LIPSCHITZ TYPE INEQUALITY
IN WEIGHTED BLOCH SPACE $B_q$

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ABSTRACT. Let $B$ be the open unit ball with center 0 in the complex space $\mathbb{C}^n$. For each $q > 0$, $B_q$ consists of holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z \in B} (1 - \|z\|^2)^q \|\nabla f(z)\| < \infty.$$ 

In this paper, we will show that functions in weighted Bloch spaces $B_q$ ($0 < q < 1$) satisfies the following Lipschitz type result for Bergman metric $\beta$:

$$|f(z) - f(w)| < C\beta(z, w)$$

for some constant $C$.

1. Introduction

Throughout this paper, $\mathbb{C}^n$ will be the Cartesian product of $n$ copies of complex plane $\mathbb{C}$. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$ and the norm by $\|z\|^2 = \langle z, z \rangle$.

Let $B$ be the open unit ball with center 0 in the complex space $\mathbb{C}^n$. The boundary of $B$ is the unit sphere $S = \{z \in \mathbb{C}^n : \|z\| = 1\}$. For $z \in B, \xi \in \mathbb{C}^n$, set

$$b_B^2(z, \xi) = \frac{n + 1}{(1 - \|z\|^2)^2} \left[ (1 - \|z\|^2)\|\xi\|^2 + \|\langle z, \xi \rangle\|^2 \right].$$

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If $\gamma : [0, 1] \rightarrow B$ is a $C^1$-curve, the Bergman length of $\gamma$ is defined by

$$|\gamma|_B = \int_0^1 b_B(\gamma(t), \gamma'(t)) dt.$$ 

For $z, w \in B$, define

$$\beta(z, w) = \inf\{ |\gamma|_B : \gamma(0) = z, \gamma(1) = w \},$$

where the infimum is taken over all $C^1$-curves from $z$ to $w$. $\beta$ is called the Bergman metric on $B$.

For $a$ in $B$ and $r > 0$, let $E(a, r) = \{ z \in B : \beta(a, z) < r \}$ be the open ball in the Bergman metric with center $a$ and radius $r$. Let $\nu$ be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. Let $|E(a, r)|$ be the $d\nu$-volume measure of $E(a, r)$. Given a function $f$ in $L^2(B, d\nu)$, let

$$\hat{f}_r(z) = \frac{1}{|E(z, r)|} \int_{E(z, r)} f(w) d\nu(w)$$

be the mean of $f$ over $E(z, r)$. The mean oscillation of $f$ in the Bergman metric is the function $MO_r f(z)$ defined on $B$ by

$$MO_r f(z) = \left[ \frac{1}{|E(z, r)|} \int_{E(z, r)} |f(w) - \hat{f}_r(z)|^2 d\nu(w) \right]^{\frac{1}{2}}.$$

We define $BMO_r(B)$ to be the space of all $f$ such that $MO_r f$ is bounded on $B$. We equip $BMO_r(B)$ with the semi-norm

$$\| f \|_r = \sup\{ MO_r f(z) : z \in B \}.$$ 

It was proved in [3] that $BMO_r(B)$ is independent of $r$ and all the semi-norm $\| \|_r$ are mutually equivalent. Thus we simply write $BMO$ for $BMO_r(B)$.

BMO in the Bergman metric was first exhibited in [2, 3] where BMO was used to characterize the boundedness of Hankel operators on the Bergman spaces. Suppose $f$ is in $L^1(B, d\nu)$. The Berezin transform of $f$ is defined by

$$\hat{f}(z) = \frac{1}{K(z, z)} \int_B |K(z, w)|^2 f(w) d\nu(w),$$
where $K(z, w)$ is the Bergman reproducing kernel. It was proved in [3] that for $f \in L^2(B, d\nu)$, we have $f \in BMO$ if and only if the function $|\overline{f}|^2(z) - |\bar{f}(z)|^2$ is bounded in $B$. Moreover

$$
\| f \|_{BMO} = \sup \{ \left( |\overline{f}|^2(z) - |\bar{f}(z)|^2 \right)^{\frac{1}{2}} : z \in B \}
$$

is a complete and invariant semi-norm on BMO.

Let $H(B)$ be the space of all holomorphic functions on $B$. In Section 2, we will show that if $f \in L^1(B, d\nu) \cap H(B)$, then $f(z) = cf(z)$ for some constant $c$.

If $f \in H(B)$, then the quantity $Qf$ is defined by

$$
Qf(z) = \sup_{\| \xi \| = 1} \frac{|\nabla f(z) : \xi|}{b_B(z, \xi)}, \quad z \in B, \quad \xi \in \mathbb{C}^n,
$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})$ is the holomorphic gradient of $f$. The quantity $Qf$ is invariant under the group $\text{Aut}(B)$ of holomorphic automorphisms of $B$. Namely, $Q(f \circ \varphi) = (Qf) \circ \varphi$ for all $\varphi \in \text{Aut}(B)$. A holomorphic function $f : B \to \mathbb{C}$ is called a Bloch function if

$$
\sup_{z \in B} Qf(z) < \infty.
$$

Bloch functions on bounded homogeneous domains were first studied in [5]. In [12], Timoney showed that the linear space of all holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$
\sup_{z \in B} (1 - \| z \|^2) \| \nabla f(z) \| < \infty
$$

is equivalent to the space $B$ of Bloch functions on $B$.

It was shown in [3] that $\text{BMO} \cap H(B) = B(B)$. Moreover the above seminorm for the Bloch functions is equivalent to the $\text{BMO}$-norm $\| f \|_{\text{BMO}}$ for holomorphic functions.

For each $q > 0$, the weighted Bloch space of $B$, denoted by $B_q$, consists of holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$
\sup_{z \in B} (1 - \| z \|^2)^q \| \nabla f(z) \| < \infty.
$$
For each $q > 0$, we let $B_{q,0}$ denote the subspace of $B_q$ consisting of functions $f$ with

$$\lim_{\|z\| \to 1^-} (1 - \|z\|^2)^q \| \nabla f(z) \| = 0.$$ 

The family of weighted Bloch spaces $B_q$ is an increasing family with respect to $q$ in the sense that $B_{q_1} \subset B_{q_2}$ for $q_1 < q_2$. In particular, $B_1 = B$ and $B_{1,0} = B_0$. Let us define a norm on $B_q$ as follows:

$$\| f \|_q = |f(0)| + \sup \{(1 - \|w\|^2)^q \| \nabla f(w) \| : w \in B\}.$$ 

It was proved in [7] that the space $B_q$ is a Banach space with respect to the above norm for each $q > 0$, and that the little Bloch space $B_{q,0}$ associated with $B_q$ is a separable subspace of $B_q$ which is the closure of the polynomials for each $q \geq 1$.

Let $D$ be the open unit disk in the complex plane $\mathbb{C}$. It was proved in [15] that an analytic function $f$ defined on $D$ belongs to the Bloch space if and only if $|f(z) - f(w)| \leq C \delta(z, w)$ for some constant $C$ and all $z, w$ in $D$ where $\delta$ is the Bergman distance on $D$. The purpose of this paper is to extend the above Lipschitz type inequality to the case of $n$-dimensional complex space.

In particular, in Section 3, we will show that if $f \in B_q$, $0 < q < 1$, then

$$|f(z) - f(w)| \leq C \beta(z, w)$$

for some constant $C$ where $\beta$ is the Bergman metric on $B$.

2. Berezin transform of $f$ in $L^1(B, dv) \cap H(B)$

Let $a \in B$ and let $P_a$ be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by $a$, which is given by $P_0 = 0$, and

$$P_a z = \frac{\langle z, a \rangle}{\langle a, a \rangle} a \quad \text{if} \quad a \neq 0.$$ 

Let $Q_a = I - P_a$. Define $\varphi_a$ on $B$ by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - \langle z, a \rangle}.$$
It is easily shown that the mapping $\varphi_a$ belongs to $\text{Aut}(B)$ where $\text{Aut}(B)$ is the group of all biholomorphic mappings of $B$ onto itself, and satisfies $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a(\varphi_a(z)) = z$. Furthermore, for all $z, w \in \overline{B}$, we have

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - \|a\|^2)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}.$$ 

In particular, for $a \in B$, $z \in \overline{B}$,

$$1 - \|\varphi_a(z)\|^2 = \frac{(1 - \|a\|^2)(1 - \|z\|^2)}{|1 - \langle z, a \rangle|^2}$$

(See [9, Theorem 2.2.2]).

**Theorem 1.** Let $\psi$ be a biholomorphic mapping of $B$ onto itself and $a = \psi^{-1}(0)$. The determinant $J_R\psi$ of the real Jacobian matrix of $\psi$ satisfies the following identity:

$$J_R\psi(z) = |J\psi(z)|^2 = \left(\frac{1 - \|a\|^2}{|1 - \langle z, a \rangle|^2}\right)^{n+1} = \left(\frac{1 - \|\psi(z)\|^2}{1 - \|z\|^2}\right)^{n+1}.$$

**Proof.** See [9, Theorem 2.2.6].

The measure $\mu_q$ is the weighted Lebesgue measure:

$$d\mu_q = c_q(1 - \|z\|^2)^q d\nu(z),$$

where $q > -1$ is fixed, and $c_q$ is a normalization constant such that $\mu_q(B) = 1$.

**Theorem 2.** If $f \in L^1(B, \mu_q) \cap H(B), q > -1$, then

$$f(z) = c_q \int_B \frac{(1 - \|w\|^2)^q}{(1 - \langle z, w \rangle)^{n+q+1}} f(w) d\nu(w).$$
Proof. Since \( f \in H(B) \), by the mean value theorem,

\[
f(0) = \int_S f(r\zeta)d\sigma(\zeta), \quad 0 < r < 1.
\]

By integrating both sides of above equality with respect to the measure
\( 2n(1 - r^2)^q r^{2n-1}dr \) over \([0, 1]\), we have

\[
2n \int_0^1 \int_S f(r\zeta)(1 - r^2)^q r^{2n-1}d\sigma(\zeta)dr = f(0)c_q^{-1}.
\]

Namely,

\[
f(0) = c_q \int_B f(w)(1 - \|w\|^2)^q d\nu(w).
\]

Replace \( f \) by \( f \circ \varphi_z \) and apply Theorem 1. Then

\[
f(z) = c_q \int_B f(w)(1 - \|\varphi_z(w)\|^2)^q \left( \frac{(1 - \|z\|^2)}{|1 - \langle w, z \rangle|^2} \right)^{n+1} d\nu(w)
\]

\[
= c_q \int_B f(w) \left( \frac{(1 - \|z\|^2)(1 - \|w\|^2)}{|1 - \langle w, z \rangle|^2} \right)^q \times \left( \frac{(1 - \|z\|^2)}{|1 - \langle w, z \rangle|^2} \right)^{n+1} d\nu(w)
\]

\[
= c_q (1 - \|z\|^2)^{n+q+1} \int_B f(w) \frac{(1 - \|w\|^2)^q}{|1 - \langle w, z \rangle|^{2(n+q+1)}} d\nu(w)
\]

\[
= c_q (1 - \|z\|^2)^{n+q+1} \times \int_B f(w) \frac{(1 - \|w\|^2)^q}{(1 - \langle w, z \rangle)^{n+q+1}(1 - \langle z, w \rangle)^{n+q+1}} d\nu(w).
\]

Replacing \( f(w) \) again by \( f(w)(1 - \langle w, z \rangle)^{n+q+1} \), we get

\[
f(z)(1 - \|z\|^2)^{n+q+1}
\]

\[
= c_q (1 - \|z\|^2)^{n+q+1} \int_B f(w) \frac{(1 - \|w\|^2)^q}{(1 - \langle z, w \rangle)^{n+q+1}} d\nu(w).
\]

\[
f(z) = c_q \int_B \frac{(1 - \|w\|^2)^q}{(1 - \langle z, w \rangle)^{n+q+1}} f(w) d\nu(w).
\]
Theorem 3. If \( f \in L^1(B, dv) \cap H(B) \), then \( c_{n+1} \tilde{f}(z) = f(z) \). Here \( c_{n+1} \) is a normalization constant such that \( \mu_{n+1}(B) = 1 \), where \( d\mu_{n+1} = c_{n+1}(1 - \| z \|^{2})^{n+1} dv(z) \).

Proof. It is well known that

\[
K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1}}
\]
in the case of open unit ball \( B \) in \( \mathbb{C}^n \).

\[
\tilde{f}(z) = \frac{1}{K(z, z)} \int_B |K(z, w)|^2 f(w) dv(w)
\]
\[
= (1 - |z|^2)^{n+1} \int_B \frac{1}{(1 - \langle z, w \rangle)^2(n+1)} f(w) dv(w)
\]
\[
= (1 - |z|^2)^{n+1} \int_B \frac{(1 - \| w \|^2)^{n+1}}{(1 - \langle z, w \rangle)^{n+1+n+1}} \frac{f(w)}{(1 - \| w \|^2)^{n+1}} dv(w).
\]

Since \( f \in L^1(B, dv) \cap H(B) \),

\[
\frac{f(w)}{(1 - \| w \|^2)^{n+1}} \in L^1(B, \mu_{n+1}) \cap H(B).
\]

By Theorem 2,

\[
\frac{f(z)}{(1 - \| z \|^2)^{n+1}}
\]
\[
= c_{n+1} \int_B \frac{(1 - \| w \|^2)^{n+1}}{(1 - \langle z, w \rangle)^{n+1+n+1}} \frac{f(w)}{(1 - \| w \|^2)^{n+1}} dv(w).
\]

We can see that

\[
\tilde{f}(z) = (1 - \| z \|^2)^{n+1} \frac{f(z)}{(1 - \| z \|^2)^{n+1}} \frac{1}{c_{n+1}}
\]
\[
= \frac{1}{c_{n+1}} f(z).
\]

\(\square\)
3. Lipschitz type result in $B_q, 0 < q < 1$

**Theorem 4.** For $z \in B$, $c$ is real, $t > -1$, define

$$I_{c,t}(z) = \int_B \frac{(1 - \|w\|^2)^t}{|1 - \langle z, w \rangle|^{n+1+c+t}}d\nu(w), \quad z \in B.$$  

Then,

(i) $I_{c,t}(z)$ is bounded in $B$ if $c < 0$;

(ii) $I_{0,t}(z) \sim -\log(1 - \|z\|^2)$ as $\|z\| \to 1^-$;

(iii) $I_{c,t}(z) \sim (1 - \|z\|^2)^{c} \quad$ as $\|z\| \to 1^-$ if $c > 0$.

**Proof.** See [9, Proposition 1.4.10].

Let $0 < p < \infty$ and $s \in \mathbb{R}$. The holomorphic Besov p-spaces $B_p^s(B)$ with weight $s$ is defined by the space of all holomorphic functions $f$ on the unit ball $B$ such that

$$\|f\|_{p,s} = \left\{ \int_B (Qf)^p(z)(1 - \|z\|^2)^s d\lambda(z) \right\}^{\frac{1}{p}} < \infty.$$  

Here $d\lambda(z) = (1 - \|z\|^2)^{-n-1}d\nu(z)$ is an invariant volume measure with respect to the Bergman metric on $B$.

For a fixed $p \in (0, \infty)$, $B_p^s(B)$ is an increasing family of function spaces in $s$; that is, if $-\infty < s \leq t < +\infty$, then $B_p^s(B) \subset B_p^t(B)$. Similarly, for a fixed $s \in \mathbb{R}$, the family $B_p^s(B)$ is increasing with respect to $p \in (0, n-s)$. The holomorphic Besov $p$-space $B_p^s(B)$ with weight $s$ include many well known spaces as special case. $B_p^s(B)$ is the usual Hardy $p$-space $H^p(B)$ for $s = n$, the Bergman space $L_p^2(B)$ for $s = n+1$ (See [1]). In particular, the diagonal Besov space $B_p^0(B)$ are shown to be Möbius invariant subsets of the Bloch space.

**Theorem 5.** Let $0 < p < \infty$ and $s \in \mathbb{R}$. Then

$$B_q \subseteq B_p^s,$$

where $q < 1 + \frac{s-n}{p}$. 
Proof. From the fact that $Qf(z)$ and $(1-\|z\|^2)\|\nabla f(z)\|$ behave the same within constants as $\|z\|\rightarrow 1$ (See [12]), we may replace $Qf(z)$ by $(1-\|z\|^2)\|\nabla f(z)\|$ with a different constant $C$ in the definition of $\|f\|_{p,s}$. Namely,

$$\|f\|_{p,s}^p = \int_B (Qf)^p(z)(1-\|z\|^2)^s d\lambda(z)$$

$$\leq C \int_B [(1-\|z\|^2)\|\nabla f(z)\|]^{\frac{p}{q}} (1-\|z\|^2)^s d\lambda(z)$$

$$\leq C \int_B \left[(1-\|z\|^2)^a\|\nabla f(z)\|\right]^{\frac{p}{q-1}} (1-\|z\|^2)^s d\lambda(z)$$

$$\leq C\|f\|_q^p \int_B (1-\|z\|^2)^{-\frac{pq+p+s-n-1}{q-1}} d\nu(z).$$

By Theorem 4, if $q < 1 + \frac{s-n}{p}$, then

$$\|f\|_{p,s} \leq C\|f\|_q$$

which yields the desired result. \hfill \Box

**THEOREM 6.** Let $p \in (1, \infty)$ and $-p < s < 0$. Then there exists a positive constant $C$ such that

$$|f(z) - f(a)| \leq \frac{C}{\|z - a\|^s} \|f\|_{p,s}, \quad a, z \in B$$

for all $M$-harmonic functions $f$ on $B$. In particular, $f \in B^s_p$ satisfies the Lipschitz condition of order $-s/p$.

**Proof.** See [6, Theorem 1.4]. \hfill \Box

**COROLLARY 7.** Let $q \in (0,1)$. If the function $f$ in $B_q$, then there exist constants $C > 0$ and $t > 0$ such that for all $z,w \in B$,

$$|f(z) - f(w)| \leq C \|z - w\|^t \|f\|_q.$$

**Proof.** If we choose $p \in (1, \infty)$ and $s (-p < s < 0)$ such that $q < 1 + \frac{s-n}{p}$, then

$$|f(z) - f(w)| \leq C \|z - w\|^{-\frac{s}{p}} \|f\|_q$$

follows from Theorem 5 and Theorem 6. \hfill \Box
THEOREM 8. For any smooth curve $\gamma : I \to B$ and any $f$ in BMO, we have
\[
\left| \frac{d}{dt} \tilde{f}(\gamma(t)) \right| \leq 2\sqrt{2} \left( \frac{ds}{dt} \right) \| f \|_{\text{BMO}(\gamma(I))}.
\]

Proof. See [3]. \qed

COROLLARY 9. For $f$ in BMO,
\[
|\tilde{f}(a) - \tilde{f}(b)| \leq 2\sqrt{2} \| f \|_{\text{BMO}} \beta(a, b).
\]

Proof. Choose $\gamma$ in Theorem 8 to be a geodesic joining $a$ to $b$ of length $\beta(a, b)$. \qed

THEOREM 10. For $f$ in $B_q$, $0 < q < 1$,
\[
|f(a) - f(b)| \leq c_{n+1} \beta(a, b).
\]

Proof. If $f \in B_q$, $0 < q < 1$, then
\[
|f(z)| \leq |f(0)| + C \| z \|_t \| f \|_q
\]
for some constant $C > 0$ and $t > 0$ by Corollary 7. Since
\[
|f(z)| \leq |f(0)| + C \| f \|_q
\]
for all $z \in B$, $f \in L^1(B, dv) \cap H(B)$. By Theorem 3,
\[
|f(a) - f(b)| = |c_{n+1} \tilde{f}(a) - c_{n+1} \tilde{f}(b)|
\leq c_{n+1} \beta(a, b).
\]

References


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