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OPERATORS FROM CERTAIN BANACH SPACES TO BANACH SPACES OF COTYPE $q \geq 2$

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ABSTRACT. Suppose $\{X_n\}_{n=1}^{\infty}$ is a sequence of finite dimensional Banach spaces and suppose that X is either a closed subspace of $(\sum_{n=1}^{\infty} X_n)_{c_0}$ or a closed subspace of $(\sum_{n=1}^{\infty} X_n)_p$ with $p > 2$. We show that every bounded linear operator from X to a Banach space Y of cotype q $(2 \leq q < p)$ is compact.

1. Introduction

It is well known that every bounded linear operator from ℓ_p to ℓ_r $(1 \leq r < p < \infty)$ is compact [2, p. 76]. The same conclusion is true in some other situations. In Corollary 3, we will prove that if $\{X_n\}_{n=1}^{\infty}$ is a sequence of finite dimensional Banach spaces and if X is either a closed subspace of $(\sum_{n=1}^{\infty} X_n)_{c_0}$ or a closed subspace of $(\sum_{n=1}^{\infty} X_n)_p$ with $p > 2$, then every bounded linear operator from X to a Banach space Y of cotype q $(2 \leq q < p)$ is compact. As a consequence, in Corollary 4 we have a new proof of the fact that every bounded linear operator from ℓ_p to ℓ_r $(1 \leq r < p < \infty)$ is compact.

For $1 \leq p < \infty$, the ℓ_p-sum $(\sum_{n=1}^{\infty} X_n)_p$ of Banach spaces X_n's is the Banach space of all sequences $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X_n$ with the norm defined by $\|x\| = (\sum_{n=1}^{\infty} \|x_n\|^p)^{1/p} < \infty$. The c_0-sum $(\sum_{n=1}^{\infty} X_n)_{c_0}$ of X_n's is the Banach space of all null sequences $x = \{x_n\}_{n=1}^{\infty}$, $x_n \in X_n$ with the norm $\|x\| = \sup_n \|x_n\|$. We can easily see that $(\sum_{n=1}^{\infty} X_n)_{c_0}^* = (\sum_{n=1}^{\infty} X_n^*)_1$ and $(\sum_{n=1}^{\infty} X_n)_p^* = (\sum_{n=1}^{\infty} X_n^*)_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and X^* is the dual space of a Banach space X.

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Recall that a (Schauder) basis $\{x_n\}_{n=1}^\infty$ of a Banach space X and a basis $\{y_n\}_{n=1}^\infty$ of a Banach space Y are said to be equivalent if the map $x_n \leftrightarrow y_n$ linearly extends to an isomorphism between X and Y. Therefore, $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are equivalent if and only if there exists a linear operator T from X onto Y and constants $A > 0$, $B > 0$ such that

$$A\|x\| \le \|Tx\| \le B\|x\|$$

for all $x \in X$.

For each $n = 1, 2, 3, \cdots$, the Rademacher function r_n on $[0, 1]$ is defined by $r_n(t) = \operatorname{sgn}(\sin(2^n \pi t))$ for $0 \le t \le 1$. A Banach space X is said to be of cotype q ($q \ge 2$) if there exists a constant $0 < M < \infty$ such that for every finite subset $\{x_j\}_{j=1}^n$ of vectors in X, we have

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\| dt \ge M^{-1} \Big(\sum_{j=1}^n \|x_j\|^q \Big)^{1/q}.$$

2. Results

The following proposition which will be used in the proof of Theorem 2 can easily be proved by a well known gliding hump argument in Banach space theory. Therefore, we only sketch a proof of it.

PROPOSITION 1. *Suppose $\{X_n\}_{n=1}^\infty$ is a sequence of finite dimensional Banach spaces and suppose that X is either a closed subspace of $(\sum_{n=1}^\infty X_n)_{c_0}$ or a closed subspace of $(\sum_{n=1}^\infty X_n)_p$ $(1 \le p < \infty)$. If $\{y_n\}_{n=1}^\infty$ is a weakly null sequence in X such that $\limsup \|y_n\| > 0$, then there exists a subsequence $\{y_{n_k}\}_{k=1}^\infty$ which is equivalent to the unit vector basis of c_0 or ℓ_p.*

SKETCH OF PROOF. For each k, let Q_k be the natural projection from X onto X_k picking up the k-th term of each $x = \{x_n\}_{n=1}^\infty \in X$, and let $\{y_n\}_{n=1}^\infty$ be a weakly null sequence in X such that $\limsup \|y_n\| > 0$. Since $\{Q_k y_n\}_{n=1}^\infty$ is a weakly null sequence in X_k and $\dim X_n < \infty$, $\|Q_k y_n\| \to 0$ as $n \to \infty$. Now we can use the standard gliding hump argument used in [2, p. 7], and Proposition 2.a.1 of [2] to get a subsequence $\{y_{n_k}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$, which is equivalent to the unit vector basis $\{e_n\}_{n=1}^\infty$ of c_0 or ℓ_p. □

THEOREM 2. *Suppose X is either a closed subspace of $(\sum_{n=1}^\infty X_n)_{c_0}$ or a closed subspace of $(\sum_{n=1}^\infty X_n)_p$ $(2 < p < \infty$, $\dim X_n < \infty$ for all $n)$, and suppose Y is a Banach space of cotype q $(2 \le q < p)$. Then* *every bounded linear operator from X to Y carries a weakly convergent sequence in X to a norm convergent sequence in Y.*

PROOF. Let T be a bounded linear operator from a closed subspace X of $(\sum_{n=1}^{\infty} X_n)_p$ $(2 < p < \infty)$ to a Banach space Y of cotype q $(2 \leq q < p)$. Let $\{x_n\}_{n=1}^{\infty}$ be a weakly null sequence in X. It suffices to show that $\|Tx_n\| \to 0$ as $n \to 0$. Suppose $\|Tx_n\| \nrightarrow 0$ as $n \to \infty$. By passing to a subsequence we may assume that $\|Tx_n\| \geq \delta > 0$ for all n. By Proposition 1, we can get a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of ℓ_p. Again, we may assume that $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of ℓ_p. There exists $\alpha > 0$ such that $\|\sum_{n=1}^{N} \pm e_n\|_{\ell_p} \geq \alpha \|\sum_{n=1}^{N} \pm x_n\|$ for all N and all choices of signs.

Since $\|T\| N^{1/p} = \|T\| \|\sum_{n=1}^{N} \pm e_n\|_{\ell_p} \geq \alpha \|\sum_{n=1}^{N} \pm Tx_n\|$ for all N and all choices of signs, we have

$$\begin{aligned} \|T\| N^{1/p} &\geq \alpha \operatorname*{Ave}_{\pm} \Big\| \sum_{n=1}^{N} \pm Tx_n \Big\| \\ &= \alpha \int_0^1 \Big\| \sum_{n=1}^{N} r_n(t) Tx_n \Big\| dt \\ &\geq \alpha M^{-1} \Big(\sum_{n=1}^{N} \|Tx_n\|^q \Big)^{1/q} \\ &\geq \alpha M^{-1} N^{1/q} \delta \quad \text{for all } N, \end{aligned}$$

which contradicts to the fact that $p > q$. Therefore, $\|Tx_n\| \to 0$ as $n \to \infty$.

Suppose T is a bounded linear operator from a closed subspace X of $(\sum_{n=1}^{\infty} X_n)_{c_0}$ to a Banach space Y of cotype $q \geq 2$ and $\{x_n\}_{n=1}^{\infty}$ is a weakly null sequence in X. By the same argument employed above we may assume that $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of c_0. Since $\|\sum_{n=1}^{N} \pm e_n\|_{c_0} = 1$ for all N and all choices of signs, we have

$$\|T\| \geq \alpha \operatorname*{Ave}_{\pm} \Big\| \sum_{n=1}^{N} \pm Tx_n \Big\| \geq \alpha M^{-1} N^{1/q} \delta$$

for all N, which is impossible. Therefore, $\|Tx_n\| \to 0$ as $n \to \infty$. □

Recall that a bounded linear operator T from a reflexive Banach space X to a Banach space Y is compact if and only if $\|Tx_n - Tx\| \to 0$ as

$n \to \infty$ whenever $x_n \to x$ weakly as $n \to \infty$ in X [4, p. 113]. Since the spaces $L_r(\mu)$, $1 \le r < \infty$ are of cotype $\max\{2, r\}$ [3, p. 73], we have the following corollary.

COROLLARY 3. *Suppose $p > 2$ and X is a closed subspace of $(\sum_{n=1}^{\infty} X_n)_p$ ($\dim X_n < \infty$). Then for $1 \le r < p$ every bounded linear operator from X to $L_r(\mu)$ is compact.*

From Corollary 3 we can deduce the following well known fact.

COROLLARY 4. *For $1 \le r < p < \infty$ every bounded linear operator from ℓ_p to ℓ_r is compact.*

PROOF. The case $p > 2$ is contained in Corollary 3. If $1 < r < p \le 2$ and $T : \ell_p \to \ell_r$ is bounded linear, then $T^* : \ell_{r'} \to \ell_{p'}$ $(2 \le p' < r' < \infty)$ is compact and hence T is compact. If $r = 1$, simply notice that ℓ_1 has the Schur property, that is, every weakly convergent sequence in ℓ_1 converges in norm [1, p. 296]. □

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