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THE HYERS-ULAM STABILITY OF THE QUADRATIC FUNCTIONAL EQUATIONS ON ABELIAN GROUPS

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ABSTRACT. In this paper, we investigate the problem of stability of the quadratic functional equation $f(x+y+z)+f(x-y)+f(y-z)+f(x-z) = 3f(x)+3f(y)+3f(z)$ on abelian group.

1. Introduction

The problem of the stability of functional equations has originally been stated by S. M. Ulam [9]. In paper D. H. Hyers [5] has proved the stability of additive mapping.

Th. M. Rassias [7] gave a generalized solution to Ulam's problem for approximately linear mappings and he proved a new generalization of the Hyers-Ulam stability theorem when one considers the most general Hyers-Ulam sequence [8]. Let G_i, $i = 1, 2$ be groups. A function $f : G_1 \to G_2$ satisfying the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called the quadratic functional equation. It is well known that a function $f : E_1 \to E_2$ between vector spaces is quadratic if and only if there exists a unique symmetric function $B : E_1 \times E_1 \to E_2$, which is additive in x for each fixed y, such that $f(x) = B(x, x)$ for any $x \in E_1$ (see [1]). The Hyers-Ulam stability of the quadratic equation (1.1) on normed spaces has been studied in [4].

Consider the following functional equations:

$$f(x-y-z)+f(x)+f(y)+f(z) = f(x-y)+f(y+z)+f(z-x) \tag{1.2}$$

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and

$$(1.3)\quad f(x+y+z)+f(x-y)+f(y-z)+f(x-z)=3f(x)+3f(y)+3f(z).$$

Recently, the first-named author investigated the generalized Hyers-Ulam stability of the equation (1.2) and (1.3) on a normed space([2], [3]). The purpose of this note is to investigate the Hyers-Ulam stability of the equation (1.3) on *abelian groups.*

Let X be a commutative semigroup with zero in which the following law of cancellation holds:

$$(1.4)\qquad a+c=b+c \text{ implies } a=b \text{ for all } a,b,c\in X.$$

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers. A multiplication by nonnegative real scalars is defined on X as follows:

$$(1.5)\qquad \alpha(a+b)=\alpha a+\alpha b,$$

$$(1.6)\qquad \alpha a+\beta a=(\alpha+\beta)a,$$

$$(1.7)\qquad \alpha(\beta a)=(\alpha\beta)a,$$

$$(1.8)\qquad 1a=a,$$

for all $a,b\in X$ and $\alpha,\beta\in\mathbb{R}_+$.

Moreover let (X,d) be a metric space such that

$$(1.9)\qquad d(x+y,x+z)=d(y,z) \text{ for all } x,y,z\in X,$$

$$(1.10)\qquad d(tx,ty)=td(x,y) \text{ for all } x,y\in X,\ t\in\mathbb{R}_+.$$

We define a norm $\|\cdot\|$ on X by

$$(1.11)\qquad \|x\|:=d(x,0), x\in X.$$

A commutative semigroup $(X,\|\cdot\|)$ with zero and metric d satisfying the conditions (1.4)–(1.11) is called a quasi-normed space.

2. The results

Let E_1 be a group. Let $h : E_1 \times E_1 \times E_1 \to \mathbb{R}_+$ be a given function. We denote

$$\begin{aligned} H(x,y,z) &:= h(x,y,z) + h(x,0,0) + h(y,0,0) + h(z,0,0) + 2h(0,0,0) \\ K(x,y,z) &:= 3h(x,y,z) + h(x+y+z,0,0) + h(x-y,0,0) \\ &\quad + h(y-z,0,0) + h(x-z,0,0) + h(0,0,0) \end{aligned}$$

for all $x, y, z \in E_1$.

LEMMA 2.1. *Let E_1 be a group and E_2 a quasi-normed space. If the functions $F, G : E_1 \to E_2$ fulfill the inequality*

$$\begin{aligned} &d[F(x+y+z) + F(x-y) + F(y-z) + F(x-z), \\ &\quad G(x) + G(y) + G(z)] \le h(x,y,z) \end{aligned} \tag{2.1}$$

for all $x, y, z \in E_1$, then we have

$$\begin{aligned} &d[F(x+y+z) + F(x-y) + F(y-z) + F(x-z) + 5F(0), \\ &\quad 3F(x) + 3F(y) + 3F(z)] \le H(x,y,z) \end{aligned} \tag{2.2}$$

$$\begin{aligned} &d[G(x+y+z) + G(x-y) + G(y-z) + G(x-z) + 5G(0), \\ &\quad 3G(x) + 3G(y) + 3G(z)] \le K(x,y,z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in E_1$.

Proof. A simple inspection shows that for all $x, y, z \in E_1$,

$$\begin{aligned} &d[G(x+y+z) + G(x-y) + G(y-z) + G(x-z) + 5G(0), \\ &\quad 3G(x) + 3G(y) + 3G(z)] \\ &\le d[G(x+y+z) + G(x-y) + G(y-z) + G(x-z) + 2G(0), \\ &\quad 3F(x+y+z) + G(x-y) + G(y-z) + G(x-z) + F(0)] \\ &\quad + d[G(x-y) + 2G(0), 3F(x-y) + F(0)] \\ &\quad + d[G(y-z) + 2G(0), 3F(y-z) + F(0)] \\ &\quad + d[G(x-z) + 2G(0), 3F(x-z) + F(0)] \\ &\quad + [3F(x+y+z) + 3F(x-y) + 3F(y-z) + 3F(x-z), \\ &\qquad 3G(x) + 3G(y) + 3G(z)] \\ &\quad + d[4F(0), 3G(0)] \\ &\le h(x+y+z,0,0) + h(x-y,0,0) + h(y-z,0,0) \\ &\quad + h(x-z,0,0) + 3h(x,y,z) + h(0,0,0) \\ &= K(x,y,z), \end{aligned}$$

which implies that the inequality (2.3) holds. The inequality (2.2) can be proved quite similarly as (2.3). □

Let $\mathbb{N}$ denote the set of all natural numbers.

LEMMA 2.2. *Let E_1 be a group and E_2 a quasi-normed space. If the functions $F, G : E_1 \to E_2$ satisfy the inequality* (2.1), *then*

$$\begin{aligned} &d[F(k^n x) + (k^{2n} - 1)F(0), k^{2n}F(x)] \\ &\leq k^{2(n-1)} \sum_{i=1}^{k-1} \sum_{j=0}^{n-1} b_i H((k-i)k^j x, k^j x, 0)k^{-2j} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &d[G(k^n x) + (k^{2n} - 1)G(0), k^{2n}G(x)] \\ &\leq k^{2(n-1)} \sum_{i=1}^{k-1} \sum_{j=0}^{n-1} b_i K((k-i)k^j x, k^j x, 0)k^{-2j} \end{aligned} \tag{2.5}$$

for all $x \in E_1$ and $n, k \in \mathbb{N}$, where $k \geq 2$, $b_i = 2b_{i-1} + b_{i-2}$ ($b_1 = 1$, $b_2 = 2$).

Proof. Our proof depends on induction. For the case $n = 1$ in (2.4) we need to show the inequality

$$d[F(kx) + (k^2 - 1)F(0), k^2 F(x)] \leq \sum_{i=1}^{k-1} b_i H((k-i)x, x, 0) \tag{2.6}$$

for all $x \in E_1$ and all integer $k \geq 2$.

Putting $z = 0$ and $y = x$ in (2.2) yields

$$d[F(2x) + 3F(0), 4F(x)] \leq H(x, x, 0)$$

for all $x \in E_1$. Assume that (2.6) is true for some $k \geq 2$ and all $x \in E_1$.

Then we have

$$\begin{aligned}
&d[F((k+1)x) + ((k+1)^2-1)F(0), (k+1)^2F(x)] \\
&\le d[F((k+1)x) + F((k-1)x) + (k^2+2k)F(0), \\
&\qquad (k^2+2k-2)F(0) + 2F(kx) + 2F(x)] \\
&\quad + d[2F(kx) + 2F(x) + (k^2+2k-2)F(0), \\
&\qquad F((k-1)x) + (k+1)^2F(x)] \\
&\le H(kx,x,0) + d[2F(kx) + (k^2+2k-2)F(0), \\
&\qquad F((k-1)x) + (k^2+2k-1)F(x)] \\
&\le H(kx,x,0) + d[2F(kx) + 2(k^2-1)F(0), 2k^2F(x)] \\
&\quad + d[(k-1)^2F(x), F((k-1)x) + ((k-1)^2-1)F(0)] \\
&\le H(kx,x,0) + \sum_{i=1}^{k-1} 2b_iH((k-i)x,x,0) + \sum_{i=1}^{k-2} b_iH((k-i-1)x,x,0) \\
&\le H(kx,x,0) + 2b_1H((k-1)x,x,0) + \sum_{i=3}^{k} 2b_{i-1}H((k+1-i)x,x,0) \\
&\quad + \sum_{i=3}^{k} b_{i-2}H((k+1-i)x,x,0) \\
&= \sum_{i=1}^{k} b_iH((k+1-i)x,x,0).
\end{aligned}$$

Hence by induction, the inequality (2.6) is valid for all $k \ge 2$ and all $x \in E_1$. Now we assume that (2.4) holds true for all $n \ge 1$. For $n+1$ we get the inequality

$$\begin{aligned}
&d[F(k^{n+1}x) + (k^{2(n+1)}-1)F(0), k^{2(n+1)}F(x)] \\
&\le d[F(k^{n+1}x) + (k^2-1)F(0), k^2F(k^nx)] \\
&\qquad + k^2d[F(k^nx) + (k^{2n}-1)F(0), k^{2n}F(x)] \\
&\le \sum_{i=1}^{k-1} b_iH((k-i)k^nx, k^nx, 0) + k^{2n}\sum_{i=1}^{k-1}\sum_{j=0}^{n-1} b_iH((k-i)k^jx, k^jx, 0)k^{-2j} \\
&= k^{2n}\sum_{i=1}^{k-1}\sum_{j=0}^{n} b_iH((k-i)k^jx, k^jx, 0)k^{-2j}.
\end{aligned}$$

Consequently this means that (2.4) is proved by induction. The inequality (2.5) can be proved by the same way as above. □

Our main result is as follows:

THEOREM 2.3. *Let E_1 be an abelian group and E_2 a Banach space. Let the functions $F, G : E_1 \to E_2$ satisfy the inequality*

$$\|F(x+y+z) + F(x-y) + F(y-z) + F(x-z) - G(x) - G(y) - G(z)\| \le h(x,y,z)$$

for all $x, y, z \in E_1$. Suppose that the series

$$\sum_{j=0}^{\infty} h((k-i)k^j x, k^j x, 0)k^{-2j} \quad \text{and} \quad \sum_{j=0}^{\infty} h(k^j x, 0, 0)k^{-2j} \tag{2.7}$$

are convergent for all $x \in E_1$ and the condition

$$\liminf_{n\to\infty} h(k^n x, k^n y, k^n z)k^{-2n} = 0 \tag{2.8}$$

for all $x, y, z \in E_1$, then there exists exactly one quadratic function $A : E_1 \to E_2$ such that

$$\|A(x) + F(0) - F(x)\| \le k^{-2} \sum_{i=1}^{k-1} \sum_{j=0}^{\infty} b_i H((k-i)k^j x, k^j x, 0)k^{-2j} \tag{2.9}$$

and

$$\|3A(x) + G(0) - G(x)\| \le k^{-2} \sum_{i=1}^{k-1} \sum_{j=0}^{\infty} b_i K((k-i)k^j x, k^j x, 0)k^{-2j} \tag{2.10}$$

for all $x \in E_1$.

If moreover, E_1 is a linear topological space and F is measurable (i.e., $F^{-1}(U)$ is a Borel set in E_1 for every open set U in E_1) or the function $t \to F(tx)$ is continuous for each fixed $x \in E_1$, then

$$A(tx) = t^2 A(x) \tag{2.11}$$

for all $x \in E_1$, $t \in \mathbb{R}$.

Proof. Let $A_n(x) := k^{-2n}F(k^n x)$ for all $x \in E_1$ and $n \in \mathbb{N}$. We shall prove that $\{A_n(x)\}$ is a Cauchy sequence for all $x \in E_1$. In fact, by Lemma 2.2, (1.10) and (2.7) we have for $n > r$ and $x \in E_1$,

$$\begin{aligned}
&\|A_n(x), A_r(x)\| \\
&= k^{-2n}\|F(k^n x) - k^{2(n-r)}F(k^r x)\| \\
&\leq k^{-2r}\|F(0)\| + k^{-2n}\|F(k^{n-r}k^r x) \\
&\qquad + (k^{2(n-r)} - 1)F(0) - k^{2(n-r)}F(k^r x)\| \\
&\leq k^{-2r}\|F(0)\| + k^{-2}\sum_{i=1}^{k-1}\sum_{j=0}^{n-r-1} b_i H((k-i)k^{j+r}x, k^{j+r}x, 0)k^{-2(j+r)} \\
&= k^{-2r}\|F(0)\| + k^{-2}\sum_{i=1}^{k-1}\sum_{j=r}^{n-1} b_i H((k-i)k^j x, k^j x, 0)k^{-2j} \\
&\to 0 \quad \text{as} \quad r \to \infty.
\end{aligned}$$

Since E_2 is complete, there exists the limit

$$A(x) := \lim_{n\to\infty} A_n(x) \tag{2.12}$$

for all $x \in E_1$. We can also check that $\{k^{-2n}G(k^n x)\}$ is a Cauchy sequence by the similar method as above. Now we claim that

$$3A(x) = \lim_{n\to\infty} k^{-2n}G(k^n x) \tag{2.13}$$

for all $x \in E_1$. By (2.1) and (1.11), we obtain that

$$\begin{aligned}
&\|3A(x) - k^{-2n}G(k^n x)\| \\
&\leq \|3A(x) - 3\cdot k^{-2n}F(k^n x)\| \\
&\quad + \|3\cdot k^{-2n}F(k^n x) + k^{-2n}F(0) - k^{-2n}G(k^n x) + 2k^{-2n}G(0)\| \\
&\quad + \|k^{-2n}G(k^n x) + 2k^{-2n}G(0) - k^{-2n}G(k^n x)\| + k^{-2n}\|F(0)\| \\
&\leq 3\|A(x) - A_n(x)\| + k^{-2n}h(k^n x, 0, 0) + 2k^{-2n}\|G(0)\| + k^{-2n}\|F(0)\|,
\end{aligned}$$

and hence from (2.8) and (2.12), it follows that the relation

$$\lim_{n\to\infty} \|3A(x) - k^{-2n}G(k^n x)\| = 0$$

for all $x \in E_1$, which gives (2.13).

The function A is quadratic. In fact,

$$\begin{aligned}&\|A_n(x+y+z)+A_n(x-y)+A_n(y-z)+A_n(x-z)\\&-k^{-2n}G(k^nx)+k^{-2n}G(k^ny)+k^{-2n}G(k^nz)\|\\&\le k^{-2n}h(k^nx,k^ny,k^nz),\end{aligned}$$

and so letting $n \to \infty$ in view of (2.8) and (2.13) we obtain the equality

$$\begin{aligned}&A(x+y+z)+A(x-y)+A(y-z)+A(x-z)\\&=3A(x)+3A(y)+3A(z)\end{aligned}$$

for all $x, y, z \in E_1$.

By (2.4) and (2.5), we can easily verify the validity of the inequality (2.9) and (2.10). To prove the uniqueness assume that there exist two quadratic functions $C_m : E_1 \to E_2, m = 1, 2$ such that

$$\begin{aligned}&\|C_m(x)+F(0)-F(x)\|\\&\le k^{-2}a_m\sum_{i=1}^{k-1}\sum_{j=0}^{\infty}b_iH((k-i)k^jx,k^jx,0)k^{-2j},\end{aligned}$$

where $x \in E_1$ and $a_m \geq 0$, $(m = 1, 2)$ are real constants.

Since C_m are quadratic functions, we see that for $m = 1, 2$

$$C_m(k^nx)=k^{2n}C_m(x)$$

for all $x \in E_1, n \in \mathbb{N}$. Hence we obtain for $x \in E_1$

$$\begin{aligned}&k^{-2n}\|C_1(k^nx)-C_2(k^nx)\|\\&\le k^{-2n}\left(\|C_1(k^nx)+F(0)-F(k^nx)\|+\|F(k^nx)-C_2(k^nx)+F(0)\|\right)\\&\le(a_1+a_2)k^{-2}\sum_{i=1}^{k-1}\sum_{j=0}^{\infty}b_iH((k-i)k^{j+n}x,k^{j+n}x,0)k^{-2(j+n)}\\&=(a_1+a_2)k^{-2}\sum_{i=1}^{k-1}\sum_{j=n}^{\infty}b_iH((k-i)k^jx,k^jx,0)k^{-2j}.\end{aligned}$$

In view of the convergence of the series (2.7), the righthand side of the last inequality can be made as small as we wish as taking n sufficiently large. Hence $C_1(x) = C_2(x)$ for all $x \in E_1$.

To end the proof, let L be any continuous linear functional defined on the space E_2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be given by

$$\varphi(t) := L[A(tx)]$$

for all $x \in E_1$, $t \in \mathbb{R}$, where x is fixed. Then φ is a quadratic function and, moreover, as the pointwise limit of the sequence

$$\varphi_n(t) = k^{-2n} L[F(k^n tx)], \quad t \in \mathbb{R}$$

is also measurable and hence has the form $\varphi(t) = t^2 \varphi(1)$ for $t \in \mathbb{R}$ (see [6]). Therefore for all $t \in \mathbb{R}$ and all $x \in E_1$

$$L[A(tx)] = \varphi(t) = t^2 \varphi(1) = L[t^2 A(x)],$$

which implies the condition (2.11). This completes the proof. □

The next lemma and theorem are to present the results concerning the case $F(0) \neq 0$.

LEMMA 2.4. *Let E_1 be a group divisible by k and E_2 a quasi-normed space. If $F, G : E_1 \to E_2$ satisfy the inequality* (2.1), *then*

$$\begin{aligned} &[F(x) + (k^{2n} - 1)F(0), k^{2n} F(k^{-n} x)] \\ &\le k^{-2} \sum_{i=1}^{k-1} \sum_{j=1}^{n} b_i H((k-i)k^{-j} x, k^{-j} x, 0) k^{2j} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &d[G(x) + (k^{2n} - 1)G(0), k^{2n} G(k^{-n} x)] \\ &\le k^{-2} \sum_{i=1}^{k-1} \sum_{j=1}^{n} b_i K((k-i)k^{-j} x, k^{-j} x, 0) k^{2j} \end{aligned} \tag{2.15}$$

for all $x \in E_1$ and $n, k \in \mathbb{N}$, where $k \ge 2$, $b_i = 2b_{i-1} + b_{i-2}$ ($b_1 = 1$, $b_2 = 2$).

Proof. Setting $x = k^{-n} t$ into (2.4) and (2.5) it is immediate to obtain (2.14) and (2.15). □

THEOREM 2.5. *Let E_1 be an abelian group divisible by k, $k \geq 2$ and E_2 a complete quasi-normed space. Suppose that the series*

$$\sum_{j=1}^{\infty} h((k-i)k^{-j}x, k^{-j}x, 0)k^{2j} \text{ and } \sum_{j=1}^{\infty} h(k^{-j}x, 0, 0)k^{2j}$$

are convergent for all $x \in E_1$ and the condition

$$\liminf_{n\to\infty} h(k^{-n}x, k^{-n}y, k^{-n}z)k^{2n} = 0$$

for all $x, y, z \in E_1$, then there exists exactly one quadratic function $A : E_1 \to E_2$ such that

$$d[B(x), F(x)] \leq k^{-2} \sum_{i=1}^{k-1} \sum_{j=1}^{\infty} b_i H((k-i)k^{-j}x, k^{-j}x, 0)k^{2j} \tag{2.16}$$

and

$$d[B(x), G(x)] \leq k^{-2} \sum_{i=1}^{k-1} \sum_{j=1}^{\infty} b_i K((k-i)k^{-j}x, k^{-j}x, 0)k^{2j} \tag{2.17}$$

for all $x \in E_1$.

Proof. Since the series $\sum_{j=0}^{\infty} h(0,0,0)k^{2j}$ is convergent, it follows that $h(0,0,0) = 0$, and from (2.1) we obtain $F(0) = G(0) = 0$. Put $B_n(x) := k^{2n}F(k^{-n}x)$ for all $x \in E_1$ and $n \in \mathbb{N}$. In view of Lemma 2.4 we can prove that for all $x \in E_1$ the sequence $\{B_n(x)\}$ is a Cauchy sequence. Let us define

$$B(x) := \lim_{n\to\infty} B_n(x) \tag{2.18}$$

for all $x \in E_1$. Then it is easy to verify that

$$3B(x) = \lim_{n\to\infty} k^{2n}G(k^{-n}x) \tag{2.19}$$

for all $x \in E_1$. Since

$$\begin{aligned} &d[B_n(x+y+z) + B_n(x-y) + B_n(y-z) + B_n(x-z), \\ &\qquad k^{2n}G(k^{-n}x) + k^{2n}G(k^{-n}y) + k^{2n}G(k^{-n}z)] \\ &\leq k^{2n}h(k^{-n}x, k^{-n}y, k^{-n}z), \end{aligned}$$

according to (2.18) and (2.19) we obtain the relation

$$\begin{aligned} &B(x+y+z)+B(x-y)+B(y-z)+B(x-z) \\ &= 3B(x)+3B(y)+3B(z) \end{aligned}$$

for all $x, y, z \in E_1$ by taking limits of both sides as $n \to \infty$. Hence B is a quadratic function. Taking into account (2.14) and (2.15) we get immediately the estimations (2.16) and (2.17). To prove the uniqueness we can proceed similarly as in the proof of Theorem 2.3. We complete the proof. □

REMARK. If h is constant we have Hyers-Ulam stability, whereas for $h(x,y,z) = \|x\|^p + \|y\|^p + \|z\|^p$, $x, y, z \in E$ (E : normed space) Rassias type of stability.

References

[1] J. Aczél, and J. Dhombres, *Functional equations in several variables*, Cambridge Univ. Press, Cambridge, 1989.
[2] J.-H. Bae, *On a Ulam stability problem of a Euler-Lagrange functional equations*, Korean. J. Comput. & Appl. Math. (Series A) **7** (2000), 693–701.
[3] J.-H. Bae and I.-S. Chang, *On the Ulam stability problem of a quadratic functional equation*, Korean. J. Comput. & Appl. Math. (Series A) **8** (2001), 561–567.
[4] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.
[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.
[6] S. Kurepa, *On the quadratic functional*, Publ. Inst. Math. Acad. Serbe. Sci. Beograd **13** (1959), 57–72.
[7] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.
[8] Th. M. Rassias and P. Semrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.
[9] S. M. Ulam, *Problems in Modern Mathematics.*, Chap. VI, Wiley, New York, 1960.

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