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# FUZZY SUB-IMPLICATIVE IDEALS OF BCI-ALGEBRAS

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ABSTRACT. We consider the fuzzification of sub-implicative ideals in BCI-algebras, and investigate some related properties. We give conditions for a fuzzy ideal to be a fuzzy sub-implicative ideal. We show that (1) every fuzzy sub-implicative ideal is a fuzzy ideal, but the converse is not true, (2) every fuzzy sub-implicative ideal is a fuzzy positive implicative ideal, but the converse is not true, and (3) every fuzzy $p$-ideal is a fuzzy sub-implicative ideal, but the converse is not true. Using a family of sub-implicative ideals of a BCI-algebra, we establish a fuzzy sub-implicative ideal, and using a level set of a fuzzy set in a BCI-algebra, we give a characterization of a fuzzy sub-implicative ideal.

## 1. Introduction

The study of BCK/BCI-algebras was initiated by Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. For the general development of BCK/BCI-algebras the ideal theory plays an important role. Zadeh, in his classic paper [4], introduced the notion of fuzzy sets and fuzzy set operations. Since then the fuzzy set theory developed by Zadeh and others has evoked great interest among researchers working in different branches of mathematics. In [1], Jun and Meng considered the fuzzification of $p$-ideals in BCI-algebras. In [3], Liu and Meng introduced the notion of fuzzy positive implicative, and investigate some

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of their properties. Liu and Meng [2] also introduced the notion of sub-implicative ideals in BCI-algebras. In this paper we consider the fuzzification of sub-implicative ideals in BCI-algebras, and investigate some related properties. We give conditions for a fuzzy ideal to be a fuzzy sub-implicative ideal. We show that (1) every fuzzy sub-implicative ideal is a fuzzy ideal, but the converse is not true, (2) every fuzzy sub-implicative ideal is a fuzzy positive implicative ideal, but the converse is not true, and (3) every fuzzy $p$-ideal is a fuzzy sub-implicative ideal, but the converse is not true. Using a family of sub-implicative ideals of a BCI-algebra, we establish a fuzzy sub-implicative ideal, and using a level set of a fuzzy set in a BCI-algebra, we give a characterization of a fuzzy sub-implicative ideal.

## 2. Preliminaries

For the sake of convenience we set out the former concepts and results which will be used in this paper.

By a *BCI-algebra* we mean an algebra $X$ of type (2,0) satisfying the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$. If we define a relation $\leq$ on $X$ as follows:

$$x \leq y \text{ if and only if } x * y = 0,$$

then $(X, \leq)$ is a partially ordered set. A BCI-algebra $X$ is said to be *implicative* if $(x*(x*y))*(y*x) = y*(y*x)$ for all $x, y \in X$. A mapping $f : X \to Y$ of BCI-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

In any BCI-algebra $X$, the following hold:

(1) $(x * y) * z = (x * z) * y$,
(2) $x * (x * (x * y)) = x * y$,
(3) $((x * z) * (y * z)) * (x * y) = 0$,
(4) $x * 0 = x$,
(5) $0 * (x * y) = (0 * x) * (0 * y)$,
(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

In what follows, $X$ shall mean a BCI-algebra unless otherwise specified. A non-empty subset $A$ of $X$ is called an *ideal* of $X$ if

(I1) $0 \in A$,
(I2) $x * y \in A$ and $y \in A$ imply $x \in A$.

A non-empty subset $A$ of $X$ is called a *positive implicative ideal* of $X$ if

(I1) $0 \in A$,
(I3) $((x * z) * z) * (y * z) \in A$ and $y \in A$ imply $x * z \in A$.

We now review some fuzzy logic concepts. A fuzzy set in a set $X$ is a function $\mu : X \rightarrow [0, 1]$. For a fuzzy set $\mu$ in $X$ and $t \in [0, 1]$ define $U(\mu; t)$ to be the set $U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$, which is called a *level set* of $\mu$.

A fuzzy set $\mu$ in $X$ is said to be a *fuzzy ideal* of $X$ if

(F1) $\mu(0) \geq \mu(x)$,
(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

for all $x, y \in X$.

## 3. Fuzzy sub-implicative ideals

For any elements $x$ and $y$ of a BCI-algebra, $x^n * y$ denotes

$$x * (\cdots * (x * (x * y)) \cdots)$$

in which $x$ occurs $n$ times.

DEFINITION 3.1 ([2, Definition 3.1]). A non-empty subset $A$ of $X$ is called a *sub-implicative ideal* of $X$ if

(I1) $0 \in A$,
(I5) $((x^2 * y) * (y * x)) * z \in A$ and $z \in A$ imply $y^2 * x \in A$.

We consider the fuzzification of a sub-implicative ideal.

DEFINITION 3.2. A fuzzy set $\mu$ in $X$ is called a *fuzzy sub-implicative ideal* of $X$ if

(F1) $\mu(0) \geq \mu(x)$,
(FI1) $\mu(y^2 * x) \geq \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\}$

for all $x, y, z \in X$,

EXAMPLE 3.3. Consider a BCI-algebra $X = \{0, 1, 2\}$ with Cayley table as follows:

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 2 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 2 | 0 |

Let $\mu$ be a fuzzy set in $X$ defined by $\mu(0) = \mu(1) = 0.6$ and $\mu(2) = 0.2$. It is easy to verify that $\mu$ is a fuzzy sub-implicative ideal of $X$.

For any fuzzy set $\mu$ in $X$ satisfying the condition (F1), if $\mu$ is a fuzzy sub-implicative ideal of $X$ then by taking $z = 0$ in (FI1) and using (F1) and (4) we get

$$\begin{aligned}\mu(y^2 * x) &\geq \min\{\mu(((x^2 * y) * (y * x)) * 0), \mu(0)\} \\ &= \min\{\mu((x^2 * y) * (y * x)), \mu(0)\} \\ &= \mu((x^2 * y) * (y * x)).\end{aligned}$$

We state this result as a theorem.

THEOREM 3.4. *Let $\mu$ be a fuzzy set in $X$ satisfying the condition* (F1). *If $\mu$ is a fuzzy sub-implicative ideal of $X$, then $\mu$ satisfies the following inequality:*
(FI2) $\mu(y^2 * x) \geq \mu((x^2 * y) * (y * x))$
*for all $x, y \in X$.*

THEOREM 3.5. *Every fuzzy sub-implicative ideal is a fuzzy ideal.*

*Proof.* Let $\mu$ be a fuzzy sub-implicative ideal of $X$. Putting $y = x$ in (FI1), we obtain

$$\begin{aligned}\mu(x) &= \mu(x^2 * x) \\ &\geq \min\{\mu(((x^2 * x) * (x * x)) * z), \mu(z)\} \\ &= \min\{\mu(x * z), \mu(z)\}\end{aligned}$$

for all $x, z \in X$. Hence $\mu$ is a fuzzy ideal of $X$. □

The following example shows that the converse of Theorem 3.5 may not be true.

EXAMPLE 3.6. Consider a BCI-algebra $X = \{0, a, b, c\}$ with Cayley table as follows:

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $c$ |
| $a$ | $a$ | $0$ | $0$ | $c$ |
| $b$ | $b$ | $b$ | $0$ | $c$ |
| $c$ | $c$ | $c$ | $c$ | $0$ |

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.7$ and $\mu(x) = 0.2$ for all $x \neq 0$. Then $\mu$ is a fuzzy ideal of $X$, but it is not a fuzzy sub-implicative ideal of $X$ because

$$\mu(a^2 * b) \not\geq \min\{\mu(((b^2 * a) * (a * b)) * 0), \mu(0)\}.$$

We now give a condition for a fuzzy ideal to be a fuzzy sub-implicative ideal.

THEOREM 3.7. *Every fuzzy ideal $\mu$ of $X$ satisfying the condition* (FI2) *is a fuzzy sub-implicative ideal of $X$.*

*Proof.* Let $\mu$ be a fuzzy ideal of $X$ satisfying the condition (FI2). For any $x, y, z \in X$, we get

$$\begin{aligned}\mu(y^2 * x) &\geq \mu((x^2 * y) * (y * x)) \qquad \text{[by (FI2)]} \\ &\geq \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\}, \qquad \text{[by (F2)]}\end{aligned}$$

which proves the condition (FI1). This completes the proof. □

COROLLARY 3.8. *In an implicative BCI-algebra every fuzzy ideal is a fuzzy sub-implicative ideal.*

Let $X$ be a $p$-semisimple BCI-algebra, i.e., $B(X) := \{x \in X \mid 0 \leq x\} = \{0\}$. Then $x^2 * y = y$ for all $x, y \in X$. Thus if $\mu$ is a fuzzy ideal of $X$, then

$$\begin{aligned}\mu(y^2 * x) &= \mu(x) \\ &\geq \min\{\mu(x * z), \mu(z)\} \qquad \text{[by (F2)]} \\ &= \min\{\mu((y^2 * x) * z), \mu(z)\} \\ &= \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\},\end{aligned}$$

which shows that $\mu$ is a fuzzy sub-implicative ideal of $X$. Hence we have the following theorem.

THEOREM 3.9. *In a p-semisimple BCI-algebra the notion of fuzzy ideals and fuzzy sub-implicative ideals coincide.*

DEFINITION 3.10 ([3]). A fuzzy set $\mu$ in $X$ is called a *fuzzy positive implicative ideal* of $X$ if

(F1) $\mu(0) \geq \mu(x)$,
(FI3) $\mu(x * z) \geq \min\{\mu(((x * z) * z) * (y * z)), \mu(y)\}$

for all $x, y, z \in X$.

LEMMA 3.11 ([3, Theorem 3.5]). *Let $\mu$ be a fuzzy ideal of $X$. Then the following are equivalent:*

(i) *$\mu$ is a fuzzy positive implicative ideal of $X$,*
(ii) *$\mu((x * y) * z) \geq \mu(((x * z) * z) * (y * z))$ for all $x, y, z \in X$,*
(iii) *$\mu(x * y) \geq \mu(((x * y) * y) * (0 * y))$ for all $x, y \in X$.*

THEOREM 3.12. *Every fuzzy sub-implicative ideal is a fuzzy positive implicative ideal.*

*Proof.* Let $\mu$ be a fuzzy sub-implicative ideal of $X$. Then $\mu$ is a fuzzy ideal of $X$, and so it is sufficient to show that $\mu$ satisfies Lemma 3.11(iii). Note from Theorem 3.4 that $\mu(b^2 * a) \geq \mu((a^2 * b) * (b * a))$ for all $a, b \in X$. Substituting $x * y$ for $a$ and $x$ for $b$ gives the following:

$$\begin{aligned}
\mu(x * y) &= \mu(x * (x * (x * y))) && \text{[by (2)]} \\
&= \mu(b^2 * a) \\
&\geq \mu((a^2 * b) * (b * a)) \\
&= \mu(((x * y) * ((x * y) * x)) * (x * (x * y))) \\
&= \mu(((x * y) * (x * (x * y))) * ((x * y) * x)) && \text{[by (1)]} \\
&= \mu(((x * (x * (x * y))) * y) * ((x * x) * y)) && \text{[by (1)]} \\
&= \mu(((x * y) * y) * (0 * y)). && \text{[by (2) and (III)]}
\end{aligned}$$

Hence $\mu$ is a fuzzy positive implicative ideal of $X$. □

We can easily check that the fuzzy set $\mu$ in Example 3.6 is a fuzzy positive implicative ideal of $X$. Hence we know that the converse of Theorem 3.12 may not be true.

DEFINITION 3.13 ([1]). A fuzzy set $\mu$ in $X$ is called a *fuzzy p-ideal* of $X$ if

(F1) $\mu(0) \geq \mu(x)$,

(FP1) $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$
for all $x, y, z \in X$.

Note from [1, Theorem 2.4 and Remark 2.5] that every fuzzy $p$-ideal is a fuzzy ideal, but the converse does not hold.

LEMMA 3.14 ([1, Proposition 2.3]). *Every fuzzy $p$-ideal $\mu$ in $X$ satisfies the inequality $\mu(x) \geq \mu(0^2 * x)$ for all $x \in X$.*

THEOREM 3.15. *Every fuzzy $p$-ideal is a fuzzy sub-implicative ideal.*

*Proof.* Let $\mu$ be a fuzzy $p$-ideal of $X$. Then $\mu$ is a fuzzy ideal of $X$, and hence it is sufficient to verify, by means of Theorem 3.7, that $\mu$ satisfies (FI2). Note that

$$\begin{aligned}
&(0^2 * (y^2 * x)) * ((x^2 * y) * (y * x)) \\
&= (0 * ((x^2 * y) * (y * x))) * (0 * (y^2 * x)) \\
&= ((0 * (x^2 * y)) * (0 * (y * x))) * ((0 * y) * (0 * (y * x))) \\
&= (((0 * x) * (0 * (x * y))) * (0 * (y * x))) * ((0 * y) * (0 * (y * x))) \\
&\leq ((0 * x) * (0 * (x * y))) * (0 * y) \qquad [\text{by } (3)] \\
&= ((0 * x) * (0 * y)) * (0 * (x * y)) \qquad [\text{by } (1)] \\
&= 0. \qquad [\text{by } (5) \text{ and (III)}]
\end{aligned}$$

Since every fuzzy ideal is order reversing, it follows from (F1), (F2) and Lemma 3.14 that

$$\begin{aligned}
&\mu(y^2 * x) \\
&\geq \mu(0^2 * (y^2 * x)) \\
&\geq \min\{\mu((0^2 * (y^2 * x)) * ((x^2 * y) * (y * x))), \mu((x^2 * y) * (y * x))\} \\
&\geq \min\{\mu(0), \mu((x^2 * y) * (y * x))\} \\
&= \mu((x^2 * y) * (y * x)).
\end{aligned}$$

Hence $\mu$ is a fuzzy sub-implicative ideal of $X$. □

In the following example, we see that the converse of Theorem 3.15 may not be true.

EXAMPLE 3.16. Consider a BCI-algebra $X = \{0, a, 1, 2, 3\}$ with Cayley table as follows:

| $*$ | 0 | $a$ | 1 | 2 | 3 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 3 | 2 | 1 |
| $a$ | $a$ | 0 | 3 | 2 | 1 |
| 1 | 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 2 | 1 | 0 | 3 |
| 3 | 3 | 3 | 2 | 1 | 0 |

Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(0) = 0.7$, $\mu(a) = 0.5$, and $\mu(1) = \mu(2) = \mu(3) = 0.2$. Then $\mu$ is a fuzzy ideal of $X$ in which the inequality

$$\mu(y^2 * x) \geq \mu((x^2 * y) * (y * x))$$

holds for all $x, y \in X$. Using Theorem 3.7, we see that $\mu$ is a fuzzy sub-implicative ideal of $X$. But $\mu$ is not a fuzzy $p$-ideal of $X$ because

$$\mu(a) \not\geq \min\{\mu((a * 1) * (0 * 1)), \mu(0)\}.$$

THEOREM 3.17. *For any fuzzy sub-implicative ideal $\mu$ of $X$, the set*

$$X_\mu := \{x \in X \mid \mu(x) = \mu(0)\}$$

*is a sub-implicative ideal of $X$.*

*Proof.* Clearly $0 \in X_\mu$. Let $x, y, z \in X$ be such that

$$((x^2 * y) * (y * x)) * z \in X_\mu \quad \text{and} \quad z \in X_\mu.$$

Then, by (FI1), we have

$$\mu(y^2 * x) \geq \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\} = \mu(0),$$

which implies from (F1) that $\mu(y^2 * x) = \mu(0)$, i.e., $y^2 * x \in X_\mu$. Therefore $X_\mu$ is a sub-implicative ideal of $X$. □

Applying Theorems 3.15 and 3.17, we have the following corollary.

COROLLARY 3.18. *If $\mu$ is a fuzzy $p$-ideal of $X$, then the set*

$$X_\mu := \{x \in X \mid \mu(x) = \mu(0)\}$$

*is a sub-implicative ideal of $X$.*

THEOREM 3.19. *Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a collection of sub-implicative ideals of $X$ such that $X = \cup_{\lambda\in\Lambda} A_\lambda$ and for all $\alpha, \beta \in \Lambda$, $\alpha > \beta$ if and only if $A_\alpha \subset A_\beta$, where $\Lambda$ is a non-empty subset of $[0,1]$. Define a fuzzy set $\mu$ in $X$ by*

$$\mu(x) = \sup\{\lambda \in \Lambda \mid x \in A_\lambda\}$$

*for all $x \in X$. Then $\mu$ is a fuzzy sub-implicative ideal of $X$.*

*Proof.* Since $0 \in A_\lambda$ for all $\lambda \in \Lambda$, clearly $\mu(0) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $\mu(((x^2 * y) * (y * x)) * z) = \delta_1$ and $\mu(z) = \delta_2$. Without loss of generality, we may assume that $\delta_1 < \delta_2$. To prove that $\mu$ satisfies the condition (FI1), we consider the following three cases:

$$\text{(i) } \lambda \leq \delta_1, \text{ (ii) } \delta_1 < \lambda \leq \delta_2 \text{ and (iii) } \lambda > \delta_2.$$

Case (i) implies that $((x^2 * y) * (y * x)) * z \in A_\lambda$ and $z \in A_\lambda$. Since $A_\lambda$ is a sub-implicative ideal of $X$, we have $y^2 * x \in A_\lambda$ and so

$$\begin{aligned}\mu(y^2 * x) &= \sup\{\lambda \in \Lambda \mid y^2 * x \in A_\lambda\} \\ &\geq \delta_1 = \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\}.\end{aligned}$$

For the case (ii), we have $((x^2 * y) * (y * x)) * z \notin A_\lambda$ and $z \in A_\lambda$. It follows that either $y^2 * x \in A_\lambda$ or $y^2 * x \notin A_\lambda$. If $y^2 * x \in A_\lambda$, then $\mu(y^2 * x) = \delta_2 \geq \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\}$. If $y^2 * x \notin A_\lambda$, then there exists $\eta \in \Lambda$ such that $\eta < \lambda$ and $y^2 * x \in A_\eta \setminus A_\lambda$. Hence

$$\mu(y^2 * x) > \delta_1 = \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\}.$$

Finally case (iii) implies $((x^2 * y) * (y * x)) * z \in A_\lambda$ and $z \notin A_\lambda$. Then we also know that either $y^2 * x \in A_\lambda$ or $y^2 * x \notin A_\lambda$. If $y^2 * x \in A_\lambda$, then obviously $\mu(y^2 * x) \geq \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\}$. If $y^2 * x \notin A_\lambda$, then $y^2 * x \in A_\zeta \setminus A_\lambda$ for some $\zeta < \lambda$, hence

$$\mu(y^2 * x) \geq \delta_1 = \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\}.$$

This completes the proof. □

THEOREM 3.20. *Let $f : X \to Y$ be an onto homomorphism of BCI-algebras. If $\mu$ is a fuzzy sub-implicative ideal of $Y$, then $\mu \circ f$ is a fuzzy sub-implicative ideal of $X$.*

*Proof.* For any $x \in X$ we get $(\mu \circ f)(x) = \mu(f(x)) \leq \mu(0) = \mu(f(0)) = (\mu \circ f)(0)$. Let $x, y \in X$. Then

$$\begin{aligned}
&(\mu \circ f)(y^2 * x) = \mu(f(y^2 * x)) = \mu((f(y))^2 * f(x)) \\
&\geq \min\{\mu((((f(x))^2 * f(y)) * (f(y) * f(x))) * z), \mu(z)\} \text{ for every } z \in Y \\
&= \min\{\mu((((f(x))^2 * f(y)) * (f(y) * f(x))) * f(w)), \mu(f(w))\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{where } f(w) = z \\
&= \min\{\mu((f(x^2 * y) * f(y * x)) * f(w)), \mu(f(w))\} \\
&= \min\{\mu(f(((x^2 * y) * (y * x)) * w), \mu(f(w))\} \\
&= \min\{(\mu \circ f)(((x^2 * y) * (y * x)) * w), (\mu \circ f)(w)\}.
\end{aligned}$$

Since $f$ is onto and since $z$ is arbitrary, the above inequality is valid for all $x, y, w \in X$. Hence $\mu \circ f$ is a fuzzy sub-implicative ideal of $X$. □

In Theorem 3.20, if $f$ is a self-map of $X$ then we will denote $\mu \circ f$ by $\mu^f$.

DEFINITION 3.21. A sub-implicative ideal $A$ of $X$ is said to be *characteristic* if $f(A) = A$ for all $f \in \text{Aut}(X)$, where $\text{Aut}(X)$ is the set of all automorphisms of $X$.

DEFINITION 3.22. A fuzzy sub-implicative ideal $\mu$ of $X$ is said to be *fuzzy characteristic* if $\mu^f(x) = \mu(x)$ for all $x \in X$ and $f \in \text{Aut}(X)$.

THEOREM 3.23. *Let $\mu$ be a fuzzy set in $X$ and let $t \in Im(\mu)$. Then $\mu$ is a fuzzy (characteristic) sub-implicative ideal of $X$ if and only if the level set $U(\mu; t)$ of $\mu$ is a (characteristic) sub-implicative ideal of $X$.*

We then call $U(\mu; t)$ a *level (characteristic) sub-implicative ideal of* $X$.

*Proof.* Let $\mu$ be a fuzzy (characteristic) sub-implicative ideal of $X$. Clearly $0 \in U(\mu; t)$. Let $x, y, z \in X$ be such that $((x^2 * y) * (y * x)) * z \in U(\mu; t)$ and $z \in U(\mu; t)$. Then $\mu(((x^2 * y) * (y * x)) * z) \geq t$ and $\mu(z) \geq t$. It follows from (FI1) that

$$\mu(y^2 * x) \geq \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\} \geq t$$

so that $y^2 * x \in U(\mu; t)$. Hence $U(\mu; t)$ is a sub-implicative ideal of $X$. Now let $x \in U(\mu; t)$ and $f \in \text{Aut}(X)$. Then $\mu^f(x) = \mu(x) \geq t$, i.e., $\mu(f(x)) \geq t$ and so $f(x) \in U(\mu; t)$. Thus $f(U(\mu; t)) \subseteq U(\mu; t)$.

Next let $x \in U(\mu;t)$ and let $y \in X$ be such that $f(y) = x$. Then $\mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq t$; hence $y \in U(\mu;t)$. It follows that $x = f(y) \in f(U(\mu;t))$ so that $U(\mu;t) \subseteq f(U(\mu;t))$. Consequently, $f(U(\mu;t)) = U(\mu;t)$, and thus $U(\mu;t)$ is characteristic.

Conversely, suppose that each level set $U(\mu;t)$ of $\mu$ is a (characteristic) sub-implicative ideal of $X$. If there exists $x_0 \in X$ such that $\mu(x_0) > \mu(0)$, then $\mu(0) < 0.5(\mu(0) + \mu(x_0)) < \mu(x_0)$ and hence $x_0 \in U(\mu;s)$ where $s = 0.5(\mu(0) + \mu(x_0))$. Since $0 \in U(\mu;s)$, we have $\mu(0) \geq s$, a contradiction. Assume that there exist $u, v, w \in X$ such that

$$\mu(v^2 * u) < \min\{\mu(((u^2 * v) * (v * u)) * w), \mu(w)\}.$$

Taking $p := 0.5(\mu(v^2 * u) + \min\{\mu(((u^2 * v) * (v * u)) * w), \mu(w)\})$, we get

$$\mu(v^2 * u) < p < \min\{\mu(((u^2 * v) * (v * u)) * w), \mu(w)\}$$

and so $((u^2 * v) * (v * u)) * w \in U(\mu;p)$, $w \in U(\mu;p)$ and $v^2 * u \notin U(\mu;p)$. This is a contradiction. Hence $\mu$ is a fuzzy sub-implicative ideal of $X$. Let $x \in X$, $f \in \mathrm{Aut}(X)$ and $\mu(x) = t$. Then $x \in U(\mu;t)$ and $x \notin U(\mu;s)$ for all $s > t$. It follows that $f(x) \in F(U(\mu;t)) = U(\mu;t)$ so that $\mu^f(x) = \mu(f(x)) \geq t$. Let $s = \mu^f(x)$ and assume that $s > t$. Then $f(x) \in U(\mu;s) = f(U(\mu;s))$, which implies from the injectivity of $f$ that $x \in U(\mu;s)$, a contradiction. Therefore $\mu^f(x) = \mu(f(x)) = t = \mu(x)$ showing that $\mu$ is fuzzy characteristic. □

LEMMA 3.24. *Let $N$ be a non-empty subset of $X$ and let $\mu_N$ be a fuzzy set in $X$ defined by*

$$\mu_N(x) := \begin{cases} t & \text{if } x \in N, \\ s & \text{otherwise} \end{cases}$$

*for all $x \in X$ and $s, t \in [0,1]$ with $s < t$. Then $\mu_N$ is a fuzzy sub-implicative ideal of $X$ if and only if $N$ is a sub-implicative ideal of $X$. Moreover, in this case, $X_{\mu_N} = N$.*

*Proof.* Assume that $\mu_N$ is a fuzzy sub-implicative ideal of $X$. Using (F1), clearly $\mu_N(0) = t$ and so $0 \in N$. Let $x, y, z \in X$ be such that $z \in N$ and $((x^2 * y) * (y * x)) * z \in N$. Then

$$\mu_N(y^2 * x) \geq \min\{\mu_N(((x^2 * y) * (y * x)) * z), \mu_N(z)\} = t,$$

which implies $\mu_N(y^2 * x) = t$, i.e., $y^2 * x \in N$. Hence $N$ is a sub-implicative ideal of $X$. Conversely, suppose that $N$ is a sub-implicative ideal of $X$. Since $0 \in N$, we have $\mu_N(0) = t \geq \mu_N(x)$ for all $x \in X$. Let $x, y, z \in X$. If $((x^2 * y) * (y * x)) * z \in N$ and $z \in N$, then $y^2 * x \in N$ by (I5). Thus

$$\mu_N(y^2 * x) = t = \min\{\mu_N(((x^2 * y) * (y * x)) * z), \mu_N(z)\}.$$

If $((x^2 * y) * (y * x)) * z \notin N$ or $z \notin N$, then clearly

$$\mu_N(y^2 * x) \geq s = \min\{\mu_N(((x^2 * y) * (y * x)) * z), \mu_N(z)\}.$$

This shows that $\mu_N$ is a fuzzy sub-implicative ideal of $X$. Using Theorem 3.17, we have

$$X_{\mu_N} = \{x \in X \mid \mu_N(x) = \mu_N(0)\} = \{x \in X \mid \mu_N(x) = t\} = N.$$

This completes the proof. □

THEOREM 3.25. *Let $\mu$ be a fuzzy sub-implicative ideal with* $\mathrm{Im}(\mu) = \{t_i \mid i \in \Lambda\}$ *and* $\mathcal{B} = \{U(\mu; t_i) \mid i \in \Lambda\}$. *Then*

(i) *there exists a unique $i_0 \in \Lambda$ such that $t_{i_0} \geq t_i$ for all $i \in \Lambda$;*

(ii) $X_\mu = \bigcap_{i \in \Lambda} U(\mu; t_i) = U(\mu; t_{i_0})$;

(iii) $X = \bigcup_{i \in \Lambda} U(\mu; t_i)$;

(iv) *the members of $\mathcal{B}$ form a chain;*

(v) *$\mathcal{B}$ contains all level sub-implicative ideals of $\mu$ if and only if $\mu$ attains its infimum on all sub-implicative ideals of $X$.*

*Proof.* (i) Since $\mu(0) \in \mathrm{Im}(\mu)$, there exists a unique $i_0 \in \Lambda$ such that $\mu(0) = t_{i_0}$. It follows from (F1) that $\mu(x) \leq \mu(0) = t_{i_0}$ for all $x \in X$ so that $t_{i_0} \geq t_i$ for all $i \in \Lambda$.

(ii) We have

$$\begin{aligned} U(\mu; t_{i_0}) &= \{x \in X \mid \mu(x) \geq t_{i_0}\} \\ &= \{x \in X \mid \mu(x) = t_{i_0}\} \\ &= \{x \in X \mid \mu(x) = \mu(0)\} \\ &= X_\mu. \end{aligned}$$

Since $t_{i_0} \geq t_i$ for all $i \in \Lambda$, it follows that $U(\mu; t_{i_0}) \subseteq U(\mu; t_i)$ for all $i \in \Lambda$. Hence $U(\mu; t_{i_0}) \subseteq \bigcap_{i \in \Lambda} U(\mu; t_i)$ and so $U(\mu; t_{i_0}) = \bigcap_{i \in \Lambda} U(\mu; t_i)$ because $i_0 \in \Lambda$.

(iii) Clearly $\bigcup_{i\in\Lambda} U(\mu;t_i) \subseteq X$. For every $x \in X$ there exists $i(x) \in \Lambda$ such that $\mu(x) = t_{i(x)}$. This implies $x \in U(\mu;t_{i(x)}) \subseteq \bigcup_{i\in\Lambda} U(\mu;t_i)$, which proves (iii).

(iv) Since either $t_i \geq t_j$ or $t_i \leq t_j$ for all $i,j \in \Lambda$, we have either $U(\mu;t_i) \subseteq U(\mu;t_j)$ or $U(\mu;t_j) \subseteq U(\mu;t_i)$ for all $i,j \in \Lambda$.

(v) Suppose $\mathcal{B}$ contains all level sub-implicative ideals of $\mu$ and let $N$ be a sub-implicative ideal of $X$. If $\mu$ is constant on $N$, then we are done. Assume that $\mu$ is not constant on $N$. We distinguish the following two cases: (1) $N = X$ and (2) $N \subsetneq X$. For the case (1), we let $s = \inf\{t_i \mid i \in \Lambda\}$. Then $s \leq t_i$ and so $U(\mu;t_i) \subseteq U(\mu;s)$ for all $i \in \Lambda$. Note that $X = U(\mu;0) \in \mathcal{B}$ because $\mathcal{B}$ contains all level sub-implicative ideals of $\mu$. Hence there exists $j \in \Lambda$ such that $t_j \in \mathrm{Im}(\mu)$ and $U(\mu;t_j) = X$. It follows that $U(\mu;s) \supseteq U(\mu;t_j) = X$ so that $U(\mu;s) = U(\mu;t_j) = X$ because every level sub-implicative ideal of $\mu$ is a sub-implicative ideal of $X$. Now it is sufficient to show that $s = t_j$. If $s < t_j$, then there exists $k \in \Lambda$ such that $t_k \in \mathrm{Im}(\mu)$ and $s \leq t_k < t_j$. This implies that $U(\mu;t_k) \supsetneq U(\mu;t_j) = X$, a contradiction. Therefore $s = t_j$. If the case (2) holds, consider the restriction $\mu_N$ of $\mu$ to $N$. By Lemma 3.24, $\mu_N$ is a fuzzy sub-implicative ideal of $X$. Let $\Lambda_N = \{i \in \Lambda \mid \mu(y) = t_i \text{ for some } y \in N\}$ and $\mathcal{B}_N = \{U(\mu_N;t_i) \mid i \in \Lambda_N\}$. Noticing that $\mathcal{B}_N$ contains all level sub-implicative ideals of $\mu_N$, we conclude that there exists $z \in N$ such that $\mu_N(z) = \inf\{\mu_N(x) \mid x \in N\}$, which implies that $\mu(z) = \inf\{\mu(x) \mid x \in N\}$.

Conversely, assume that $\mu$ attains its infimum on all sub-implicative ideals of $X$. Let $U(\mu;t)$ be a level sub-implicative ideal of $\mu$. If $t = t_i$ for some $i \in \Lambda$, then clearly $U(\mu;t) \in \mathcal{B}$. Assume that $t \neq t_i$ for all $i \in \Lambda$. Then there does not exist $x \in X$ such that $\mu(x) = t$. Let $N = \{x \in X \mid \mu(x) > t\}$. Obviously, $0 \in N$. Let $x,y,z \in X$ be such that $((x^2*y)*(y*x))*z \in N$ and $z \in N$. Then $\mu(((x^2*y)*(y*x))*z) > t$ and $\mu(z) > t$. It follows from (FI1) that

$$\mu(y^2 * x) \geq \min\{\mu(((x^2 * y) * (y * x)) * z), \mu(z)\} > t$$

so that $y^2 * x \in N$. This shows that $N$ is a sub-implicative ideal of $X$. By hypothesis, there exists $y \in N$ such that $\mu(y) = \inf\{\mu(x) \mid x \in N\}$. Now $\mu(y) \in \mathrm{Im}(\mu)$ implies $\mu(y) = t_i$ for some $i \in \Lambda$. Hence we get $\inf\{\mu(x) \mid x \in N\} = t_i$. Obviously $t_i \geq t$, and so $t_i > t$ by assumption. Note that there does not exist $z \in X$ such that $t \leq \mu(z) < t_i$. It follows that $U(\mu;t) = U(\mu;t_i) \in \mathcal{B}$. This concludes the proof. $\square$

We have introduced the notion of fuzzy sub-implicative ideals and discussed its characterization and properties. This ideas could enable us to discuss the direct products of fuzzy sub-implicative ideals, interval-valued fuzzification of sub-implicative ideals, $\Omega$-fuzzification of sub-implicative ideals, and fuzzy sub-implicative ideals with operators.

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