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THE EIGENVALUE PROBLEM AND A WEAKER FORM OF THE PRINCIPLE OF SPATIAL AVERAGING

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ABSTRACT. In this paper, we find explicitly the eigenvalues and the eigenfunctions of Laplace operator on a triangle domain with a mixed boundary condition. We also show that a weaker form of the principle of spatial averaging holds for this domain under suitable boundary condition.

1. INTRODUCTION

In this paper, we introduce a weaker form of the principle of spatial averaging (PSA) which was introduced by Mallet-Paret & Sell [3] to prove the existence of inertial manifolds for a class of scalar-valued reaction diffusion equations

$$u_t = \nu \Delta u + f(x, u) \tag{1}$$

under suitable conditions. In the study of nonlinear dissipative evolutionary equations, the PSA is crucial to prove the existence of inertial manifold. The PSA is a property which the Laplacian over a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n, n \leq 3$, may (may not) have. However it is not clear at all for which domains and boundary conditions PSA holds; indeed, since this property depends heavily on the eigenvalues and the eigenfunctions of Laplace operator on a domain, it is hard to prove the existence of inertial manifold for various domain and various boundary condition. However, Kwean [2] found a weaker form of PSA and then proved the existence of inertial manifold for different type of domain with homogeneous boundary conditions.

The purpose of this paper is to find explicitly those information for particular domains and boundary condition and then show that a weaker PSA holds for the

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domain and boundary condition we study here. Therefore, our result helps us to extend the result of Mallet-Paret & Sell [3] for our domain and boundary condition.

The eigenvalue problem that we are concerned is of the form

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega_n, \tag{2}$$

where Ω_n is a subset of $\mathbb{R}^n, n = 2, 3$, as follows:

$$\begin{cases} \Omega_2 = \{(x_1, x_2) \in \mathbb{R}^2 : \ 0 < x_2 < \sqrt{3}x_1, \ 0 < x_1 < \frac{\pi}{2}\}, \\ \Omega_3 = \Omega_2 \times [0, L\pi]. \end{cases} \tag{3}$$

Also, we let

$$\begin{cases} S_2 = \{(x_1, x_2) \in \partial\Omega_2 : \ x_1 = \frac{\pi}{2}\}, \\ S_3 = \{(x_1, x_2, x_3) \in \partial\Omega_3 : \ x_1 = \frac{\pi}{2}\}, \\ S_n^c = \partial\Omega_n \backslash S_n. \end{cases} \tag{4}$$

Then we consider the following boundary conditions

$$\begin{cases} u & = 0 \quad \text{on} \quad S_n, \\ \dfrac{\partial u}{\partial n} & = 0 \quad \text{on} \quad S_n^c. \end{cases} \tag{5}$$

This problem is one of classical problems which many authors have been studied for various types of domains (*cf.* Courant & Hilbert [1]). However, most of previous results were concerned about the homogeneous boundary conditions. In particular, Pinsky [4] did for equilateral triangle with homogeneous Dirichlet and Neumann boundary conditions.

Here, we are concerned about a different triangle domain and a mixed boundary condition.

2. THE EIGENVALUE PROBLEM

For our purpose, we basically adapt the idea of Pinsky [4]. Consider the following eigenvalue problem: For $n = 2, 3$,

$$\begin{cases} \Delta u + \lambda u & = 0 \quad \text{in} \quad \Omega_n, \\ u & = 0 \quad \text{on} \quad S_n, \\ \dfrac{\partial u}{\partial n} & = 0 \quad \text{on} \quad S_n^c. \end{cases} \tag{6}$$

To solve this problem, first we introduce a notation. For $k = (k_1, k_2) \in \mathbb{Z}^2$, let $\tilde{f}_{(k_1,k_2)}(x_1, x_2)$ be a function defined as follows:

$$\begin{aligned}\tilde{f}_k(x_1,x_2) = e^{i/3}\left(k_2x_1 + \frac{2k_1-k_2}{\sqrt{3}}x_2\right) + e^{i/3}\left((k_1-k_2)x_1 + \frac{k_1+k_2}{\sqrt{3}}x_2\right)\\ + e^{i/3}\left((k_1-k_2)x_1 - \frac{k_1+k_2}{\sqrt{3}}x_2\right) + e^{i/3}\left(-k_1x_1 + \frac{k_1-2k_2}{\sqrt{3}}x_2\right)\\ + e^{i/3}\left(-k_1x_1 + \frac{2k_2-k_1}{\sqrt{3}}x_2\right) + e^{i/3}\left(k_2x_1 + \frac{k_2-2k_1}{\sqrt{3}}x_2\right). \quad (7)\end{aligned}$$

Then we obtain the following result.

Theorem 1. *Let $\Omega_n \subset \mathbb{R}^n$, $n = 2, 3$ be given in (3). Then the eigenvalues and the eigenfunctions of $-\Delta$ for the given boundary conditions in (5) are of the forms:*

For $n = 2$,

$$\begin{cases}\lambda_k = \dfrac{4}{27}(k_1^2 + k_2^2 - k_1k_2),\\ f_{(k_1,k_2)}(x_1,x_2) = \tilde{f}_{(k_1,k_2)}(x_1,x_2) - \tilde{f}_{(k_2,k_1)}(x_1,x_2),\end{cases} \quad (8)$$

where $\tilde{f}_{(k_1,k_2)}(x_1,x_2)$ is given in (7) and $k = (k_1,k_2) \in \mathbb{Z}^2$ and, for $n = 3$,

$$\begin{cases}\lambda_k = \dfrac{4}{27}(k_1^2 + k_2^2 - k_1k_2) + \dfrac{k_3^2}{L^2},\\ f_{(k_1,k_2,k_3)}(x_1,x_2,x_3) = \cos\dfrac{k_3}{L}x_3 f_{(k_1,k_2)}(x_1,x_2)\end{cases} \quad (9)$$

(for $n = 3$, $k = (k_1,k_2,k_3) \in \mathbb{Z}^3$, $k_3 \geq 0$) satisfies that k_1 and k_2 are multiples of 6.

Proof. It suffices to prove the case $n = 2$ and then to apply separation of variable to obtain (9) for $n = 3$. Now, we consider the parallelogram

$$\tilde{\Omega} = \left\{(x_1,x_2) \in \mathbb{R}^2 : \ 0 < x_2 < 3\sqrt{3}\pi, \ \frac{x_2}{\sqrt{3}} < x_1 < 6\pi + \frac{x_2}{\sqrt{3}}\right\},$$

and the reflection operators

$$\begin{cases}R_1: \ (x_1,x_2) \ \to \ (\pi - x_1, x_2),\\ R_2: \ (x_1,x_2) \ \to \ (x_1, -x_2),\\ R_3: \ (x_1,x_2) \ \to \ \left(-\dfrac{x_1}{2} + \dfrac{x_2}{\sqrt{3}}2, \ \dfrac{x_1\sqrt{3}}{2} + \dfrac{x_2}{2}\right).\end{cases} \quad (10)$$

Then there is an isomorphism between $L^2(\Omega)$ and the following subspace of $L^2(\tilde{\Omega})$,

$$V = \{\tilde{f} \in L^2(\tilde{\Omega}) : \ R_1\tilde{f} = -\tilde{f}, \ \text{and} \ R_i\tilde{f} = \tilde{f} \text{ for } i = 2,3\},$$

obtained by $\tilde{f} \to \tilde{f}|_\Omega$ for $\tilde{f} \in V$. Then, every eigenfunction of $-\Delta$ on Ω can be obtained by solving the equation on V and the restriction to Ω satisfies the boundary conditions in (6). By the classical method, we can obtain a complete list of the eigenfunctions of $-\Delta$ on Ω given by linear combination

$$\tilde{f}(x_1, x_2) = \exp(i(\alpha x_1 + \beta x_2)),$$

where (α, β) are in the dual lattice satisfying

$$3\alpha\pi + 3\sqrt{3}\beta\pi = 2k_1\pi, \quad 6\pi\alpha = 2k_2\pi$$

for $(k_1, k_2) \in \mathbb{Z}^2$. Therefore, $\alpha = k_2/3$, $\beta = (2k_1 - k_2)/3\sqrt{3}$ and hence

$$\lambda_{k_1k_2} = \alpha^2 + \beta^2 = \frac{4}{27}(k_1^2 + k_2^2 - k_1k_2).$$

The corresponding eigenfunction is of the form

$$\tilde{f} = \sum_{(k_1,k_2)} A_{k_1,k_2} e^{1/3}\left(k_2x_1 + \frac{(2k_1 - k_2)}{\sqrt{3}}x_2\right), \tag{11}$$

where the sum is taken over (k_1, k_2) with $\lambda_{k_1k_2} = \lambda$. Now applying the reflection operator to the function in (11) in order to find functions satisfying the reflection conditions, we obtain some conditions on A_{k_1,k_2} such that

$$\begin{cases} R_1\tilde{f} = -\tilde{f} & \Rightarrow \; A_{k_1,k_2} = -A_{k_1-k_2,-k_2}e^{(-n\pi i/3)}, \\ R_2\tilde{f} = \tilde{f} & \Rightarrow \; A_{k_1,k_2} = A_{k_2-k_1,k_1}, \\ R_3\tilde{f} = \tilde{f} & \Rightarrow \; A_{k_1,k_2} = A_{k_1,k_1-k_2}. \end{cases} \tag{12}$$

If for fixed (k_1, k_2) we have $A_{k_1,k_2} = A$, then by iterating the operators R_2, R_3 we obtain that

$$A = A_{k_1,k_2} = A_{k_2-k_1,k_2} = A_{k_2-k_1,-k_1} = A_{-k_2,-k_1} = A_{-k_2,k_1-k_2} = A_{k_1,k_1-k_2}.$$

Therefore, for each $k = (k_1, k_2) \in \mathbb{Z}^2$, the function $\tilde{f}$ defined in (7) satisfies $R_i\tilde{f} = \tilde{f}$ for $i = 2, 3$. On the other hand, if we apply the operator R_1 to the function $\tilde{f}$ for $k = (k_1, k_2)$, then by the first restriction of (12), we have $R_1\tilde{f}_{(k_1,k_2)} = \tilde{f}_{(k_2,k_1)}$ whenever k_1 and k_2 are the multiples of 6. Therefore, for each $k = (k_1, k_2) \in \mathbb{Z}^2$, the solution of the first equation of (6) with respect to the eigenvalue λ_k on the space V has of the form in (8). Then we can easily check $f_{(k_1,k_2)} = \tilde{f}_{(k_1,k_2)} - \tilde{f}_{(k_2,k_1)}$ satisfies the equation (6). □

3. A WEAKER FORM OF PSA

Let $-A = \Delta$ be the Laplace operator on $\Omega \subset \mathbb{R}^n$ with a choice of boundary condition and let $\{\lambda_m\}_{m=1}^{\infty}$ denote the eigenvalues of A ordered (with multiplicities) so that

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \to \infty \quad \text{as} \quad m \to \infty,$$

and let $\{e_m\}_{m=1}^{\infty} \subset L^2$ be the set of corresponding eigenfunctions. The for each $\lambda > 0$, set

$$P_\lambda = \text{span}\{e_m : \ \lambda_m < \lambda\}, \quad Q_\lambda = \text{closure}(\text{span}\{e_m : \ \lambda_m > \lambda\}).$$

Also, let P_λ and $Q_\lambda = I - P_\lambda$ denote the orthogonal projections onto these subspaces.

Definition. For a given (bounded Lipschitz) domain $\Omega \subset \mathbb{R}^n, n \leq 3$, and choice of boundary conditions for Laplace operator, we say that the weaker PSA holds if there exists a quantity $\xi > 0$ such that for every $\epsilon > 0, \kappa > 0$ and any bounded subset $V \subset H^2$, there exist arbitrarily large $\lambda = \lambda(V) > \kappa$ such that

$$\|(P_{\lambda+\kappa} - P_{\lambda-\kappa})(B_v - \tilde{v}I)(P_{\lambda+\kappa} - P_{\lambda-\kappa})\|_{\text{op}} \leq \epsilon \tag{13}$$

holds for any $v \in V$; and such that

$$\lambda_{m+1} - \lambda_m \geq \ \xi \tag{14}$$

where m satisfies $\lambda_m \leq \lambda < \lambda_{m+1}$ and $\|\cdot\|_{\text{op}}$ is the norm of an operator on L^2 and $\tilde{v} = (\text{vol}\,\Omega)^{-1} \int_\Omega v dx$, and B_v is an operator on L^2 given by $(B_v u)(x) = v(x)u(x)$, $v \in L^2$, $x \in \Omega$.

Theorem 2. *The weaker PSA holds for the domains given in* (3) *with the boundary condition given in* (5).

When the space is of dimension 2, it is directly consequence of the following proposition.

Proposition 3. *Let T be a function defined on $\mathbb{Z}^2$ by $T(k_1, k_2) = \lambda_{k_1 k_2}$ for each $k = (k_1, k_2) \in \mathbb{Z}^2$. Then for any given $h > 0$ there exists arbitrarily large $m > 0$ such that*

$$T(k_1, k_2) \notin [m, m+h] \quad \textit{for} \quad (k_1, k_2) \in \mathbb{Z}^2.$$

Proof. The proof of this result follows immediately from Mallet-Paret & Sell [3]. □

Remark. Since $h > 0$ is given arbitrarily, Proposition 3 says that for sufficiently large $h > 0$, there exists an integer $m > 0$ such that the difference of two consecutive eigenvalues λ_{m+1} and λ_m satisfies

$$\lambda_{m+1} - \lambda_m > h.$$

This means that the eigenvalues have arbitrarily large gap. Hence, when $\epsilon > 0, \kappa > 0$, and a bounded set $V \subset H^2$ are given, then for sufficiently large $h > 0$ with $h > 2\kappa$, we can choose $m > 0$ satisfying the conclusion of Proposition 3. Then there exist an eigenvalue λ_m so that

$$\lambda_m \leq m < m + h < \lambda_{m+1}.$$

If we choose $\lambda = m + \frac{h}{2}$ and ξ, $0 < \xi < h$, then $(\lambda - \kappa, \lambda + \kappa] \subset [m, m + h]$ implies that

$$P_{\lambda+\kappa} - P_{\lambda-\kappa} = 0 \quad \text{and} \quad \lambda_{m+1} - \lambda_m > \xi.$$

For the case $n = 3$, we consider the following proposition.

Proposition 4. *Assume that L^2 is a rational number. For $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ let $\|k\|^2 = \frac{4}{27}(k_1^2 + k_2^2 - k_1k_2) + \frac{k_3^2}{L^2}$. Then there exists $\xi > 0$ such that for any $\kappa > 1$ and $d > 0$, there exists an arbitrarily large $\lambda > 0$ satisfying two conditions;*

(i) *whenever $\|k\|^2$, $\|l\|^2 \in (\lambda - \kappa,\ \lambda + \kappa]$ with k, $l \in \mathbb{Z}^3$, one has either $k = l$ or $\|k - l\| \geq d$, and*

(ii) *$\|k\|^2 \notin (\lambda - \frac{\xi}{2}, \lambda + \frac{\xi}{2})$ for each $k \in \mathbb{Z}^3$.*

Proof. Since the proof is exactly the similar as one in Kwean [2] and Mallet-Paret & Sell [3], we omit the proof. □

From the result of Proposition 4, we can obtain the following.

Proposition 5. *Let Ω_3 be given in (3). Fix the boundary condition (5). Let V be a bounded subset of $H^2(\Omega_3)$. Then for any $\epsilon > 0$ and $\kappa > 1$, there exists arbitrarily large $\lambda = \lambda(V) > \kappa$ such that*

$$\left| \int_{\Omega_3} (v - \tilde{v})\rho^2 dx \right| \leq \epsilon \tag{15}$$

for any $v \in V$ and $\rho \in$ Range $(P_{\lambda+\kappa} - P_{\lambda-\kappa}) \subset L^2$ with $\|\rho\|_{L^2} = 1$.

Proof. Note that the product of any two eigenfunctions of the form in (9) is also a finite combination of eigenfunctions of $-\Delta$. Then by combining the property (i) of Proposition 4 and the fact that any bounded set of H^2 is a compact subset of L^2

for $n \leq 3$, the inequality (15) can be obtained. For more detailed proof, we refer to Mallet-Paret & Sell [3] and Kwean [2]. □

Proof of **Theorem 2**. Fix a quantity $\xi > 0$ satisfying property (ii) of Proposition 4. Let $\epsilon > 0, \kappa > 0$, and a bounded subset $V \subset H^2(\Omega_3)$ be given. Then we have arbitrarily large $\lambda > \kappa$ satisfying the property (i) of Proposition 4 and the inequality (15). Therefore the inequalities (13) and (14) can be satisfied by the choice of $\xi > 0$ and $\lambda > 0$. □

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