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ON FARTHEST POINTS IN METRIC SPACES

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Abstract. For a bounded subset G of a metric space (X, d) and $x \in X$, let f_G be the real-valued function on X defined by $f_G(x) = \sup\{d(x,g) : g \in G\}$, and $F(G,x) = \{z \in X : \sup_{g\in G} d(g,z) = \sup_{g\in G} d(g,x) + d(x,z)\}$. In this paper we discuss some properties of the map f_G and of the set $F(G,x)$ in convex metric spaces. A sufficient condition for an element of a convex metric space X to lie in $F(G,x)$ is also given in this paper.

1. Introduction

Let G be a bounded set in a metric spaces (X,d) and $x \in X$. The deviation of G from x is the number $\delta(x,G) = \sup\{d(x,g) : g \in G\}$ and any $g_0 \in G$ for which the supremum is attained, i. e., such that $d(x,g_0) = \sup\{d(x,g) : g \in G\}$ is called a farthest point to x in G. We shall denote by $F_G(x)$ the set of all farthest points to x in G, i. e.,

$$F_G(x) = \{g_0 \in G : d(x,g_0) = \delta(x,G)\}. \tag{1}$$

The map $F_G : X \longrightarrow 2^G$ (the collection of all subsets of G) defined by (1), is called the *farthest point map.*

The set G is said to be

(a) *remotal* if for each $x \in X$, the set $F_G(x)$ is non-empty,
(b) *uniquely remotal* if for each $x \in X$, the set $F_G(x)$ consists of exactly one element, and
(c) nearly compact (*cf.* Ahuja, Narang & Trehan [1]) or ^-compact (*cf.* Blatter [2]) or sup compact (*cf.* Govindarajulu & Pai [5]) or M-compact (*cf.* Panda & Kapoor [10]) if for each $x \in X$, the sequence $\langle g_n\rangle$ in G satisfying $d(x,g_n) \longrightarrow F_G(x)$ contains a subsequence converging to an element of G.

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Obviously, every compact set in a metric space is nearly compact but a nearly compact set need not be compact, e. g., the set G consisting of the open unit square together with its corners in the 2-dimensional Euclidean space $\mathbb{R}^2$ is nearly compact but not compact (see Panda & Kapoor [10]). The set G is not even closed.

Since nearly compact sets in a metric space are remotal (see Ahuja, Narang & Trehan [1]), this example shows that a remotal set need not be closed. It is easy to see that if G is a closed set in a metric space (X, d) then the set $F_G(x)$ is closed. The following example shows that if G is a remotal set then $F_G(x)$ need not be closed.

Definition. Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\} \cup \{x_0\}$, where $x_0 \notin \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Define a matric d on X by

(i) $d(1/n, 0) = d(0, 1/n) = 1/n$,
(ii) $d(1/n, 1/m) = |1/n - 1/m|$,
(iii) $d(x_0, 0) = d(0, x_0) = 1$, and
(iv) $d(x, x) = 0$ for all $x \in X$.

This d defines a metric on X. Take $G = \{1/n : n \in \mathbb{N}\} \cup \{x_0\}$. Then G is remotal but $F_G(x_0) = \{1/n : n \in \mathbb{N}\}$ is not closed.

2. Main results

Concerning nearly compact sets, we have

Proposition 1. *The closure of a nearly compact set in a metric space is nearly compact.*

For normed linear spaces this result is proved in Panda & Kapoor [10] and it can be easily seen that the proof given in Panda & Kapoor [10] works in metric spaces too.

Before proving our next result we recall the following.

Definition. Let (X, d) be a metric space and $I = [0, 1]$ be the closed unit interval. A continuous mapping $W : X \times X \times I \longrightarrow X$ is said to be a *convex structure* on X if for all $x, y \in X$, $\lambda \in I$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y) \quad \text{for all } u \in X.$$

The metric space (X, d) together with a convex structure is called a convex metric space (*cf.* Takahashi [11]).

For a bounded subset G of a metric space (X, d), consider a function f_G on X defined by $f_G(x) = \delta(x, G) := \sup\{d(x, g) : g \in G\}$.

In a convex metric spaces, we have the following proposition.

Proposition 2. *If G is a bounded subset of a convex metric space (X, d) then f_G is a convex 1-Lipschitz function, i. e., a uniformly Lipschitz continuous convex function with Lipschitz constant 1.*

Proof. Let $x, y \in X$ and $0 \leq \lambda \leq 1$. Consider

$$\begin{aligned} f_G[W(x, y, \lambda)] &= \sup_{g \in G} d(W(x, y, \lambda), g) \\ &\leq \sup_{g \in G} d(x, g) + (1 - \lambda) \sup_{g \in G} d(y, g) = \lambda f_G(x) + (1 - \lambda) f_G(y) \end{aligned}$$

showing thereby that f_G is convex.

Now we prove the Lipshitzian property of f_G. For any element z in G, consider

$$d(x, z) \leq d(x, y) + d(y, z)$$

and so

$$\sup_{z \in G} d(x, z) \leq d(x, y) + \sup_{z \in G} d(y, z),$$

i. e., $f_G(x) \leq d(x, y) + f_G(y)$.

This implies $|f_G(x) - f_G(y)| \leq d(x, y)$ for all $x, y \in X$. □

Remark 1. For normed linear spaces this result is given in Miyajima & Wada [8].

Remark 2. If G is a non-empty closed subset of a Banach space X, then a nearest point in G is defined similarly as in the case of a farthest points and the distance function d_G is defined by

$$d_G(x) = \inf\{d(x, z) : z \in G\}.$$

It was shown in Borwein & Fitzpatrick [3] that if G is a non-empty closed subset of a Banach space X such that X/G is convex then d_G is concave on X/G.

Does a similar result hold in metric spaces or in convex metric spaces? The following remarks were made in Miyajima & Wada [8].

Remark 3. Suppose G is a non-empty bounded closed subset of a Banach space X. If $z \in G$ is a farthest point from an $x \in X$, then z is also a nearest point in G. Indeed z is a nearest point in G from any point which is on the line connecting x and z and lies on the opposite side of z to x. Therefore, if there exist no nearest point in G, there exist also no farthest point in G.

Do we have a similar situation in metric spaces? For the results on farthest points in metric spaces we refer Narang [9].

Remark 4. Consider the problem of choosing an element of X which best represents the set G. If x is any particular element of X chosen to represent the set G, the error incurred will be $\sup\{d(x,y) : y \in G\}$. An $x_0 \in X$ will best represent the set G when this error is minimum. Such elements x_0 are called centres or Chebyshev centres of the set G. Since the function $f_G(x)$ is convex and (Lipschitz) continuous on X, the set $E(G)$ (the collection of Chebyshev centres of G) should be, as in normed linear spaces (see Holmes [6, p. 179]), a bounded closed convex subset of the convex metric space X.

Now we consider a set somewhat similar to the set $F_G(x)$ and study some properties of this set. Let G be a non-empty bounded subset of a metric space (X,d) and $x_0 \in X$. For each $z \in X$ we know that

$$\sup_{g\in G} d(g,z) \leq \sup_{g\in G} d(g,x_0) + d(x_0,z).$$

Let us define the set $F(G,x_0)$ as

$$F(G,x_0) = \{z \in X : \sup_{g\in G} d(g,z) = \sup_{g\in G} d(g,x_0) + d(x_0,z)\}.$$

Then $F(G,x_0)$ is a non-empty (since $x_0 \in F(G,x_0)$) closed subset of X. In normed linear spaces this set was considered in Elumalai & Ravi [4].

The following results give some simple properties of the set $F(G,x_0)$ in metric spaces.

Proposition 3. *Let $g_n \in G$ be such that $\sup_{g\in G} d(g,z) = \lim_{n\to\infty} d(g_n,z)$ for each $z \in F(G,x_0)\backslash\{x_0\}$. Then $\sup_{g\in G} d(g,x_0) = \lim_{n\to\infty} d(g_n,x_0)$.*

Corollary. *For each $z \in F(G,x_0)\backslash\{x_0\}$, we have $F_G(z) \subseteq F_G(x_0)$.*

Proposition 4. *Let $z \in F(G,x_0)$ and $y \in F(G,z)$ then $d(x_0,y) = d(x_0,z)+d(z,y)$.*

Proposition 5. *Let $z \in F(G,x_0)$ then $F(G,z) \subseteq F(G,x_0)$.*

Proposition 6. *Let $G \subseteq G_1$, and $x_0 \in X$ be such that*

$$\sup_{g\in G} d(g,x_0) = \sup_{g\in G_1} d(g,x_0)$$

then $F(G,x_0) \subseteq F(G_1,x_0)$.

All these results have been proved in Elumalai & Ravi [4] when the underlying space is a normed linear space and one can easily see that the proofs given in Elumalai & Ravi [4] work in metric spaces too.

If the metric space X is a convex metric space, we have the following proposition.

Proposition 7. *Let (X, d) be a convex metric space and $z \in F(G, x_0)$ then*

$$W(z, x_0, \lambda) \subseteq F(G, x_0) \quad \text{for every scalar } \lambda \in [0,1].$$

Proof. Let $g \in G$. Then for every scalar $\lambda \in [0,1]$,

$$d(z,g) \leq d(g, W(z,x_0,\lambda)) + d(W(z,x_0,\lambda), z)$$

implies

$$\begin{aligned}
&\sup_{g\in G} d(g, W(z,x_0,\lambda)) \\
&\qquad \geq \sup_{g\in G} d(z,g) - d(W(z,x_0,\lambda), z) \\
&\qquad = \sup_{g\in G} d(g,x_0) + d(x_0,z) - d(W(z,x_0,\lambda),z) \text{ as } z \in F(G,x_0) \\
&\qquad \geq \sup_{g\in G} d(g,x_0) + d(x_0,z) - (1-\lambda)d(x_0,z) \\
&\qquad = \sup_{g\in G} d(g,x_0) + \lambda d(x_0,z) \\
&\qquad \geq \sup_{g\in G} d(g,x_0) + d(x_0, W(z,x_0,\lambda)).
\end{aligned}$$

But $\sup_{g\in G} d(g, W(z,x_0,\lambda)) \leq \sup_{g\in G} d(g,x_0) + d(x_0, W(z,x_0,\lambda))$. So,

$$\sup_{g\in G} d(g, W(z,x_0,\lambda)) = \sup_{g\in G} d(g,x_0) + d(x_0, W(z,x_0,\lambda)).$$

Therefore $W(z,x_0,\lambda) \subseteq F(G,x_0)$ for every scalar $\lambda \in [0,1]$. □

Remark 5. For normed linear spaces this result is proved in Elumalai & Ravi [4].

Before proving our next result, we describe the space $X_0^{\#}$ discussed in Johnson [7].

Let (X,d) be a metric space and y_0 be a fixed point of X. The set

$$X_0^{\#} = \left\{ f : X \longrightarrow \mathbb{R} : \sup_{\substack{x\neq y \\ x,y\in X}} \frac{|f(x)-f(y)|}{d(x,y)} < \infty, \quad f(y_0) = 0 \right\}$$

with the usual operations of addition and multiplication by real scalars, normed by

$$\|f\|_X = \sup_{\substack{x\neq y \\ x,y\in X}} \frac{|f(x)-f(y)|}{d(x,y)}, \quad f \in X_0^{\#}$$

is a Banach space (even a conjugate Banach space (*cf.* Johnson [7])).

The space $X_0^\#$ plays, with respect to X, in many ways, the same role as the conjugate space E^* of a normed linear space E, with respect to E.

Proposition 8. *Let G be a bounded subset of a metric space (X,d) and $x_0, z_0 \in X$, $x_0 \neq z_0$ with $F_G(z_0) \neq \varnothing$. Then $z_0 \in F(G,x_0)$ if there exists an $f \in X_0^\#$ such that*

(i) $\|f\|_X = 1$,
(ii) $f(x_0) + \sup_{g\in G} d(g,x_0) \leq \sup_{g\in G} f(g)$, *and*
(iii) $f(x_0) - f(z_0) = d(x_0,z_0)$.

Proof. For any $g \in G$, we have

$$\begin{aligned} d(g,z_0) &\geq \frac{|f(g)-f(z_0)|}{\|f\|_X} \\ &\geq f(g) - f(z_0) \\ &= f(g) - f(x_0) + f(x_0) - f(z_0) \\ &= f(g) - f(x_0) + d(x_0,z_0). \end{aligned}$$

This implies

$$\begin{aligned} \sup_{g\in G} d(g,z_0) &\geq \sup_{g\in G} f(g) - f(x_0) + d(x_0,z_0) \\ &\geq \sup_{g\in G} d(g,x_0) + d(x_0,z_0) \\ &\geq \sup_{g\in G} d(g,z_0). \end{aligned}$$

This implies that

$$\sup_{g\in G} d(g,z_0) = \sup_{g\in G} d(g,x_0) + d(x_0,z_0),$$

i. e., $z_0 \in F(G,x_0)$. □

Remark 6. For normed linear spaces above the result and also its converse are proved in Elumalai & Ravi [4]. It is not known whether the converse part is true in metric spaces. Some more results concerning $F(G,x_0)$ have been proved in normed linear spaces in Elumalai & Ravi [4]. It will be interesting to prove those results too in metric spaces.

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