S*
(3) *$B \cap I \cap Q \subseteq (QIB]$ for each quasi-ideal Q, each ideal I and each bi-ideal B of S.*
(4) *$B \cap I \cap L \subseteq (BIL]$ for each bi-ideal B, each ideal I and each left ideal L of S.*
(5) *$Q \cap I \cap L \subseteq (QIL]$ for each quasi-ideal Q, each ideal I and each left ideal L of S.*
(6) *$R \cap I \cap B \subseteq (RIB]$ for each right ideal R, each ideal I and each bi-ideal B of S.*
(7) *$R \cap I \cap Q \subseteq (RIQ]$ for each right ideal R, each ideal I and each quasi-ideal B of S.*
(8) *$R \cap X \cap L \subseteq (RXL]$ for each right ideal R, each subset X and each left ideal L.*

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