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ON THE HAUSDORFF MEASURE FOR A TRAJECTORY OF A BROWNIAN MOTION IN l_2

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ABSTRACT. We consider the Hausdorff measure for Brownian motion(BM) in l_2. Several path properties of BM in l_2 are used to show the upper bound of Hausdorff measure. We also show the lower bound of it applying a law of iterated logarithm for the occupation time of BM in l_2.

1. Introduction

Let l_2 be as usual. In the previous paper [1] we considered the total occupation time of Brownian motion in l_2 starting from the origin. Let $T(a, \omega)$ denote the total time spent in a sphere of radius a with center 0 by a particular Brownian path $\omega = \beta(\cdot)$ in l_2. As a corollary of the theorems we proved in [1], we get that $\limsup_{a\to 0^+} \frac{T(a,\omega)}{a^2 \log\log a^{-1}} = C$ for some constant $C > 0$. Then this gives us a motivation to think of the Hausdorff measure of Brownian motion in l_2, since this limit contributes to prove a kind of density theorem and leads to show the lower bound of Hausdorff measure of Brownian motion. Let $\mathcal{S}_0$ be the covering family of all open spheres in l_2. We define a Hausdorff measure for $E \subset l_2$ using $\mathcal{S}_0$ i.e.,

$$\begin{aligned}&\phi - m(E)\\ &= \liminf_{\epsilon\to 0} \left\{ \sum \phi(\mathrm{diam} S_i) : E \subset \cup S_i,\ \sup_i(\mathrm{diam} S_i) \le \epsilon, \cup S_i \in \mathcal{S}_0 \right\},\end{aligned}$$

where ϕ is increasing and continuous with $\phi(s) \to 0$ as $s \to 0$ and satisfies a smoothness condition; that is, there exists $C' > 0$ such that $\phi(2x) < C'\phi(x)$ for $0 < x < \frac{1}{2}$.

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If $\phi(t) = t^2 \log\log t^{-1}$ and $E^d(\omega)$ denotes the trajectory of Brownian motion in R^d, for $0 \le t \le 1$ then Levy [7] showed that $\phi - m(E^d(\omega)) < C$ for some constant C with probability 1 and conjectured that $\phi - m(E^d(\omega)) > 0$ also with probability 1. This was proved by Z. Ciesielski and S. Taylor [2] using a density theorem obtained by Rogers and Taylor [8] for general completely additive set functions.

Let $\beta(t,\omega)$ be a standard Brownian motion in l_2 starting from the origin and let

$$E(\omega) = \{x \in l_2 | x = \beta(t,\omega), t \in [0,1]\},$$

i.e., $E(\omega)$ denotes the range of sample path for $0 \le t \le 1$ of a Brownian motion process in l_2. We want to show that $\phi(t) = t^2 \log\log t^{-1}$ is the right function such that

$$0 < \phi - m(E(\omega)) < \infty, \quad \text{with probability 1.}$$

Even if this problem has a long history, as far as we know, there is no analogous result for a Brownian motion in l_2.

Now we review a Brownian motion in l_2. Let β_t be a standard l_2-valued Brownian motion defined over $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\beta_0 = 0$. We have

$$\mathbf{E}(\beta_t, h) = 0 \quad \text{and} \quad \mathbf{E}(\beta_t, g)(\beta_s, h) = (t \wedge s)(Tg, h)$$

for all $g, h \in l_2$, where $T : l_2 \to l_2$ is a nuclear (trace class) covariance operator. The existence of such a Brownian motion is well known ([5, 6]). Let $\{e_i\}_{i=1}^{\infty}$ be the usual orthonormal set in l_2 and suppose that T has the orthonormal eigensystem $\{e_i, \xi_i\}$ so that

$$T(e_i) = \xi_i e_i, \; i = 1, 2, \cdots,$$

where $\xi_i > 0$, $\xi_1 \ge \xi_2 \ge \xi_3 \ge \cdots$, and $\sum_{i=1}^{\infty} \xi_i < \infty$. Then the following representation holds almost surely:

$$\beta_t = \sum_{i=1}^{\infty} \sqrt{\xi_i} B_t^i e_i,$$

where $B_t^i, i = 1, 2, \cdots$ are independent, identically distributed, standard Brownian motions in one dimension.

2. Upper bound

If we want to prove that $\phi - m(E(\omega)) < \infty$ with probability 1 it is sufficient to find $K < \infty$ such that for each $n > 0$ there is a covering $\mathcal{S}$ of $E(\omega)$ for which $\sup_{S_i \in \mathcal{S}}(\text{diam} S_i) < 2^{-n}$ and $\sum_{S_i \in \mathcal{S}} \phi(\text{diam} S_i) \leq K$ with probability 1. Let's consider two random times determined by ω :

$$P(a) = P(a, \omega) = \inf\{t; \|\beta(t, \omega)\| \geq a\},$$
$$T(a) = T(a, \omega) = \int_0^\infty \chi_{B(0,a)}(\beta(t, \omega))dt, \tag{2.1}$$

where $\chi_{B(0,a)}$ is the indicator function of the closed sphere of radius a. Note that $P(a)$ is the first passage time process and $T(a)$ is the sojourn time process. We shall apply the following theorem in the proof of Lemmas 2.1 and 2.2.

THEOREM 3.1 [3]. *Let $\beta(t)$ be the Brownian motion in l_2 with covariance operator T. Let $\gamma_0 = \|T\|$ and $\gamma_1 = tr(T)$. Then, for every $r > 0$,*

$$\mathbf{P}\Big\{ \sup_{0 \leq s \leq t} \|\beta(s)\| \geq r \Big\} \leq \exp\Big\{ - \frac{r^2 - 2\gamma_1 t)}{4\gamma_0 t} \Big\} \quad \textit{for every } t.$$

LEMMA 2.1. *There exists a positive constant C_1 such that for $\lambda \geq \lambda_0 > 0$*

$$\mathbf{P}\{P(a) \geq \lambda a^2\} = \mathbf{P}\{P(1) \geq \lambda\} \geq \exp(-C_1 \lambda).$$

PROOF. Let $\delta^2 > 2\gamma_1$ and consider the following;

$$\begin{aligned}
\mathbf{P}\{\|\beta(0\| < \delta, \|\beta(1)\| < \delta\} &\geq \mathbf{P}\Big\{ \sup_{0 \leq s \leq 1} \|\beta(s)\| < \delta \Big\} \\
&= 1 - \mathbf{P}\Big\{ \sup_{0 \leq s \leq 1} \|\beta(s)\| > \delta \Big\} \\
&\geq 1 - \exp\Big(- \frac{\delta^2 - 2\gamma_1}{4\gamma_0} \Big),
\end{aligned}$$

by Theorem 3.1 [3]. Let $C_0 = 1 - \exp(-\frac{\delta^2 - 2\gamma_1}{4\gamma_0})$. Also we can choose r_0 (a fixed sufficiently large number, $r_0 > \delta + (4\gamma_0 \ln(\frac{C_0}{2})^{-1} + 2\gamma_1)^{\frac{1}{2}}$) satisfying

$$\mathbf{P}\{ \sup_{0 < s < 1} \|\beta(s)\| \geq r_0 - \delta\} < \frac{1}{2} C_0.$$

Now, we have

$$\begin{aligned}C_0 &\le \mathbf{P}\{\|\beta(0)\| < \delta,\ \|\beta(1)\| < \delta\}\\ &= \mathbf{P}\{\|\beta(0)\| < \delta,\ \|\beta(1)\| < \delta,\ \sup_{0\le s\le 1}\|\beta(s)\| < r_0\}\\ &\quad + \mathbf{P}\{\|\beta(0)\| < \delta,\ \|\beta(1)\| < \delta,\ \sup_{0\le s\le 1}\|\beta(s)\| \ge r_0\}\\ &\le \mathbf{P}\{\|\beta(0)\| < \delta,\ \|\beta(1)\| < \delta,\ \sup_{0\le s\le 1}\|\beta(s)\| < r_0\}\\ &\quad + \mathbf{P}\{\sup_{0\le s\le 1}\|\beta(s)\| \ge r_0 - \delta\}\\ &\le \mathbf{P}\{\|\beta(0)\| < \delta,\ \|\beta(1)\| < \delta,\ \sup_{0\le s\le 1}\|\beta(s)\| < r_0\} + \frac{1}{2}C_0.\end{aligned}$$

Hence

$$\mathbf{P}\{\|\beta(0)\| < \delta,\ \|\beta(1)\| < \delta,\ \sup_{0\le s\le 1}\|\beta(s)\| < r_0\} > \frac{1}{2}C_0. \tag{2.2}$$

We can now use the Markov property, restarting the process at $t = 1, 2, \dots k-1$, and estimating the probability by assuming that $\|\beta(i-1)\| < \delta$, $\|\beta(i)\| < \delta$ and $\beta(t)$ remains in the sphere of radius r_0 for $i-1 \le t \le i$, $i = 1, 2 \dots k$. By (2.2),

$$\mathbf{P}\{\sup_{0\le s\le k}\|\beta(s)\| < r_0\} > \Big(\frac{1}{2}C_0\Big)^k (= \exp\{-k\log(2C_0^{-1})\}).$$

Thus

$$\begin{aligned}\mathbf{P}\{P(1)\ge\lambda\} &= \mathbf{P}\{\sup_{0\le s\le \lambda}\|\beta(s)\| \le 1\}\\ &= \mathbf{P}\{\sup_{0\le s\le r_0^2\lambda}\|\beta(s)\| \le r_0\}\\ &> \exp\{-\lambda r_0^2\log(2C_0^{-1})\}.\end{aligned}$$

Letting $C_1 = r_0^2\log(2C_0^{-1})$, we get the result. □

To overcome independence difficulties we consider a sparse sequence $t_k = e^{-k^2}$, $k = 1, 2 \dots$. Let

$$\begin{aligned}\psi(t) &= t^{\frac{1}{2}}(\log\log t^{-1})^{-\frac{1}{2}},\\ M(t) &= \sup_{0\le s\le t}\|\beta(s)\|,\\ \bar{M}(t_k) &= \sup_{t_{k+1}\le t\le t_k}\|\beta(t)\|,\end{aligned}$$

and

$$M'(t_k) = \sup_{t_{k+1} \le t \le t_k} \|\beta(t) - \beta(t_{k+1})\|.$$

(Note that $\phi(\psi(s)) \sim s$ and $\psi(\phi(s)) \sim s$ as $s \to 0_+$.) The following Lemma 2.2 and Lemma 2.3 are adapted from the part of proof for Theorem 4 in [9].

LEMMA 2.2. *Let* $C_2 = (3C_1)^{\frac{1}{2}}$. *Then*

$$\mathbf{P}\Big\{ \inf_{\tau \le t \le t_m} \frac{M(t)}{\psi(t)} > 2C_2 \Big\} < \exp\{-m^{\frac{1}{4}}\}, \tag{2.3}$$

provided $0 \le \tau \le t_{2m}$ *for sufficiently large* m.

PROOF. Note that $\bar{M}(t_k) \le M'(t_k) + \|\beta(t_{k+1})\|$. If we let

$$D_k = \Big\{ \frac{\bar{M}(t_k)}{\psi(t_k)} > 2C_2 \Big\},\ G_k = \Big\{ \frac{M'(t_k)}{\psi(t_k)} > C_2 \Big\},\ H_k = \Big\{ \frac{\|\beta(t_{k+1}\|}{\psi(t_k)} > C_2 \Big\}$$

we have $D_k \subset G_k \cup H_k$ and

$$\cap_{k=m}^{2m} D_k \subset (\cap_{k=m}^{2m} G_k) \cup (\cup_{k=m}^{2m} H_k), \tag{2.4}$$

where the events G_k are now independent while the H_k have very small probability when k is large. Put $\mathbf{P}(G_k) = 1 - p_k$, $\mathbf{P}(H_k) = q_k$. Then by Lemma 2.1

$$\begin{aligned}
p_k &\ge \mathbf{P}\Big\{ \frac{M(t_k)}{\psi(t_k)} \le C_2 \Big\} \\
&= \mathbf{P}\{P(C_2\psi(t_k)) \ge t_k\} \\
&= \mathbf{P}\{P(1) \ge C_2^{-2}(\log\log t_k^{-1})\} \\
&\ge \exp\Big\{ -\frac{1}{3}\log\log(\exp k^2) \Big\} \\
&= k^{-\frac{2}{3}}, \\
q_k &= \mathbf{P}\{\sqrt{t_{k+1}}\|\beta(1)\| > C_2\sqrt{t_k} \cdot (\log\log(t_k)^{-1})^{-\frac{1}{2}}\} \\
&\le \mathbf{P}\Big\{ \|\beta(1)\| > \frac{C_2 e^k}{\sqrt{2\log k}} \Big\} \quad \Big(\text{since } \frac{t_k}{t_{k+1}} > e^{2k}\Big) \\
&\le \mathbf{P}\Big\{ \|\beta(1)\| > \frac{C_2 e^k}{k} \Big\} \\
&\le \exp\Big\{ -\frac{(C_2^2(e^k/k)^2 - 2\gamma_1)}{4\gamma_0} \Big\} \quad \Big(\text{by Theorem 3.1 in [3]}\Big) \\
&\le \exp\Big\{ -\frac{e^{2k}}{4\gamma_0 k^2} \Big\} \exp\Big\{ \frac{\gamma_1}{2\gamma_0} \Big\}.
\end{aligned}$$

Therefore $\sum_m^{2m} q_k \leq e^{-m}$ for large m. Since $\{\bar{M}(t_{m+i}) \geq 2C_2\psi(t_m)\} \subset D_{m+i}$ for every $i = 0, 1, 2....$, using this estimates with (2.4),

$$\begin{aligned}\mathbf{P}\{(\cap_{i=0}^m D_{m+i}\} &\leq \Pi_{k=m}^{2m}(1-p_k) + \sum_{k=m}^{\infty} q_k \\ &\leq \exp\Big\{-\sum_{k=m}^{2m} p_k\Big\} + e^{-m} \\ &< \exp\{-m^{\frac{1}{4}}\}.\end{aligned}$$

Hence we get

$$\mathbf{P}\Big\{\inf_{\tau \leq t \leq t_m} \frac{M(t)}{\psi(t)} \geq 2C_2\psi(t_m)\Big\} < \exp\{-m^{\frac{1}{4}}\}$$

in case that $0 \leq \tau \leq t_{2m}$. □

LEMMA 2.3. *There are positive constants C_3, C_4 such that*

$$\mathbf{P}\Big\{\sup_{2^{-6k} \leq a \leq 2^{-k}} \frac{P(a)}{\phi(a)} < C_3\Big\} < \exp\{-C_4 k^{\frac{1}{8}}\}. \tag{2.5}$$

PROOF. If we put $a_m = \psi(t_m)$, then $(a_m)^4 < a_{2m}$. Note that

$$\begin{aligned}\mathbf{P}\Big\{\frac{M(t_m)}{\psi(t_m)} > 2C_2\Big\} &= \mathbf{P}\{P(2C_2\psi(t_m)) < t_m\} \\ &\geq \mathbf{P}\Big\{\frac{P(a_m)}{\phi(a_m)} < \frac{1}{8}C_2^{-2}\Big\}.\end{aligned}$$

By (2.3),

$$\mathbf{P}\Big\{\sup_{\lambda \leq a \leq a_m} \frac{P(a)}{\phi(a)} < \frac{1}{8}C_2^{-2}\Big\} < \exp\{-m^{\frac{1}{4}}\},$$

provided $0 < \lambda \leq a_m^4$ and m is large. If λ is small enough and $m = [(\frac{2}{5}\log\lambda^{-1})^{\frac{1}{2}}]$ (the integer part of $(\cdot)^{\frac{1}{2}}$), then we have $\lambda \leq a_m^4 < a_m < \lambda^{\frac{1}{6}}$ since

$$\begin{aligned}a_m &= [\log m^2 \exp(m^2)]^{-\frac{1}{2}} \\ &\sim (\log[(-\frac{2}{5}\log\lambda)^{\frac{1}{2}}])^{-\frac{1}{2}} \cdot \lambda^{\frac{1}{5}} \\ &< \lambda^{\frac{1}{6}}.\end{aligned}$$

If we take $\lambda = 2^{-6k}$, then $m^{\frac{1}{4}} \sim [(\frac{2}{5}\log\lambda^{-1})^{\frac{1}{2}}]^{\frac{1}{4}} \sim (\frac{12}{5}\log 2)^{\frac{1}{8}} \cdot k^{\frac{1}{8}}$. Let $C_3 = \frac{1}{8}C_2^{-2}$ and $C_4 = (\frac{12}{5}\log 2)^{\frac{1}{8}}$, then we get (2.5). □

In the following lemma, we can show that for each n there exists a covering of $E(\omega)$ in l_2, denote Λ_n, which consists of spheres with radius 2^{-n} centered at $\beta(t,\omega)$ for some $t \in [0,1]$, and $\mathbf{E}[N_n(\omega)] \le C_5 2^{2n}$ where $N_n(\omega)$ is the number of these spheres and C_5 is some constant.

LEMMA 2.4. *There exists $\Lambda_n(\omega)$, a collection of spheres of radius 2^{-n}, centered at $\beta(t,\omega)$ for some $t \in [0,1]$, which coveres $E(\omega)$. Let $N_n(\omega)$ be the number of these spheres. Then there is a constant C_5 which is independent of n such that $\mathbf{E}[N_n(\omega)] \le C_5 2^{2n}$.*

PROOF. Let $\sigma_0 = 0$ and for $k \ge 1$, let

$$\begin{aligned}\tau_k &= \inf\{s \ge \sigma_{k-1} : \|\beta(s) - \beta(\sigma_{k-1})\| > 2^{-n}\}\\ \sigma_k &= \min\{\tau_k, \sigma_{k-1} + 2^{-n}\}.\end{aligned}$$

Then $Y_k = Y_k(2^{-n}) = \sigma_k - \sigma_{k-1}$ is a sequence of independent identically distributed random variables. If $\eta = \min\{k : \sigma_k \ge 1\}$ then let

$$\Lambda_n(\omega) = \{B(\beta(\sigma_k), 2^{-n})\}_{k=1}^{\eta},$$

where $B(\beta(\sigma_k), 2^{-n})$ is the sphere of radius 2^{-n} centered at $\beta(\sigma_k)$,

$$\begin{aligned}\mathbf{E}[\sigma_\eta] &= \mathbf{E}\Big[\sum_{k=1}^{\eta}(\sigma_k - \sigma_{k-1})\Big]\\ &= \mathbf{E}[\eta]\mathbf{E}[\sigma_1 - \sigma_0]\\ &= \mathbf{E}[\eta]\mathbf{E}[Y_1],\end{aligned}$$

and

$$\mathbf{E}[\sigma_\eta] \le \mathbf{E}[1 + 2^{-n}] \le 2.$$

Now $Y_1(2^{-n})$ and $2^{-2n}Y_1(1)$ have the same distribution by the scaling property so that $\mathbf{E}[Y_1(2^{-n})] = 2^{-2n}\mathbf{E}[Y_1(1)]$. Thus

$$\mathbf{E}[N_n(\omega)] = \mathbf{E}[\eta(\omega)] = \mathbf{E}[Y_1(1)]^{-1} \cdot 2^{2n}\mathbf{E}[\sigma_\eta] \le 2[\mathbf{E}(Y_1(1)]^{-1} \cdot 2^{2n}.$$

Let $C_5 = \frac{2}{\mathbf{E}[Y_1(1)]}$. Then we get the result. □

THEOREM 2.5.

$$\phi - m(E_\omega) < K_0 \quad \textit{with probability 1,}$$

where K_0 is a constant.

PROOF. Let Λ_{6h} be a family of sphere $B(\beta(\sigma_k), 2^{-6h})$ for some positive integer h, constructed by Lemma 2.5. In some sense we want to give the lattice points on $E(\omega)$, by $\beta(\sigma_k)$. As the notation (2.1), we denote (2.6)

$$\mu(B(\beta(\sigma_k), a)) = \int_0^\infty \chi_{B(\beta(\sigma_k),a)}(\beta(t,\omega))dt,$$
$$P(B(\beta(\sigma_k), a)) = \inf\{t : \beta(t,\omega) \notin B(\beta(\sigma_k), a) \quad \text{after hitting } \beta(\sigma_k)\}.$$

We call $\beta(\sigma_k)$ bad if

$$\sup_{2^{-6h} \le a \le 2^{-h}} \frac{\mu(B(\beta(\sigma_k), a))}{\phi(a)} \le C_3,$$

where $\phi(a) = a^2 \log\log a^{-1}$. Otherwise it is good. By Lemma 2.3,

$$\begin{aligned}\mathbf{P}\Big\{\beta(\sigma_k) \text{ is bad } \Big\} &\sim \mathbf{P}\Big\{ \sup_{2^{-6h} \le a \le 2^{-h}} \frac{T(B(\beta(\sigma_k), a))}{\phi(a)} \le C_3\Big\} \\ &\le \mathbf{P}\Big\{ \sup_{2^{-6h} \le a \le 2^{-h}} \frac{P(B(\beta(\sigma_k), a))}{\phi(a)} \le C_3\Big\} \\ &\le \exp(-C_4 h^{\frac{1}{8}}).\end{aligned}$$

We cover the bad point $\beta(\sigma_k)$ on $E(\omega)$ by (so called) 'bad' sphere, $B(\beta(\sigma_k), 2^{-6h})$ in Λ_{6h}. Then the expectation of the number of the bad spheres, say M_{6h},

$$\mathbf{E}[M_{6h}] < \mathbf{E}[N_{6h}] \cdot \exp(-C_4 h^{\frac{1}{8}}) \le C_5 2^{12h} \cdot \exp(-C_4 h^{\frac{1}{8}}).$$

Now

$$\begin{aligned}&\mathbf{P}\Big\{M_{6h} > (C_5 2^{12h} \cdot \frac{\exp(-C_4 h^{\frac{1}{8}})}{\phi(2^{-6h})})^{\frac{1}{2}}\Big\} \\ &\le \frac{C_5 2^{12h} \cdot \exp(-C_4 h^{\frac{1}{8}})}{[C_5 2^{12h} \cdot \exp(-C_4 h^{\frac{1}{8}})/\phi(2^{-6h})]^{\frac{1}{2}}} \\ &\sim (C_5 2^{12h} \cdot \exp(-C_4 h^{\frac{1}{8}}) 2^{-12h} \cdot \log 6h)^{\frac{1}{2}} \\ &\sim \exp(-C_4 h^{\frac{1}{8}})^{\frac{1}{2}}.\end{aligned}$$

An application of the Borell-Cantelli Lemma now shows that, with probability 1, there exists an integer $h_1 = h_1(\omega)$ such that for $h \geq h_1$

$$M_{6h}(\omega) \leq \left(C_5 2^{12h} \cdot \frac{\exp(-C_4 h^{\frac{1}{8}})}{\phi(2^{-6h})}\right)^{\frac{1}{2}}.$$

Using this, we can show that the contribution of the bad spheres to the sum is negligible, that is, let

$$\mathcal{S}_1 = \{B(\beta(\sigma_k), 2^{-6h}), \beta(\sigma_k) \quad \text{is a bad point}\}.$$

Then

$$\sum_{S_i \in \mathcal{S}_1} \phi(\text{diam} S_i) \leq M_{6h}\phi(\text{diam} S_i)$$
$$\leq (C_5 2^{12h} \exp(-C_4 h^{\frac{1}{8}})\phi(2^{-6h}))^{\frac{1}{2}}.$$

Thus this converges to 0 a.s. as $h \to \infty$. Hence it becomes enough to consider a covering of good points of $E(\omega)$.

If $\beta(\sigma_k)$ is good then there exists $a_k \in [2^{-6h}, 2^{-h}]$ such that

$$\frac{\mu(B(\beta(\sigma_k)), a_k)}{\phi(a_k)} > C_3.$$

Assuming $\lambda_0 \equiv \beta(\sigma_0) = 0$ is good, then there exists minimal $a_0 \in [2^{-6h}, 2^{-h}]$ such that $\mu(B(\lambda_0, a_0)) \geq C_3\phi(a_0)$. Denote $S_0 = B(\lambda_0, a_0)$ and cover $\beta(0)$ by S_0. Let

$$\lambda_1 \equiv \inf\{\sigma_k > \lambda_0 : \beta(\sigma_k) \in S_0^c, \beta(\sigma_k) \,\text{is good}\}.$$

Then there exists minimal $a_1 \in [2^{-6h}, 2^{-h}]$ such that $\mu(B(\beta(\lambda_1), a_1)) \geq C_3\phi(a_1)$. Cover $\beta(\lambda_1)$ by $S_1 \equiv B(\beta(\lambda_1), a_1)$. By induction, let

$$\lambda_n \equiv \inf\{\sigma_k > \lambda_{n-1} : \beta(\sigma_k) \in (\cup_{j \leq n-1} S_j)^c, \beta(\sigma_k) \quad \text{is good}\}$$

and choose $a_n \in [2^{-6h}, 2^{-h}]$ such that $\mu(B(\beta(\lambda_n), a_n)) \geq C_3\phi(a_n)$, and cover $\beta(\lambda_n)$ by $S_n = B(\beta(\lambda_n), a_n)$. Note that we have covered the bad points by $\mathcal{S}_1 = \{(B(\beta(\sigma_k), 2^{-6h})\}$. Let $\mathcal{S}_2 = \cup S_n$. Then $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ is a covering for $E(\omega)(\equiv E_\omega)$ with $\text{diam} S_i \leq 2 \cdot 2^{-h}$ for every $S_i \in \mathcal{S}$. If any S_i is a good sphere satisfying $S_i \cap E_\omega \subset S_j \cap E_\omega$, $i \neq j$, then throw S_i away. Also if there exists j and k satisfying $S_i \cap E_\omega \subset (S_{i-j} \cap E_\omega) \cup (S_{i+k} \cap E_\omega)$, throw S_i away. Let $K(h)$ be the maximal number of $S_i \in \mathcal{S}_2$ which a point of $E(\omega)$ belongs to, then

$$K(h) = \sup_{0 \le s \le 1} \sharp\{\lambda_k : \beta(s) \in S_k,\ S_k \in \mathcal{S}_2\}.$$

Thus we can find a subcovering of $\mathcal{S}_2$ such that no point of $E(\omega)$ is in more than $K(h) < \infty$ spheres of S_i. Now $K(h) < N(6h)$ a.s. and $\mathbf{E}(N_{6h}) < C_5 2^{12h}$. Hence (for fixed h) $K(h)$ is bounded a.s. If $\beta(s) \in B(\beta(\sigma_{k_0}), a_{k_0})$ for some k_0, then the number of spheres which $B(\beta(\sigma_{k_0}), a_{k_0})$ meets is decreasing as $h \to \infty$, and hence $K(h)$ is non-increasing as $h \to \infty$, i.e. $\beta(s)$, $s \in [0,1]$ belongs to less spheres as radius becomes smaller.

Up to now we covered good points by good spheres and then obtained an economical covering, say $\mathcal{S}_2'$ such that

$$\begin{aligned}\sum_{S_i \in \mathcal{S}_2'} \phi(\text{diam} S_i) &\le \sum_{S_i \in \mathcal{S}_2'} \phi(2a_i) \le \sum_{S_i \in \mathcal{S}_2'} 4\phi(a_i) \\ &\le \frac{4}{C_3} \sum_{S_i \in \mathcal{S}_2'} \mu(S_i) \\ &\le \frac{4}{C_3} \cdot K(h) \quad \text{a.s.}\end{aligned}$$

Let $K = \lim_{h \to \infty} \frac{4}{C_3} K(h)$. Then

$$\sum_{S_i \in \mathcal{S}_2'} \phi(\text{diam} S_i) < K \quad \text{a.s.}$$

Hence, almost surely

$$\liminf_{h \to \infty} \left\{ \sum \phi(\text{diam} S_i) : E_\omega \subset \cup S_i, \text{diam} S_i \le 2^{-h}, \cup S_i \in \mathcal{S}_0 \right\} < K. \ \square$$

3. Lower bound

Let $\mathcal{M}$ be the set of σ-finite measures on l_2.

LEMMA 3.1. *Let $\{\beta(s), 0 \le s < \infty\}$ be Brownian motion in l_2 starting from the origin and for any Borel set, $B \subset l_2$,*

$$\sigma(B,\omega) = \int_0^\infty \chi_B(\beta(s,\omega))ds.$$

Let Φ be a continuous function in the vague topology on $\mathcal{M}$. Then for almost all Brownian paths ω,

$$\limsup_{a \to 0+} \frac{\Phi\left(\sigma(\cdot,\omega)\right)}{a^2 \log\log a^{-1}} = C \tag{3.1}$$

for some constant C.

PROOF. This is a corollary of Theorem 3.1 and 3.2 in [1]. □

REMARK 3.2. Let $T(a,\omega)$ be defined as (2.1). For $\sigma \in \mathcal{M}$, let $\Phi(\sigma) = \sigma(B(0,1))$ where $B(0,1)$ is the unit sphere in l_2 with center at 0. Applying the above lemma we get for some constant $C < \infty$

$$\limsup_{a \to 0+} \frac{T(a,\omega)}{a^2 \log\log a^{-1}} = C \quad \text{a.s.}$$

COROLLARY 3.3. *For fixed t_0,*

$$\limsup_{a \to 0+} \frac{\sigma(B(\beta(t_0),a),\omega)}{\phi(a)} = C. \quad \text{a.s.}$$

PROOF. Let $\tilde{\beta}(s) = \beta(t_0 + s) - \beta(t_0), 0 \le s < \infty$. Then it defines a version of the Brownian motion for which $\tilde{\beta}(0) = 0$. □

The following is a generalization of density theorem proved by Rogers and Taylor [8].

LEMMA 3.4. *Suppose F is any finite completely additive measure defined for all Borel subsets of l_2. Let $\phi(t)$ be a continuous monotone increasing function of t with $\lim_{t\to 0^+}\phi(t)=0$ and for any $k>0$ define $D_\phi F(x)$ and E_k as the following:*

$$D_\phi F(x)=\limsup_{h\to 0^+}\frac{F(B(x,2^{-h}))}{\phi(2\cdot 2^{-h})},\quad E_k=\{x: D_\phi F(x)>k\}.$$

If E is a set of l_2 such that $E\cap E_k=\emptyset$ then

$$F(E)\le k(\phi-m(E)).$$

PROOF. This is a routine extension of Theorem [B] in [2] applying the above corollary. □

THEOREM 3.5.

$$\phi-m(E_\omega)>0\quad \textit{with probability 1.}$$

PROOF. Since $\beta(t,\omega)$ is continuous, we may define a set function $F_\omega(A)$ for every Borel set A in l_2 by

$$F_\omega(A)=m\{t\in[0,1]:\beta(t,\omega)\in A\},$$

where m is the Lebesque measure in R^1. Let us consider

$$D_\phi F_\omega(x)=\limsup_{a\to 0^+}\frac{F_\omega(B(x,a))}{\phi(2a)}.$$

If $x\neq\beta(t,\omega)$ for any t, $0\le t\le 1$, then since the path is a closed set we have $D_\phi F_\omega(x)=0$.

If $x=\beta(t_0,\omega)$ for some t_0, $0\le t_0\le 1$, then with probability 1

$$\begin{aligned}D_\phi F_\omega(x)&\le\limsup_{a\to 0^+}\frac{\sigma(B(x,a),\omega)}{\phi(a)}\\&=C\end{aligned}$$

for some constant C by Corollary 3.3. By setting up a product measure in $[0,1]\times\Omega$ and applying Fubini theorem, it follows from that with probability 1

$$D_\phi F_\omega(x)\le C,\quad \text{for almost all } t \text{ in } [0,1], \tag{3.2}$$

where $x=\beta(t,\omega)$. Let $E_c=\{x: D_\phi F_\omega(x)>C\}$. Then $E_\omega\cap E_c=\emptyset$ from (3.2). Therefore by Lemma 3.4 $F_\omega(E(\omega))\le C\cdot\phi-m(E(\omega))$. Therefore we showed that with probability 1

$$\phi-m(E(\omega))\ge C^{-1}.$$

□

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