FIXED POINT THEOREMS FOR FUZZY MAPPINGS SATISFYING AN IMPLICIT RELATION

SUSHIL SHARMA

ABSTRACT In this paper, we obtain the common fixed point for fuzzy mappings satisfying an implicit relation. We improve earlier results of this line.

1. Introduction

In 1981, Heilpern [7] introduced the concept of fuzzy mappings. In 1987, Bose and Sahani [5] gave an improved version of Heilpern. Fixed point theorems for fuzzy mappings have been studied by Butnariu [3], Chang [6], Chitra [1], Weiss [4], Lee and Cho [2] and Arora and Sharma [8]. In the present paper we improve results of Arora and Sharma [8].

2. Terminology

The definitions and terminology for further discussions are taken from Heilpern [7].

Let \((X, d)\) be metric linear space, \(F(X)\) the collection of all fuzzy sets in \(X\) and \(W(X)\) the collection of all those fuzzy sets \(A\) of \(F(X)\) whose \(\alpha\)-level sets.

\[ A_\alpha = \{x \in X : A(x) \geq \alpha\} \]
for each \( \alpha \in [0,1] \) and
\[
A_0 = \{ x \in X : A(x) > 0 \}
\]
are compact and convex with \( \sup_{x \in X} A(x) = 1 \). \( A(x) \) being the grade of membership of \( x \) in \( A \). The members of \( W(X) \) are called the approximate quantities. If \( A, B \in W(X) \) then we say \( A \subseteq B \) if and only if \( A(x) \leq B(x) \) for each \( x \in X \).

By a fuzzy map \( F \) on \( X \), we mean a mapping \( F : X \to W(X) \). A point \( x \in X \) is a common fixed point of a family \( f \) of fuzzy maps if \( \{x\} \subseteq F_i(x) \) for all \( F_i \in f \).

If \( A, B \in W(X) \) and \( \alpha \in [0,1] \), then denote
\[
p_\alpha(A, B) = \inf_{x \in A} \inf_{y \in B} d(x, y)
\]
\[
D_\alpha(A, B) = H(A, B),
\]
where \( H \) denotes the Hausdorff distance.

Also
\[
D(A, B) = \sup_a D_\alpha(A, B),
\]
\[
p(A, B) = \sup_a p_\alpha(A, B),
\]
\[
D_\alpha(A, B) = H(A, B),
\]
where \( H \) denotes the Hausdorff distance.

For the proof of our theorems we need following lemmas due to Heilpern [7].

**Lemma 1.** Let \( x \in X, A \in W(X) \) and \( \{x\} \) be a fuzzy set with membership function equal to the characteristic function of set \( \{x\} \). If \( \{x\} \subseteq A \), then \( p_\alpha(x, A) = 0 \) for each \( \alpha \in [0,1] \).

**Lemma 2.** \( p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A) \) for any \( x, y \in X \).

**Lemma 3.** If \( \{x_0\} \subseteq A \) then \( p_\alpha(x_0, B) \leq D_\alpha(A, B) \) for each \( B \in W(X) \).
**Lemma 4.** [8] Let \((X,d)\) be metric linear space \(F : X \to W(X)\) be a fuzzy map and \(x_0 \in X\), then there exists \(x_1 \in X\) such that \(\{x_1\} \subset F(x_0)\).

**Implicit Relation**

Let \(\psi\) be the set of all continuous functions \(\varphi : R^6_+ \to R\) satisfying the following conditions:

(\(\psi_1\)) \(\varphi(t_1, \ldots, t_6)\) is decreasing in variables \(t_2, \ldots, t_6\).

(\(\psi_2\)) there exists \(h \in (0, 1)\) such that the inequalities:

(i) \(u \leq t\) and \(\varphi(t, v, v, u, u + v, 0) \leq 0\) or

(ii) \(u \leq t\) and \(\varphi(t, v, u, v, 0, u + v) \leq 0\) implies \(t \leq hv\).

**Example 1.** \(\varphi(t_1, \ldots, t_6) = t_1 - \{ct_2t_3 + bt_4t_5 + ct_6\}^{\frac{1}{3}}\) where \(a, b, c > 0\) and \(a + b < 1\).

(\(\psi_1\)) : Obviously

(\(\psi_2\)) : Let \(u > 0\), \(u \leq t\) and \(\varphi(t, v, v, u, u + v, 0) = t - \{av^2 + buv + 0\}^{\frac{1}{2}} \leq 0\).

If \(v \leq u\) then \(u \leq t \leq (a + b)^{\frac{1}{2}}v < u\), a contradiction. Thus \(u < v\) and \(t \leq (a + b)^{\frac{1}{2}}v = hv\), where \(h = (a + b)^{\frac{1}{2}}\). Similarly, \(u \leq t\) and \(\varphi(t, v, u, v, 0, u + v) \leq 0\) implies \(t \leq hv\). If \(u = 0\), then \(u \leq v\) and \(t \leq (a + b)^{\frac{1}{2}}v = hv\).

**Example 2**

\(\varphi(t_1, \ldots, t_6) = t_1^3 - m\max\{t_2t_3^2, t_4^2t_5, t_6^2t_5, t_6^2\}\), where \(m \in (0, 1)\).

(\(\psi_1\)) : Obviously

(\(\psi_2\)) : Let \(u > 0\), \(u \leq t\) and \(\varphi(t, v, v, u, u + v, 0) = t^3 - m\max\{uv^2, v^2u, 0, 0\} \leq 0\). If \(v \leq u\) then \(u \leq t \leq m^{\frac{1}{2}}u < u\), a contradiction.

Thus \(u < v\) and \(t \leq m^{\frac{1}{2}}v = hv\), where \(h = m^{\frac{1}{2}}\). Similarly, \(u \leq t\) and \(\varphi(t, v, u, v, 0, u + v, 0) \leq 0\) implies \(t \leq hv\). If \(u = 0\), then \(u \leq v\) and \(t \leq m^{\frac{1}{2}}v = hv\).

**Example 3.** \(\varphi(t_1, \ldots, t_6) = t_1 - m\max\{t_2, \frac{1}{2}(t_3 + t_4), \frac{1}{2}(t_5 + t_6)\}\), where \(m \in (0, 1)\).

(\(\psi_1\)) : Obviously.
(ψ2): Let u > 0, u ≤ t and ϕ(t, v, v, u, u+v, 0) = t - m max{v, 1/2(u+v)} ≤ 0. If v ≤ u then u ≤ t ≤ mu < u, a contradiction. Thus u < v and t ≤ mv = hv, where h = m ∈ (0,1). Similarly, u ≤ t and ϕ(t, v, u, v, 0, u+v) ≤ 0 implies t ≤ mv = hv. If u = 0, then u ≤ v and t ≤ mv = hv.

Main Results

Theorem 1. Let (X, d) be a complete metric linear space and F_i : X → W(X) be fuzzy mappings for i = 1, 2 such that for all x, y ∈ X

(1.1)
ϕ(D(F_1x, F_2y), d(x, y), p(x, F_1x), p(y, F_2y), p(x, F_2y), p(y, F_1x)) ≤ 0

Then F_1 and F_2 have a common fixed point.

Proof. Let x_0 ∈ X. Then by lemma 4 there exists x_1 ∈ X such that \{x_1\} ⊂ F_1(x_0). For x_1 ∈ X, by lemma 4 the 1-level set F_2(x_1)_1 of F_2(x_1) is a compact nonempty subset of X. Thus, there exists x_2 ∈ F_2(x_1)_1 such that

\[ d(x_1, x_2) = \inf d(x_1, x) \]
\[ x ∈ F_2(x_1) \]

By Lemma 3, we have

\[ d(x_1, x_2) = p_1(x_1, F_2x_1) \]
\[ ≤ D_1(F_1x_0, F_2x_1) \]
\[ ≤ D(F_1x_0, F_2x_1) \]

Similarly, for x_2 ∈ X, there exists x_3 ∈ F_1(x_2)_1 such that

\[ d(x_2, x_3) = p_1(x_2, F_1x_2) \]
\[ ≤ D_1(F_2x_1, F_1x_2) \]
\[ ≤ D(F_2x_1, F_1x_2) \]
Continuing this way, we can obtain a sequence \( \{x_n\} \) of \( X \) such that

\[
\begin{align*}
\{x_{2n+1}\} & \subset F_1x_{2n} \\
\{x_{2n+2}\} & \subset F_2x_{2n+1}, \quad n = 1, 2, \ldots \text{with}
\end{align*}
\]

\[
d(x_{2n}, x_{2n+1}) = p_1(x_{2n}, F_1x_{2n}) \\
\leq D_1(F_1x_{2n}, F_2x_{2n-1}) \\
\leq D(F_1x_{2n}, F_2x_{2n-1})
\]

and

\[
d(x_{2n+1}, x_{2n+2}) = p_1(x_{2n+1}, F_2x_{2n+1}) \\
\leq D_1(F_1x_{2n}, F_2x_{2n+1}) \\
\leq D(F_1x_{2n}, F_2x_{2n+1})
\]

By (1.1), we write

\[
\varphi(D(F_1x_{2n}, F_2x_{2n+1}), d(x_{2n}, x_{2n+1}), p(x_{2n}, F_1x_{2n}), p(x_{2n+1}, F_2x_{2n+1}),
\]

\[
p(x_{2n}, F_2x_{2n+1}), p(x_{2n+1}, F_1x_{2n})) \leq 0
\]

\[
\varphi(D(F_1x_{2n}, F_2x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}),
\]

\[
d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})) \leq 0
\]

\[
\varphi(d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}),
\]

\[
d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0) \leq 0
\]

By implicit relation (i), we have

\[
(1.2) \quad d(x_{2n+1}, x_{2n+2}) \leq hd(x_{2n}, x_{2n+1})
\]

Similarly, by (1.1) and implicit relation (ii), we have

\[
(1.3) \quad d(x_{2n+1}, x_{2n}) \leq hd(x_{2n}, x_{2n-1})
\]
and so

$$d(x_{2n+1}, x_{2n+2}) \leq h^{2n+1}d(x_0, x_1)$$

Since $h \in (0,1)$ it follows from (1.4) that $\{x_n\}$ is a Cauchy sequence and hence convergent in $X$. Let $\lim_{n \to \infty} x_n = z \in X$.

We claim that $z$ is a fixed point of both $F_1$ and $F_2$.

Now by (1.1) we write

$$\varphi(D(F_1z, F_2x_{2n+1}), d(z, x_{2n+1}), p(z, F_1z), p(x_{2n+1}, F_2x_{2n+1}),$$

$$p(z, F_2x_{2n+1}), p(x_{2n+1}, F_1z)) \leq 0$$

$$\varphi(p(F_1z, x_{2n+2}), d(z, x_{2n+1}), p(z, F_1z), d(x_{2n+1}, x_{2n+2}),$$

$$d(z, x_{2n+2}), p(x_{2n+1}, F_1z)) \leq 0$$

Letting $n \to \infty$, we obtain

$$\varphi(p(F_1z, z), 0, p(z, F_1z), 0, 0, p(z, F_1z)) \leq 0$$

By implicit relation (ii) we see that $\{z\} \subset F_1z$. Proceeding similarly, it can be verified that $p(z, F_2z) = 0$. Hence $\{z\} \subset F_2z$, i.e. $z$ is a common fixed point of $F_1$ and $F_2$.

This completes the proof of the theorem.

Let us replace $F_1$ by $F_0$ and $F_2$ by $F_n(n \neq 0)$ and as done in Theorem 1, choose the sequence $\{x_n\}$ as

$$x_0 \in X, \{x_1\} \subset F_0(x_0), \{x_2\} \subset F_n(x_1), \{x_3\} \subset F_0(x_2), \ldots,$$

$$\{x_{2n-1}\} \subset F_0(x_{2n-2}), \{x_{2n}\} \subset F_n(x_{2n-1}), \ldots$$

Following the procedure of Theorem 1, we get a common fixed point for each pair $(F_0, F_i)$, $i = 1, 2, \ldots$. Thus we state
THEOREM 2. Let \{F_n : n \in \mathbb{Z}^+\} be a collection of fuzzy maps from \(X \rightarrow W(X)\), \(X\) being a complete metric linear space, and for all \(x, y \in X, n = 1, 2, \ldots\)

\[ \varphi(D(F_0x, F_ny), d(x, y), p(x, F_0x), p(y, F_ny), p(x, F_y), p(y, F_0x)) \leq 0 \]

Then there exists a fixed point of the family \(\{F_n : n \in \mathbb{Z}^+\}\). Letting \(F_1 = F_2 = F\) in Theorem 1 and \(x_0 \in X\), by Lemma 4 we can obtain a sequence \(\{x_n\}\) of \(X\) such that for all \(n = 1, 2, \ldots\)

\[ \{x_n\} \subset F(x_{n-1}) \]

and

\[ d(x_n, x_{n+1}) \leq D(Fx_{n-1}, Fx_n). \]

Now as \(F\) satisfies

(1.5) \( \varphi(D(Fx, Fy), d(x, y), p(x, Fx), p(y, Fy), p(x, Fy), p(y, Fx)) \leq 0 \)

for every \(x, y \in X\). It can be easily proved that \(\{x_n\}\) is a Cauchy sequence.

THEOREM 3. Let \((X, d)\) be a metric linear space and \(F : X \rightarrow W(X)\) be a fuzzy mapping satisfying (1.5). Then \(F\) has a fixed point in \(X\) if any one of the following conditions is true

(i) \(X\) is complete,
(ii) \(\{x_n\}\) converges to \(z \in X\),
(iii) \(\{x_n\}\) has a convergent subsequence \(\{x_{n_k}\}\).

The following corollaries follow immediately from the Theorems.

COROLLARY 1. Let \((X, d)\) be a complete metric linear space and \(F_i : X \rightarrow W(X)\) be fuzzy mappings for \(i = 1, 2\) such that for all \(x, y \in X\) and \(q \in (0, \frac{1}{2})\)

\[
D(F_1x, F_2y) \leq q\text{Max}\{d(x, y), p(x, F_1x), p(y, F_2y), p(x, F_2y), p(y, F_1x)\}
\]

Then \(F_1\) and \(F_2\) have a common fixed point.
Corollary 2. Let \( \{F_n : n \in \mathbb{Z}^{+}\} \) be a collection of fuzzy maps from \( X \to W(X) \), \( X \) being a complete metric linear space, and for all \( x, y \in X \), and \( q \in (0, \frac{1}{2}) \), \( n = 1, 2, \ldots \).

\[
D(F_0x, F_ny) \leq q \max \{d(x, y), p(x, F_0x), p(y, F_ny), p(x, F_ny), p(y, F_0x)\}
\]

Then there exists a fixed point of the family \( \{F_n : n \in \mathbb{Z}^{+}\} \).

Corollary 3. Let \((X, d)\) be a metric linear space and \( F : X \to W(X) \) be a fuzzy mapping satisfying :

\[
D(Fx, Fy) \leq q \max \{d(x, y), p(x, Fx), p(y, Fy), p(x, Fy), p(y, Fx)\}
\]

for all \( x, y \in X \) and some \( q \in (0, \frac{1}{2}) \). Then \( F \) has a fixed point in \( X \) if any one of the following conditions is true

(i) \( X \) is complete,
(ii) \( \{x_n\} \) converges to \( z \in X \),
(iii) \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\} \).

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References


114, A-Sector
Vivekanand Colony
Ujjain-456010, India
E-mail: sksharma2005@yahoo.com