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NOTE ON CAHEN'S INTEGRAL FORMULAS

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ABSTRACT. We present an explicit form for a class of definite integrals whose special cases include some definite integrals evaluated, over a century ago, by Cahen who made use of an appropriate contour integral for the integrand of a well-known integral representation of the Riemann Zeta function given in (3). Furthermore another analogous class of definite integral formulas and some identities involving Riemann Zeta function and Euler numbers E_n are also obtained as by-products.

We begin by recalling the definitions of Riemann Zeta function $\zeta(s)$ and Hurwitz (or generalized) Zeta function $\zeta(s,a)$:

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s} \quad (\Re(s) > 1) \tag{1}$$

and

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (\Re(s) > 1;\ a \neq 0,\ -1,\ -2,\ -3,\ \ldots). \tag{2}$$

Both functions can be continued analytically to the whole complex s-plane except for a simple pole at $s = 1$ with their residues 1 usually by the contour or some other integral representations (*cf.* [5, p. 266]; [3, p. 33]).

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The following integral representation for $\zeta(s)$ is well-known [5, p. 266]:

$$
\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \\
&= \frac{1}{\Gamma(s)} \int_0^1 \frac{(-\ln x)^{s-1}}{1-x}\, dx \quad (\Re(s) > 1),
\end{aligned} \tag{3}
$$

where $\Gamma(s)$ denotes the familiar Gamma function [3, p. 1].

Cahen integrated the function $f(z)$ defined by

$$
f(z) := \frac{(\pm \mathrm{Log}\, z)^{s-1}}{1-z} \quad (s \in \mathbb{N} \setminus \{1\},\ \mathbb{N} := \{1, 2, 3, \ldots\}) \tag{4}
$$

along the contour ($0 < \delta < 1 - \epsilon$ and $\epsilon > 0$) in Figure 1 and gave some definite integral formulas.

Here we first present an explicit formula for a class of definite integrals including Cahen's formulas [1, pp. 110-111] as special cases. In fact, we integrate $f(z)$ in (4) with plus sign, using Cauchy-Goursat theorem [2, p. 106], along the contour in Figure 1 and take the limits $\epsilon \to 0+$ and $\delta \to 0+$ in the resulting integral. One obtains

$$
\begin{aligned}
0 = &(-1)^{s-1}\,\Gamma(s)\,\zeta(s) - \frac{(i\pi)^s}{2s} \\
&+ \frac{i^{s+1}}{2} \int_0^\pi \frac{\theta^{s-1}}{\tan\frac{\theta}{2}}\, d\theta + \int_0^1 \frac{(\ln x + i\pi)^{s-1}}{1+x} dx.
\end{aligned} \tag{5}
$$

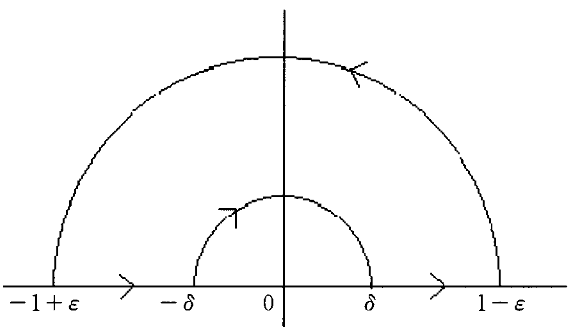

Figure 1

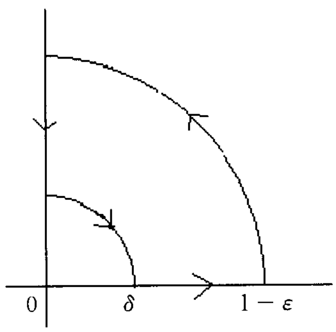

Figure 2

By applying the binomial theorem to $(\ln x+i\pi)^{s-1}$, (5) may be rewritten in an equivalent form:

$$(6)\quad \begin{aligned} i^{s+1}\int_0^{\pi}\frac{\theta^{s-1}}{\tan\frac{\theta}{2}}\,d\theta &= 2\,(-1)^s\,\Gamma(s)\,\zeta(s)+\frac{(i\,\pi)^s}{s}-2\,(i\pi)^{s-1}\ln 2\\ &\quad -2\sum_{k=1}^{s-1}\binom{s-1}{k}(i\pi)^{s-k-1}\int_0^1\frac{(\ln x)^k}{1+x}dx. \end{aligned}$$

Setting $s=2n$ in (6) and using an elementary identity to separate real and imaginary parts

$$(7)\qquad \sum_{k=1}^{2n-1}A_k=\sum_{k=1}^{n-1}A_{2k}+\sum_{k=1}^{n}A_{2k-1},$$

we, by virtue of the following known integral formulas (*cf.* [4, p. 546, Entry 4.271]):

$$(8)\qquad \int_0^1\frac{(\ln x)^{2k}}{1+x}dx=\left(1-2^{-2k}\right)\,(2k)!\,\zeta(2k+1)\quad (k\in\mathbb{N})$$

and

$$(9)\qquad \int_0^1\frac{(\ln x)^{2k-1}}{1+x}dx=\left(2^{1-2k}-1\right)\,(2k-1)!\,\zeta(2k)\quad (k\in\mathbb{N}),$$

get a class of definite integral formulas and a recurrence relation for $\zeta(s)$:

$$(10)\quad \begin{aligned} \int_0^{\pi}\frac{\theta^{2n-1}}{\tan\frac{\theta}{2}}\,d\theta &= 2\,\pi^{2n-1}\ln 2+2\sum_{k=1}^{n-1}(-1)^k\binom{2n-1}{2k}(2k)!\\ &\quad\times\pi^{2n-2k-1}\left(1-2^{-2k}\right)\zeta(2k+1)\quad (n\in\mathbb{N}), \end{aligned}$$

where the empty sum is interpreted (as usual, in what follows) to be nil;

$$(11)\quad \begin{aligned} &2\sum_{k=1}^{n}(-1)^k\binom{2n-1}{2k-1}(2k-1)!\,\pi^{2n-2k}\left(2^{1-2k}-1\right)\zeta(2k)\\ &=2\,(-1)^n\,(2n-1)!\,\zeta(2n)+\frac{\pi^{2n}}{2n}\qquad (n\in\mathbb{N}). \end{aligned}$$

If we set $s = 2n + 1$ in (6), we, similarly, obtain

$$\begin{aligned}\int_0^\pi \frac{\theta^{2n}}{\tan\frac{\theta}{2}}\,d\theta &= 2\,\pi^{2n}\,\ln 2 + 2\,(-1)^n\,(2n)!\,\zeta(2n+1)\\ &\quad + 2\sum_{k=1}^{n}(-1)^k\binom{2n}{2k}(2k)!\\ &\quad\times \pi^{2n-2k}\left(1-2^{-2k}\right)\zeta(2k+1)\quad (n\in\mathbb{N});\end{aligned}\tag{12}$$

$$\begin{aligned}\frac{1}{2n+1} &= 2\sum_{k=1}^{n}(-1)^k\binom{2n}{2k-1}(2k-1)!\\ &\quad\times \pi^{-2k}\left(2^{1-2k}-1\right)\zeta(2k)\quad (n\in\mathbb{N}).\end{aligned}\tag{13}$$

Next integrate the function $f(z)$ in (4) with minus sign along the contour in Figure 2. We get

$$0 = \Gamma(s)\,\zeta(s) + \frac{1}{2s}\left(-\frac{i\,\pi}{2}\right)^s + \frac{(-i)^{s+1}}{2}\int_0^{\frac{\pi}{2}}\frac{\theta^{s-1}}{\tan\frac{\theta}{2}}\,d\theta + I(s),\tag{14}$$

where, for convenience, we put

$$\begin{aligned}&I(s)\\ &:= (-1)^s\,i\sum_{k=0}^{s-1}\binom{s-1}{k}\left(i\frac{\pi}{2}\right)^{s-k-1}\int_0^1\frac{(\ln x)^k}{1+x^2}dx\\ &\quad + (-1)^{s+1}\sum_{k=0}^{s-1}\binom{s-1}{k}\left(i\frac{\pi}{2}\right)^{s-k-1}2^{-k-1}\int_0^1\frac{(\ln x)^k}{1+x}dx.\end{aligned}\tag{15}$$

Similarly as above, now using (8), (9), and known integral formulas (*cf.* [4, p. 546, Entry 4.271]):

$$\int_0^1\frac{(\ln x)^k}{1+x^2}dx = \frac{(-1)^k\,k!}{4^{k+1}}\left\{\zeta\left(k+1,\tfrac{1}{4}\right)-\zeta\left(k+1,\tfrac{3}{4}\right)\right\}\quad (k\in\mathbb{N});\tag{16}$$

$$\begin{aligned}\int_0^1\frac{(\ln x)^{2k}}{1+x^2}dx &= \frac{1}{2}\int_0^\infty\frac{(\ln x)^{2k}}{1+x^2}dx\\ &= \frac{(-1)^k}{2}\left(\frac{\pi}{2}\right)^{2k+1}E_{2k}\quad (k\in\mathbb{N}),\end{aligned}\tag{17}$$

where E_{2k} are Euler numbers defined by the generating function

$$\frac{2\,e^z}{e^{2z}+1} = \operatorname{sech} z = \sum_{k=0}^{\infty} E_k \frac{z^k}{k!} \quad \left(|z| < \frac{\pi}{2}\right), \tag{18}$$

we get the following identities:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\theta^{2n-1}}{\tan\frac{\theta}{2}}\,d\theta = {} & \left(\frac{\pi}{2}\right)^{2n-1} \ln 2 \\ & + 2^{1-2n} \sum_{k=1}^{n-1} (-1)^k \binom{2n-1}{2k} (2k)! \\ & \times \pi^{2n-2k-1} \left(1-2^{-2k}\right) \zeta(2k+1) \\ & - 2 \sum_{k=1}^{n} (-1)^k \binom{2n-1}{2k-1} (2k-1)! \\ & \times \frac{\pi^{2n-2k}}{2^{2n+2k}} \left\{\zeta\left(2k,\tfrac{1}{4}\right) - \zeta\left(2k,\tfrac{3}{4}\right)\right\} \quad (n \in \mathbb{N})\end{aligned} \tag{19}$$

and

(20)

$$\begin{aligned} & 2\,(-1)^n\,(2n-1)!\,\zeta(2n) \left(\frac{2}{\pi}\right)^{2n} + 1 + \frac{1}{2n} + \sum_{k=1}^{n-1} \binom{2n-1}{2k} E_{2k} \\ & = 2 \sum_{k=1}^{n} (-1)^k \binom{2n-1}{2k-1} (2k-1)!\,\pi^{-2k} \left(2^{1-2k}-1\right) \zeta(2k) \quad (n \in \mathbb{N});\end{aligned}$$

(21)

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\theta^{2n}}{\tan\frac{\theta}{2}}\,d\theta = {} & \left(\frac{\pi}{2}\right)^{2n} \ln 2 + 2\,(-1)^n\,(2n)!\,\zeta(2n+1) \\ & + \sum_{k=1}^{n} (-1)^k \binom{2n}{2k} (2k)! \\ & \times \left(\frac{\pi}{2}\right)^{2n-2k} \left(2^{-2k}-2^{-4k}\right) \zeta(2k+1) \\ & - \sum_{k=1}^{n} (-1)^k \binom{2n}{2k-1} (2k-1)! \\ & \times \left(\frac{\pi}{2}\right)^{2n-2k+1} 2^{1-4k} \left\{\zeta\left(2k,\tfrac{1}{4}\right) - \zeta\left(2k,\tfrac{3}{4}\right)\right\} \quad (n \in \mathbb{N})\end{aligned}$$

If use is made of an identity for Euler numbers:

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0 \quad (n \in \mathbb{N}), \tag{22}$$

the companion of (21) can be seen to correspond to (13).

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