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CONSTRUCTIVE AND DISCRETE VERSIONS OF THE LYAPUNOV'S STABILITY THEOREM AND THE LASALLE'S INVARIANCE THEOREM

JAEWOOK LEE

ABSTRACT. The purpose of this paper is to establish discrete versions of the well-known Lyapunov's stability theorem and LaSalle's invariance theorem for a non-autonomous discrete dynamical system. Our proofs for these theorems are constructive in the sense that they are made by explicitly building a Lyapunov function for the system. A comparison between non-autonomous discrete dynamical systems and continuous dynamical systems is conducted.

1. Introduction

Since the Lyapunov's stability theory and the LaSalle's invariance theorem have been developed, many authors have applied these powerful tools in the study of stability for non-autonomous continuous dynamical systems arising in many disciplines such as engineering and the applied sciences ([2], [6], [7], [8]). Recently, the steady improvements in the performance of digital computers has inspired the need for the stability analysis of discrete systems and a significant effort has been spent in studying theoretical aspects of discrete dynamical systems which include various numerical methods to simulate continuous systems by discretization. However, so far there seems to be no discrete versions of these two theorems, and the main aim of this paper is to extend it from non-autonomous continuous dynamical systems to non-autonomous discrete dynamical systems.

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We first give a necessary and sufficient condition for an existence of an exponentially stable fixed point of non-autonomous discrete dynamical systems. We present a "constructive" proof for this theorem in the sense that it is made by explicitly constructing a Lyapunov function for the system. Also we establish a discrete version of the well-known LaSalle's invariance theorem for locating limit sets of non-autonomous dynamical systems and conduct a comparison between non-autonomous discrete dynamical systems and continuous dynamical systems.

2. Preliminaries

In this section, we introduce some fundamental concepts in the theory of non-autonomous discrete dynamical systems which is needed in the subsequent developments ([1], [3]).

Consider a non-autonomous discrete dynamical system of the form

$$x_{k+1} = f_k(x_k), \tag{2.1}$$

where $k \in \mathbb{Z}, x_k \in \Re^n$, and the function $f_k : \Re^n \to \Re^n$ is assumed to be C^1 and invertible for each $k \in \mathbb{Z}$. We call the system (2.1) *autonomous* if $f_k(\cdot) = f(\cdot)$ for all k for some $f : \Re^n \to \Re^n$. Here the condition that f is invertible is a sufficient condition for the existence of solution which can be defined on Z. Most important of all, widely used major types of practical numerical methods for solving initial value problems of ODEs such as Runge-Kutta methods, Richardson extrapolation, and its particular implementation as the Bulirsch-Stoer method, predictor-corrector methods all satisfy this condition.

EXAMPLE 1. The fourth-order Runge-Kutta formula for ODE

$$\frac{dx}{dt} = g(x)$$

has the form of

$$x_{n+1} = x_n + k_1/6 + k_2/3 + k_3/3 + k_4/6,$$

where

$$\begin{aligned}
k_1 &= hg'(y_n, x_n),\\
k_2 &= hg'(y_n + h/2, x_n + k_1/2),\\
k_3 &= hg'(y_n + h/2, x_n + k_2/2),\\
k_4 &= hg'(y_n + h, x_n + k_3),\\
y_{n+1} &= y_n + h.
\end{aligned}$$

Usually, g' is bounded and h is sufficiently small. Let $f_n(x_n) = x_{n+1} = x_n + k_1/6 + k_2/3 + k_3/3 + k_4/6$. Then f_n is invertible for appropriate condition for g', h since $Df_n(x) = I + hDg'_n(x)$.

The solution of (2.1) starting from $x \in \Re^n$ at $k = k_0$ is called a *trajectory*, denoted by $\phi(\cdot, k_0, x) : \mathbb{Z} \to \Re^n$ (or $x_k = \phi(k, k_0, x)$). In fact, $\phi(k, k_0, x) = f_{k-1}(f_{k-2}(\cdots(f_{k_0}(x))\cdots))$.

A state vector x^* is called a *fixed point* of system (2.1) if $x^* = f_k(x^*)$ (or $\phi(k, k_0, x) = x$) for all $k, k_0 \in \mathbb{Z}$. A fixed point x^* is called *uniformly stable*, if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, independent of k_0, such that

$$\| x - x^* \| < \delta \Rightarrow \| \phi(k, k_0, x) - x^* \| < \epsilon \quad \forall k \geq k_0 \geq 0$$

and is called *asymptotically stable*, if it is uniformly stable and δ can be chosen, independent of k_0, such that

$$\| x - x^* \| < \delta \Rightarrow \lim_{k \to \infty} \phi(k, k_0, x) = x^*$$

and is called *exponentially stable*, if there exists a positive constant r, M and $0 < s < 1$ such that

$$\| \phi(k, k_0, x) - x^* \| \leq M \| x - x^* \| s^{k-k_0} \quad \forall \| x \| < r, \ \forall k \geq k_0 \geq 0.$$

A fixed point x^* is called *unstable*, if not stable.

A useful tool for the analysis of nonlinear systems is the Lyapunov function theory. If there exists a smooth function (or C^r) $V : \mathbb{Z} \times \Re^n \to \Re$ for (2.1) such that

$$V(k+1, \phi(k+1, k, x)) < V(k, x)$$

for all x which is not a fixed point. Any smooth function (or C^r) satisfying above inequality will be called a *Lyapunov function* for the nonlinear discrete dynamical system (2.1).

3. Necessary and sufficient conditions for the exponentially stable fixed point of discrete dynamical systems

In this section, we give a necessary and sufficient condition for an exponentially stable fixed point of non-autonomous discrete dynamical systems, which is a discrete version of Lyapunov's second stability theory.

THEOREM 3.1 (Sufficent Condition). *Let $x = 0$ be a fixed point for (2.1) and $D = \{x \in \Re^n : \| x \| < r\}$. Let $V(k, \cdot) : D \to \Re$ be a continuously differentiable functions for each $k \in \mathbb{Z}$ such that*

$$c_1 \| x \|^2 \leq V(k, x) \leq c_2 \| x \|^2$$

and

$$V(k+1, \phi(k+1, k, x)) - V(k, x) \leq -c_3 \| x \|^2$$

for all $k \in \mathbb{Z}$ and all $x \in D$. Then $x = 0$ is exponentially stable.

PROOF. Let $\rho < r$, and $\Omega_{k,\rho} = \{x_k \in D : V(k, x) \leq c_1 \rho^2\}$.
(I) First, we will prove that $\{x : \| x \|^2 \leq c_1/c_2 \rho^2\} \subset \Omega_{k,\rho}$. Suppose that

$$\| x \|^2 \leq c_1/c_2 \rho^2 \text{ or } c_2 \| x \|^2 \leq c_1 \rho^2.$$

Then

$$V(k, x) \leq c_1 \rho^2.$$

Thus, the set $\Omega_{k,\rho}$ contains the ball $\{x : \| x \|^2 \leq c_1/c_2 \rho^2\}$.
(II) Next, we will prove that $\Omega_{k,\rho} \subset \{x : \| x \| \leq \rho\}$. Suppose that

$$V(k, x) \leq c_1 \rho^2,$$

then

$$c_1 \| x \|^2 \leq c_1 \rho^2 \text{ or } \| x \| \leq \rho.$$

Hence, $\Omega_{k,\rho}$ is a subset of the ball $\{x : \| x \| \leq \rho\}$ since

$$V(k, x) \leq c_1 \rho^2 \Rightarrow c_1 \| x \|^2 \leq c_1 \rho^2 \Rightarrow \| x \| \leq \rho.$$

Therefore, by (I) and (II), we should have

$$\{x : \| x \|^2 \leq c_1/c_2 \rho^2\} \subset \Omega_{k,\rho} \subset \{x : \| x \| \leq \rho\} \subset D \quad \forall k \in \mathbb{Z}^+.$$

(III) Finally, we will prove that $x = 0$ is exponentially stable. To prove this, we first note that for any $x_{k_0} \in \Omega_{k_0,\rho}$, the solution starting at x_{k_0} stays in $\Omega_{k,\rho}$ for all $k \geq k_0$ since $V(k+1, \phi(k+1, k, x)) \leq V(k, x)$. Hence, the solution starting from x_{k_0} is defined for all $k \geq k_0$ and $x_k \in D$.
For the rest of the proof we will assume that $\| x \|^2 \leq c_1/c_2 \rho^2$. Then

$$V(k+1, \phi(k+1, k, x)) - V(k, x) \leq -c_3 \| x \|^2 \leq -c_3/c_2 V(k, x),$$

which implies

$$V(k+1, \phi(k+1, k, x)) \leq (1 - c_3/c_2) V(k, x).$$

Here, we may assume $c_2 > c_3$, i.e., $0 < s = 1 - c_3/c_2 < 1$ without loss of generality. Then

$$V(n, \phi(n, k, x)) \leq s^{n-k} V(k, x)$$

and any solution starting in $\Omega_{k_0,\rho}$, thus, satisfies the inequality

$$\begin{aligned}\| \phi(n, k, x) \|^2 &\leq c_1^{-1} V(n, \phi(n, k, x)) \leq c_1^{-1} s^{n-k} V(k, x) \\ &\leq c_1^{-1} s^{n-k} c_2 \| x \|^2 .\end{aligned}$$

Therefore, $x = 0$ is exponentially stable. □

THEOREM 3.2 (Necessary condition). *Let $x = 0$ be an exponentially stable fixed point for (2.1), i.e., there exists a positive constant r_0, M and $0 < s < 1$ such that*

$$\| \phi(k, k_0, x) \| \leq M \| x \| s^{k-k_0} \quad \forall \| x \| < r_0, \quad \forall k \geq k_0 \geq 0.$$

Let $D = \{x \in \Re^n : \| x \| < r\}$ where $r = r_0 M$. Let $g_k(x) = f_k(x) - x$ and the Jacobian matrix $[\partial g_k / \partial x]$ is bounded and $\| \partial g_k/\partial x \| \leq a < 1$ on D, uniformly in k. Let $D_0 = \{x \in \Re^n : \| x \| < r_0\}$. Then, there exists a continuously differentiable functions $V(k, \cdot) : D \to \Re$ for each $k \in \mathbb{Z}$ such that

$$c_1 \| x \|^2 \leq V(k, x) \leq c_2 \| x \|^2$$

and

$$V(k+1, \phi(k+1, k, x)) - V(k, x) \leq -c_3 \| x_k \|^2$$

for all $k \in \mathbb{Z}$ and all $x \in D$, where c_1, c_2, c_3 are positive constants. Moreover, if $r = \infty$ and the origin is globally exponentially stable, then $V(k, x)$ is defined and satisfies the above inequalities on $\Re^n$. Furthermore, if the system is autonomous, V can be chosen independent of k.

PROOF. (I) First, we will prove that

$$c_1 \| x \|^2 \leq V(k, x) \leq c_2 \| x \|^2 .$$

Let $x_n = \phi(n, k, x)$ denote the solution of the system that starts at $x_k = \phi(k, k, x) = x$. For all $x_k \in D_0, x_n \in D$ for all $n \geq k$. Let

$$V(k, x) = \sum_{n=k}^{k+N} \| \phi(n, k, x) \|^2 = \sum_{n=k}^{k+N} \| x_n \|^2,$$

where N is a positive constant to be chosen later. Then we have

$$V(k,x) = \sum_{n=k}^{k+N} \| x_n \|^2 \leq \sum_{n=k}^{k+N} M^2 s^{2(n-k)} \| x_k \|^2 = c_2 \| x_k \|^2,$$

where $c_2 = M^2(1-s^{2(N+1)})/(1-s^2)$.
On the other hand, since $\| g_n(x) \| \leq \| \partial g_n(z)/\partial x \| \| x \|$,

$$\begin{aligned} \| f_n(x) \| &= \| x - g_n(x) \| \geq \| x \| - \| g_n(x) \| \\ &\geq \| x \| - \| \partial g_n(z)/\partial x \| \| x \| \\ &\geq (1-a) \| x \|, \end{aligned}$$

where z is a point on the line segment connecting x to the origin. Hence

$$\| \phi(n,k,x) \| = \| f_{n-1}(f_{n-2}(\cdots(f_k(x))\cdots)) \| \geq (1-a)^{n-k} \| x \|$$

and

$$V(k,x) = \sum_{n=k}^{k+N} \| x_n \|^2 \geq \sum_{n=k}^{k+N} M^2(1-a)^{2(n-k)} \| x_k \|^2 = c_1 \| x_k \|^2,$$

where $c_1 = M^2(1-(1-a)^{2(N+1)})/(1-(1-a)^2)$. This proves the first inequality of the theorem.
(II) Next we will prove

$$V(k+1, \phi(k+1,k,x)) - V(k,x) \leq -c_3 \| x_k \|^2 .$$

Since $\phi(n,k+1,\phi(k+1,k,x)) = \phi(n,k,x)$, we have

$$\begin{aligned} & V(k+1,\phi(k+1,k,x)) - V(k,x) \\ &= \sum_{n=k+1}^{k+1+N} \| \phi(n,k+1,\phi(k+1,k,x)) \|^2 - \sum_{n=k}^{k+N} \| \phi(n,k,x) \|^2 \\ &= \sum_{n=k+1}^{k+1+N} \| \phi(n,k,x) \|^2 - \sum_{n=k}^{k+N} \| \phi(n,k,x) \|^2 \\ &= \sum_{n=k+1}^{k+1+N} \| x_n \|^2 - \sum_{n=k}^{k+N} \| x_n \|^2 = \| x_{k+1+N} \|^2 - \| x_k \|^2 \\ &= M^2 s^{2(N+1)} \| x_k \|^2 - \| x_k \|^2 = -(1 - M^2 s^{2(N+1)}) \| x_k \|^2 . \end{aligned}$$

Choosing $N > \ln(2M)/\ln(1/s)$,

$$V(k+1,\phi(k+1,k,x)) - V(k,x) \leq -(3/4) \| x \|^2 .$$

This proves the first inequality of the theorem.
(III) If all the assumptions hold globally, then clearly r_0 can be chosen

arbitrarily large. Therefore, $V(k,x)$ is defined and satisfies the above inequalities on $\Re^n$.
(IV) If the system is autonomous, then $\phi(n,k,x)$ depends only on $(n-k)$; that is, $\phi(n,k,x) = f^{n-k}(x)$. Then

$$V(k,x) = \sum_{n=k}^{k+N} \| f^{n-k}(x) \|^2 = \sum_{n=0}^{N} \| f^n(x) \|^2$$

which is independent of k. □

REMARK 3.3. Faced with searching for a Lyapunov function, two questions come to mind.

(i) Is there a function that satisfies the condition of Lyapunov function?

(ii) How can we search for such a function?

In the case of continuous dynamical systems, Lyapunov function theory ([5]) provides an affirmative answer to the first question. The answer takes the form of a converse Lyapunov theorem which is the inverse of one of the theorem. However, there are no general method for finding a Lyapunov function for continuous dynamical systems, which gives a negative answer to the second question (note that in general, the construction of a Lyapunov function in a continuous dynamical system assumes the knowledge of the explicit solutions of a family of differential equations. ([4], [5])). On the contrary, in the case of discrete dynamical systems, the above theorems provide affirmative answers to both these questions. Interesting enough, the above theorems is proved by explicitly constructing a Lyapunov function.

4. An invariance theorem for non-autonomous discrete dynamical systems

In this section, we establish a discrete version of the well-known LaSalle's invariance theorem. Unlike the case of continuous dynamical systems, the discrete version of this theorem will be proved without using Barbalat's lemma ([5]).

THEOREM 4.1 (Invariance Theorem). *Let $D = \{x \in \Re^n : \| x \| < r\}$. Let $V(k,\cdot) : D \to \Re$ be a continuously differentiable functions for each $k \in \mathbb{Z}$ such that*

$$c_1 \| x \|^2 \leq V(k,x) \leq c_2 \| x \|^2$$

and

$$V(k+1, \phi(k+1,k,x) - V(k,x) \leq -W(x) \leq 0$$

for all $k \in \mathbb{Z}$ and all $x \in D$, where W(x) is continuous on D. Then all solutions of (2.1) with $\| x_{k_0} \|^2 \leq c_1/c_2\rho^2$ are bounded and satisfy

$$W(\phi(k, k_0, x)) \to 0 \text{ as } k \to \infty.$$

Moreover, if all the assumptions hold globally, the statement is true for all $x_{k_0} \in \Re^n$.

PROOF. Let $\rho < r$, and $\Omega_{k,\rho} = \{x \in D : V(k,x) \leq c_1\rho^2\}$. Since

$$\| x_{k_0} \|^2 \leq c_1/c_2\rho^2 \text{ or } c_2 \| x_{k_0} \|^2 \leq c_1\rho^2,$$

we have

$$V(k, x_k) \leq V(k_0, x_{k_0}) \leq c_1\rho^2$$

which implies that

$$x_k = \phi(k, k_0, x) \in \Omega_{k,\rho} \quad \forall k \geq k_0.$$

Therefore, the set $\Omega_{k,\rho}$ contains the ball $\{x : \|x\|^2 \leq c_2/c_1\rho^2\}$ and $\|x_k\| \leq \rho$ for all $k \geq k_0$. Since $V(k, x_k)$ is decreasing and bounded from below by zero, it converges as $k \to \infty$. Now,

$$\begin{aligned} 0 \leq \sum_{n=k_0}^{k} W(x_n) &\leq -\sum_{n=k_0}^{k} (V(n+1, x_n) - V(n, x_n)) \\ &= V(k_0, x_{k_0}) - V(k+1, x_{k+1}). \end{aligned}$$

Therefore $\sum_{n=k_0}^{\infty} W(x_n)$ exists and is finite. Hence $W(\phi(k, k_0, x)) \to 0$ as $k \to \infty$. □

REMARK 4.2. The limit $W(x_k) \to 0$ as $k \to \infty$ implies that x_k approaches

$$E = \{x \in D : W(x) = 0\}$$

as $k \to \infty$.

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Department of Industrial Engineering
Pohang University of Science and Technology
Pohang, Kyungbuk 790-784, Korea
E-mail: jaewookl@postech.ac.kr