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THE RANGE OF DERIVATIONS ON BANACH ALGEBRAS

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Abstract. In this paper we show that if D is a continuous linear Jordan derivation on a Banach algebra A satisfying $[[D(x^n), x^n], x^n] \in \mathrm{rad}(A)$ for a positive integer n and for all $x \in A$, then D maps A into $\mathrm{rad}(A)$.

1. Introduction

Throughout this paper R will represent an associative ring with center Z and A an associative algebra over a complex field $\mathbb{C}$. $\mathbb{Z}$ will represent the set of all integers and $\mathbb{Z}^+$ the set of all positive integers. The (Jacobson) radical of A is the intersection of all primitive ideals of A and denoted by $\mathrm{rad}(A)$. A ring R is said to be n-torsion free if $nx = 0$, $x \in R$ implies $x = 0$. The commutator $xy - yx$ will be denoted by $[x, y]$, and we make extensive use of the basic identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring R is prime if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies that $a = 0$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation D is inner if there exists an $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. An additive mapping D from R to R is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A mapping F from R to R is said to be commuting on R if $[F(x), x] = 0$ holds for all $x \in R$, and is said to be centralizing on R if $[F(x), x] \in Z$ holds for all $x \in R$. Obviously, every derivation is a Jordan derivation. The converse is in general not true. Brešar showed that every Jordan derivation on

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a 2-torsion free semiprime ring is a derivation [1]. There has been considerable interest in commuting, centralizing, and related mappings in prime and semiprime rings. K. W. Jun and B. D. Kim [7] have obtained the algebraic condition that every derivation on a Banach algebra maps into its radical. In this paper we shall give the various algebraic conditions on prime ring that every derivation on the ring is zero and using these results, we show that every continuous linear Jordan derivation with some conditions on a Banach algebra maps into its radical.

2. Preliminaries

We list a few more or less well-known results which will be needed in the sequel.

REMARK. R will represent a prime ring with center Z and extended centroid C.

1. Suppose that the elements a_i, b_i in the central closure of R satisfy $\Sigma a_i y b_i = 0$. If $b_i \neq 0$ for some i then the a_i's are C-dependent.
2. The elements a, b in the central closure of R are C-dependent if and only if $ayb = bya$ holds for all $y \in R$.

The explanation of the notions of the extended centroid and the central closure of a prime ring, as well as the proof of Remark, can be found in [5, pp. 20–31].

The following lemma is due to L. O. Chung and J. Luh [3].

LEMMA 2.1. *Let R be a $n!$-torsion free ring. Suppose that $t_1, t_2, \cdots, t_n \in R$ satisfy $kt_1 + k^2t_2 + \cdots + k^n t_n = 0$ for $k = 1, 2, \cdots, n$. Then $t_i = 0$ for all i.*

LEMMA 2.2 ([8]). *Let R be a noncommutative prime ring of $(n+1)!$-torsion free for a positive integer n. If a derivation D satisfies either $[D(x), x]x^n = 0$ or $x^n[D(x), x] = 0$ for all $x \in R$, then in both cases we have $D = 0$.*

Posner [10] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Recently, Vukman [12] has proved that in case there exists a nonzero derivation

$D : R \to R$, where R is a prime ring of characteristic different from 2 and 3, such that the mapping $x \mapsto [D(x), x]$ is commuting on R, R is commutative. We are going to generalize the result of Vukman [12, Theorem 1] as follows.

LEMMA 2.3. *Let R be a noncommutative prime ring of $(n+3)!$-torsion free for a positive integer n. Suppose that there exists a Jordan derivation $D : R \to R$ such that either $[[D(x), x], x]x^n = 0$ or $x^n[[D(x), x], x] = 0$ holds for all $x \in R$. Then we have $D = 0$ on R.*

Proof. We introduce a symmetric biadditive mapping $F : R \times R \to R$ by the relation $F(x, y) = [D(x), y] + [D(y), x]$ for all $x, y \in R$. A routine calculation shows that the relations $F(xy, z) = F(x, z)y + xF(y, z) + D(x)[y, z] + [x, z]D(y)$ is fulfilled for all $x, y, z \in R$. Let us write $f(x)$ for $F(x, x)$. Thus $f(x) = 2[D(x), x]$ for all $x \in R$. The mapping f satisfies the relation $f(x + \lambda y) = f(x) + \lambda^2 f(y) + 2\lambda F(x, y)$ for all $x, y \in R$ and $\lambda \in \mathbb{Z}$. We prove the lemma under the assumption $[[D(x), x], x]x^n = 0$. The other case is similarly proved. Now the assumption of the lemma can be written in the form

$$[f(x), x]x^n = 0, \qquad x \in R. \tag{2.1}$$

Replacing x by $x + \lambda y$ in (2.1), we get

$$0 = \lambda a_1(x, y) + \lambda^2 a_2(x, y) + \cdots + \lambda^{n+2} a_{n+2}(x, y)$$

for all $x, y \in R$, $\lambda \in \mathbb{Z}$, where $a_i(x, y)$ denotes the sum of these terms in which y appears as a term in the product i times. Applying Lemma 2.1, we have $a_1(x, y) = 0$ for all $x, y \in R$. That is,

$$\begin{aligned} 0 = {} & [f(x), x](x^{n-1}y + x^{n-2}yx + \cdots + xyx^{n-2} + yx^{n-1}) \\ & + ([f(x), y] + 2[F(x, y), x])x^n, \quad x,\ y \in R. \end{aligned} \tag{2.2}$$

Let us replace y by yx in (2.2). Then, by (2.1) and (2.2) we get

$$0 = 3[y, x]f(x)x^n + 2[[y, x], x]D(x)x^n, \quad x,\ y \in R, \tag{2.3}$$

which reduces to

$$\begin{aligned} 0 = {} & y(3xD(x)x^{n+1} - 2x^2D(x)x^n) \\ & + xy(xD(x)x^n - 3D(x)x^{n+1}) + x^2yD(x)x^n \end{aligned}$$

for all $x, y \in R$. The substitution $x^k y$ for y in (2.3) leads to

$$\begin{aligned} 0 = {} & x^k y(3xD(x)x^{n+1} - 2x^2D(x)x^n) + x^{k+1}y(xD(x)x^n \\ & - 3D(x)x^{n+1}) + x^{k+2}yD(x)x^n, \quad x, y \in R, \ k \in \mathbb{Z}^+. \end{aligned} \tag{2.4}$$

Substituting xy for y in (2.2) and using the above relation (2.2), we obtain that

$$\begin{aligned} 0 = {} & [f(x), x](x^n y + x^{n-1}yx + \cdots + xyx^{n-1}) \\ & + ([f(x), xy] + 2[F(x, xy), x])x^n \\ = {} & [f(x), x](x^n y + x^{n-1}yx + \cdots + xyx^{n-1}) \\ & + 3f(x)[y, x]x^n + 3[f(x), x]yx^n + 2D(x)[[y, x], x]x^n \\ & - x[f(x), x](x^{n-1}y + x^{n-2}yx + \cdots + yx^{n-1}) \end{aligned} \tag{2.5}$$

for all $x, y \in R$. Putting $y = x^n y$ in (2.5), we obtain by assumption, $0 = \ 3f(x)x^n[y, x]x^n + 2D(x)x^n[[y, x], x]x^n$, which leads to

$$\begin{aligned} 0 = {} & (3xD(x)x^{n+1} - 2D(x)x^{n+2})yx^n \\ & + (D(x)x^{n+1} - 3xD(x)x^n)yx^{n+1} + D(x)x^n yx^{n+2} \end{aligned} \tag{2.6}$$

for all $x, y \in R$.

Suppose $D \neq 0$. Then there exists an $x \in R$ such that $D(x)x^n \neq 0$. Let $D(x)x^n \neq 0$. We set conveniently

$$\begin{aligned} a = {} & 3xD(x)x^{n+1} - 2x^2D(x)x^n, \ b = xD(x)x^n - 3D(x)x^{n+1}, \\ c = {} & D(x)x^n, \ a_k = x^k, k \in \mathbb{Z}^+. \end{aligned}$$

First, we claim that if a_n, a_{n+1} are C-dependent, then $[D(x), x]x^{n+2} = 0$. Observe that from (2.6) a_n, a_{n+1}, a_{n+2} are C-dependent by Remark 1. Thus we have by setting $n-1$ instead of k in (2.4),

$$0 = a_{n-1}ya + a_n yb + a_{n+1}yc, \qquad y \in R. \tag{2.7}$$

Substituting $za_n y$ for y in (2.7), we obtain that $0 = a_{n-1}za_n ya + a_n za_n yb + a_{n+1}za_n yc$ for all $y, z \in R$. But on the other hand we see from (2.7) that $a_n za_n yb = -a_n za_{n-1}ya - a_n za_{n+1}yc$. Comparing the last two relations, we arrive at

$$0 = (a_{n-1}za_n - a_n za_{n-1})ya + (a_{n+1}za_n - a_n za_{n+1})yc$$

for all $y,\ z \in R$, which gives

$$0 = (a_{n-1}za_n - a_nza_{n-1})ya, \quad y, z \in R, \tag{2.8}$$

since a_n, a_{n+1} are C-dependent (*c.f.* Remark 2). Now it follows from (2.8) that we have either that a_{n-1}, a_n are C-dependent or that $a = 0$ by primeness of R. If a_{n-1}, a_n are C-dependent, then from (2.4)

$$0 = a_{n-2}ya + a_{n-1}yb + a_nyc, \quad y \in R. \tag{2.9}$$

Replacing $za_{n-1}y$ for y in (2.9), we obtain

$$0 = a_{n-2}za_{n-1}ya + a_{n-1}za_{n-1}yb + a_nza_{n-1}yc, \qquad y,\ z \in R.$$

On the other hand, we see from (2.9) that $a_{n-1}\ za_{n-1}\ yb = -a_{n-1}\ za_{n-2}\ ya - a_{n-1}\ za_nyc$. Comparing the last two relations, we arrive at $0 = (a_{n-2}za_{n-1} - a_{n-1}za_{n-2})ya + (a_nza_{n-1} - a_{n-1}za_n)yc$, which gives

$$0 = (a_{n-2}za_{n-1} - a_{n-1}za_{n-2})ya, \quad y, z \in R, \tag{2.10}$$

since a_{n-1}, a_n are C-dependent. Now it follows from (2.10) that we have either that a_{n-2}, a_{n-1} are C-dependent or that $a = 0$ by primeness of R. Continuing the process, we have either that a_1, a_2 are C-dependent or that $a = 0$ by primeness of R. If a_1, a_2 are C-dependent, then from (2.3)

$$0 = ya + a_1yb + a_2yc, \qquad y \in R. \tag{2.11}$$

Replacing za_1y for y in (2.9), we obtain that $0 = za_1ya + a_1za_1yb + a_2za_1yc$, for all $y, z \in R$. But on the other hand we see from (2.11) that $a_1za_1yb = -a_1zya - a_1za_2yc$. Comparing the last two relations, we arrive at $0 = (za_1 - a_1z)ya + (a_2za_1 - a_1za_2)yc$, which gives

$$0 = (za_1 - a_1z)ya, \quad y,\ z \in R, \tag{2.12}$$

since a_1, a_2 are C-dependent. From (2.12), we have either $a_1 \in Z$ or $a = 0$ by primeness of R. If $a = 0$, since $[D(x), x]x^{n+1} = x[D(x), x]x^n$ by assumption, we obtain

$$\begin{aligned} 0 &= [3xD(x)x^{n+1} - 2x^2D(x)x^n, x] \\ &= 3x[D(x), x]x^{n+1} - 2x[D(x), x]x^{n+1} \\ &= 3[D(x), x]x^{n+2} - 2[D(x), x]x^{n+2} \\ &= [D(x), x]x^{n+2}, \end{aligned}$$

which is also true when $a_1 \in Z$. We have thus proved that $[D(x), x]\, x^{n+2} = 0$ in case that a_n, a_{n+1} are C-dependent.

Now let us consider the case that a_n, a_{n+1} are C-independent. Since a_n, a_{n+1}, a_{n+2} are C-dependent from (2.6), we have $a_{n+2} = \lambda a_n + \mu a_{n+1}$ for some $\lambda, \mu \in C$. From (2.4) we know that

$$\begin{aligned} 0 =& a_n y a + a_{n+1} y b + a_{n+2} y c = a_n y a + a_{n+1} y b + (\lambda a_n + \mu a_{n+1}) y c \\ =& a_n y(a + \lambda c) + a_{n+1} y(b + \mu c) \qquad \text{for all } y \in R. \end{aligned}$$

Since a_n and a_{n+1} are C-independent, $a + \lambda c = 0 = b + \mu c$ by Remark 2. So we have

$$b + \mu c = a_1 c - 3ca_1 + \mu c = 0. \tag{2.13}$$

On the other hand, we get from (2.6)

$$\begin{aligned} 0 =& (3a_1 c a_1 - 2ca_2) y a_n + (ca_1 - 3a_1 c) y a_{n+1} + cy(\lambda a_n + \mu a_{n+1}) \\ =& (3a_1 c a_1 - 2ca_2 + \lambda c) y a_n + (ca_1 - 3a_1 c + \mu c) y a_{n+1}, \quad y \in R. \end{aligned}$$

Since a_n and a_{n+1} are C-independent, we obtain by Remark 2

$$0 = ca_1 - 3a_1 c + \mu c. \tag{2.14}$$

Subtracting (2.13) from (2.14), we arrive at $0 = [c, a_1]$, which yields $0 = [D(x), x]x^n$. We have thus proved that $[D(x), x]x^n = 0$ in case that a_n, a_{n+1} are C-independent. As a result, if $D(x)x^n \neq 0$, one obtains $0 = [D(x), x]x^{n+2}$ in any case. Obviously, $D(x)x^n = 0$ implies that $[D(x), x]x^{n+2} = 0$. Hence $[D(x), x]x^{n+2} = 0$ for all $x \in R$. By Lemma 2.2, $D = 0$. The proof of the lemma is complete. □

3. Main results

Using the previous results for the ring theory, we obtain the main theorems for the Banach algebra theory.

THEOREM 3.1. *Let D be a continuous linear Jordan derivation on a Banach algebra A such that either $[[D(x), x], x]x^n \in \mathrm{rad}(A)$ or $x^n[[D(x), x], x] \in \mathrm{rad}(A)$ for a positive integer n and for all $x \in A$. Then $D(A) \subseteq \mathrm{rad}(A)$.*

Proof. Let P be a primitive ideal of A. Since D is continuous, by [11, Lemma 3.2] we have $D(P) \subseteq P$. Thus we can define a Jordan derivation D_P on A/P by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$ for all $x \in A$. The factor algebra A/P is prime and semisimple since P is a primitive ideal. Thus D_P is a derivation by Brešar's result [1]. Johnson [6] has proved that every derivation on a semisimple Banach algebra is continuous. Combining this result with Singer-Wermer theorem, we obtain that there are no nonzero derivations on a commutative semisimple Banach algebra. Hence in case A/P is commutative, we have $D_P = 0$. It remains to show that $D_P = 0$ in case A/P is noncommutative. The assumption of the theorem gives either $[[D_P(\hat{x}), \hat{x}], \hat{x}]\hat{x}^n = 0$ or $\hat{x}^n[[D_P(\hat{x}), \hat{x}], \hat{x}] = 0$, $\hat{x} \in A/P$. Then the assumption of Lemma 2.3 is fulfilled and thus we have $D_P = 0$. In any case $D_P = 0$. Hence we see that $D(A) \subseteq P$. Since P is any primitive ideal, the result follows. This completes the proof. □

One can easily show the following relations by induction for a positive integer n.

$$\begin{aligned} D(x^n) &= D(x)x^{n-1} + xD(x)x^{n-2} + \cdots + x^{n-1}D(x), \\ [D(x), x^n] &= [D(x), x]x^{n-1} + x[D(x), x]x^{n-2} \\ &\quad + \cdots + x^{n-1}[D(x), x]. \end{aligned} \tag{3.1}$$

Combining the above two relations, we have the following two equations.

$$\begin{aligned} [D(x^n), x^n] = {} & [D(x), x]x^{2n-2} + 2x[D(x), x]x^{2n-3} \\ & + \cdots + nx^{n-1}[D(x), x]x^{n-1} \\ & + (n-1)x^n[D(x), x]x^{n-2} + \cdots \\ & + 2x^{2n-3}[D(x), x]x + x^{2n-2}[D(x), x], \end{aligned} \tag{3.2}$$

$$\begin{aligned} [[D(x), x], x^n] = {} & [[D(x), x], x]x^{n-1} \\ & + x[[D(x), x], x]x^{n-2} \\ & + \cdots + x^{n-1}[[D(x), x], x]. \end{aligned} \tag{3.3}$$

Using the relations (3.2) and (3.3), we obtain the following equation, which is needed to prove the next theorem.

$$
\begin{aligned}
[[D(x^n),x^n],x^n] = {} & [[D(x),x],x^n]x^{2n-2} + 2x[[D(x),x],x^n]x^{2n-3} \\
& + \cdots + nx^{n-1}[[D(x),x],x^n]x^{n-1} \\
& + (n-1)x^n[[D(x),x],x^n]x^{n-2} \\
& + \cdots + 2x^{2n-3}[[D(x),x],x^n]x \\
& + x^{2n-2}[[D(x),x],x^n].
\end{aligned} \tag{3.4}
$$

THEOREM 3.2. *Let D be a continuous linear Jordan derivation on a Banach algebra A such that $[[D(x^n),x^n],x^n] \in \operatorname{rad}(A)$ for a positive integer n and for all $x \in A$. Then D maps A into its radical.*

Proof. Let P be a primitive ideal of A. Since D is continuous, by [11, Lemma 3.2] we have $D(P) \subseteq P$. Thus we can define a Jordan derivation D_P on A/P by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$ for all $x \in A$. As in the proof of Theorem 3.1, we can see that in case A/P is commutative, we have $D_P = 0$. We claim that $D_P = 0$ in case A/P is noncommutative. Then we will see that $D(A) \subseteq P$ in any case and so the result of the theorem follows since P is any primitive ideal. Now the assumption of the theorem gives $[[D_P(\hat{x}^n),\hat{x}^n],\hat{x}^n] = 0$, $\hat{x} \in A/P$. Thus without any loss of generality we may assume that A is noncommutative primitive Banach algebra and the condition $[[D(x^n),x^n],x^n] = 0$ holds for all $x \in A$. We use the notations f, F in Lemma 2.3. The assumption of the theorem can now be written in the form by virtue of (3.3) and (3.4)

$$
\begin{aligned}
0 = {} & \sum_{k=1}^{n} \frac{k(k+1)}{2} x^{k-1}[f(x),x]x^{3n-k-2} + \sum_{k=0}^{n-3}\{(k+2) + \cdots \\
& + n + \cdots + (n-k-1)\}x^{n+k}[f(x),x]x^{2n-k-3} \\
& + \sum_{k=n}^{1} \frac{k(k+1)}{2} x^{3n-k-2}[f(x),x]x^{k-1}
\end{aligned} \tag{3.5}
$$

for all $x \in A$, where we write $x^0 = 1$ conveniently. The above relation contains the sum of these $3n-2$ terms. Replacing x by $x + \lambda y$ in (3.5) and expanding the resulting equation, we obtain

$$
0 = \lambda a_1(x,y) + \lambda^2 a_2(x,y) + \cdots + \lambda^{3n-1} a_{3n-1}(x,y),
$$

$\lambda \in \mathbb{Z},\ x, y \in A$, where $a_i(x, y)$ denotes the sum of these terms in which y appears as a term in the product i times. Applying Lemma 2.1, we have $0 = a_1(x, y)$ for all $x, y \in A$. Arranging the resulting relation, we obtain

$$
\begin{aligned}
0 &= [f(x), x](x^{3n-4}y + x^{3n-5}yx + \cdots + yx^{3n-4}) \\
&+ ([f(x), y] + 2[F(x, y), x])x^{3n-3} \\
&+ \sum_{k=2}^{n} \frac{k(k+1)}{2} \Big\{ x^{k-1}[f(x), x](x^{3n-k-3}y + x^{3n-k-4}yx \\
&\qquad + \cdots + yx^{3n-k-3}) \\
&\qquad + x^{k-1}([f(x), y] + 2[F(x, y), x])x^{3n-k-2} \\
&\qquad + (x^{k-2}y + x^{k-3}yx + \cdots + yx^{k-2}) \\
&\qquad \cdot [f(x), x]x^{3n-k-2} \Big\} \\
&+ \sum_{k=0}^{n-3} \{(k+2) + \cdots + n + \cdots + (n-k-1)\} \\
&\quad \cdot \Big\{ x^{n+k}[f(x), x](x^{2n-k-4}y + x^{2n-k-5}yx + \cdots + yx^{2n-k-4}) \\
&\qquad + x^{n+k}([f(x), y] + 2[F(x, y), x])x^{2n-k-3} \\
&\qquad + (x^{n+k-1}y + x^{n+k-2}yx + \cdots + yx^{n+k-1}) \\
&\qquad \cdot [f(x), x]x^{2n-k-3} \Big\} \\
&+ \sum_{k=n}^{2} \frac{k(k+1)}{2} \Big\{ x^{3n-k-2}[f(x), x] \\
&\qquad \cdot (x^{k-2}y + x^{k-3}yx + \cdots + yx^{k-2}) \\
&\qquad + x^{3n-k-2}([f(x), y] + 2[F(x, y), x])x^{k-1} \\
&\qquad + (x^{3n-k-3}y + x^{3n-k-4}yx + \cdots + yx^{3n-k-3}) \\
&\qquad \cdot [f(x), x]x^{k-1} \Big\} \\
&+ x^{3n-3}([f(x), y] + 2[F(x, y), x]) \\
&+ (x^{3n-4}y + x^{3n-5}yx + \cdots + yx^{3n-4})[f(x), x]
\end{aligned}
\tag{3.6}
$$

for all $x, y \in A$. Let us put x^2 instead of y in (3.6). Putting in order the resulting long relation with the use of identity $[f(x), x^2] + 2[F(x, x^2), x] =$

$3[f(x),x]x+3x[f(x),x]$, we get

$$\begin{aligned}0 = {} & 3n\sum_{k=1}^{n}\frac{k(k+1)}{2}x^{k-1}[f(x),x]x^{3n-k-1}\\ & + 3n\sum_{k=0}^{n-2}\{(k+1)+\cdots+n\\ & \qquad +\cdots+(n-k-1)\}x^{n+k}[f(x),x]x^{2n-k-2}\\ & + 3n\sum_{k=n}^{1}\frac{k(k+1)}{2}x^{3n-k-1}[f(x),x]x^{k-1}\end{aligned}$$

for all $x\in A$, which can be written in the form

$$\begin{aligned}0 = {} & 3n\Big\{\sum_{k=1}^{n}\frac{k(k+1)}{2}x^{k-1}[f(x),x]x^{3n-k-2}\\ & + \sum_{k=0}^{n-3}\{(k+2)+\cdots+n\\ & \qquad +\cdots+(n-k-1)\}x^{n+k}[f(x),x]x^{2n-k-3}\\ & + \sum_{k=n}^{1}\frac{k(k+1)}{2}x^{3n-k-2}[f(x),x]x^{k-1}\Big\}x\\ & + 3n\Big\{x^{n}[f(x),x]x^{2n-2}+2x^{n+1}[f(x),x]x^{2n-3}+\cdots\\ & \quad + nx^{2n-1}[f(x),x]x^{n-1}+(n-1)x^{2n}[f(x),x]x^{n-2}+\cdots\\ & \quad + 2x^{3n-3}[f(x),x]x+x^{3n-2}[f(x),x]\Big\}.\end{aligned}$$

Applying (3.5) to the last relation, we have

$$\begin{aligned}0 = {} & x^{n}[f(x),x]x^{2n-2}+2x^{n+1}[f(x),x]x^{2n-3}+\cdots\\ & + nx^{2n-1}[f(x),x]x^{n-1}+(n-1)x^{2n}[f(x),x]x^{n-2}\\ & +\cdots+2x^{3n-3}[f(x),x]x+x^{3n-2}[f(x),x].\end{aligned} \tag{3.7}$$

Thus the last relation contains the sum of these $2n-1$ terms, which the number of $n-1$ terms decreased by virtue of our process starting from (3.5). But the power of x in (3.7) has increased by 1 in comparison to

(3.5). We repeat similarly the process from the relation (3.5) to (3.7) as follows. Substituting $x + \lambda y$ for x in (3.7) and expanding the resulting equation, we obtain

$$0 = \lambda b_1(x, y) + \lambda^2 b_2(x, y) + \cdots + \lambda^{3n} b_{3n}(x, y),$$

$\lambda \in \mathbb{Z},\ x, y \in A$, where $b_i(x, y)$ denotes the sum of these terms in which y appears as a term in the product i times. By Lemma 2.1 we have $0 = b_1(x, y)$ for all $x, y \in A$. Thus

(3.8)

$$\begin{aligned}
0 = \sum_{k=1}^{n} k\Big\{ & x^{n+k-1}[f(x), x](x^{2n-k-2}y + x^{2n-k-3}yx + \cdots + yx^{2n-k-2}) \\
& + x^{n+k-1}([f(x), y] + 2[F(x, y), x])x^{2n-k-1} \\
& + (x^{n+k-2}y + x^{n+k-3}yx + \cdots + yx^{n+k-2})[f(x), x]x^{2n-k-1}\Big\} \\
+ \sum_{k=n-1}^{2} k\Big\{ & x^{3n-k-1}[f(x), x](x^{k-2}y + x^{k-3}yx + \cdots + yx^{k-2}) \\
& + x^{3n-k-1}([f(x), y] + 2[F(x, y), x])x^{k-1} \\
& + (x^{3n-k-2}y + x^{3n-k-3}yx + \cdots + yx^{3n-k-2}) \\
& \cdot [f(x), x]x^{k-1}\Big\} \\
+ x^{3n-2}&([f(x), y] + 2[F(x, y), x]) \\
+ (x^{3n-3}&y + x^{3n-4}yx + \cdots + yx^{3n-3})[f(x), x]
\end{aligned}$$

for all $x, y \in A$. Putting x^2 instead of y in (3.8) and using the identity $[f(x), x^2] + 2[F(x, x^2), x] = 3[f(x), x]x + 3x[f(x), x]$, we obtain the resulting long relation

$$\begin{aligned}
0 &= \sum_{k=0}^{n-1} \{(3k+2)n + (2k+1)\} x^{n+k}[f(x), x]x^{2n-k-1} \\
&\quad + \sum_{k=n-1}^{0} \{(k+1)3n + (2k+1)\} x^{3n-k-1}[f(x), x]x^{k} \\
&= \sum_{k=0}^{n-1} \{(k+1)(2n+1) + (n+1)k\} x^{n+k}[f(x), x]x^{2n-k-1} \\
&\quad + \sum_{k=n-1}^{0} \{k(2n+1) + (k+1)(n+1) + 2n\} x^{3n-k-1}[f(x), x]x^{k}
\end{aligned}$$

for all $x \in A$, which can be rearranged in the form

$$\begin{aligned} 0 = (2n+1)\Big\{ & x^n[f(x),x]x^{2n-2} + 2x^{n+1}[f(x),x]x^{2n-3} \\ & + \cdots + nx^{2n-1}[f(x),x]x^{n-1} \\ & + (n-1)x^{2n}[f(x),x]x^{n-2} \\ & + \cdots + 2x^{3n-3}[f(x),x]x + x^{3n-2}[f(x),x]\Big\}x \\ + (n+1)x\Big\{ & x^n[f(x),x]x^{2n-2} + 2x^{n+1}[f(x),x]x^{2n-3} \\ & + \cdots + nx^{2n-1}[f(x),x]x^{n-1} \\ & + (n-1)x^{2n}[f(x),x]x^{n-2} \\ & + \cdots + 2x^{3n-3}[f(x),x]x + x^{3n-2}[f(x),x]\Big\} \\ + 2n\Big\{ & x^{2n}[f(x),x]x^{n-1} + x^{2n+1}[f(x),x]x^{n-2} \\ & + \cdots + x^{3n-1}[f(x),x]\Big\} \end{aligned} \tag{3.9}$$

for all $x \in A$. Comparing (3.7) with the above relation, we get

$$0 = x^{2n}[f(x),x]x^{n-1} + x^{2n+1}[f(x),x]x^{n-2} + \cdots + x^{3n-1}[f(x),x] \tag{3.10}$$

for all $x \in A$. The last relation contains the sum of these n terms, which the number of $n-1$ terms has decreased by virtue of our process starting from (3.7). We repeat once more the process from the relation (3.7) to (3.10) as follows. Taking the place of x by $x + \lambda y$ in (3.10), we get

$$0 = \lambda c_1(x,y) + \lambda^2 c_2(x,y) + \cdots + \lambda^{3n+1} c_{3n+1}(x,y),$$

$\lambda \in \mathbb{Z}$, $x, y \in A$, where $c_i(x,y)$ denotes the sum of these terms in which y appears as a term in the product i times. Applying Lemma 2.1 and expanding the resulting equation, we have $0 = c_1(x,y)$ for all $x, y \in A$.

Therefore
(3.11)

$$
\begin{aligned}
0 = \sum_{k=0}^{n-2} \Big\{ & x^{2n+k}[f(x),x](x^{n-k-2}y + x^{n-k-3}yx + \cdots + yx^{n-k-2}) \\
& + x^{2n+k}([f(x),y] + 2[F(x,y),x])x^{n-k-1} \\
& + (x^{2n+k-1}y + x^{2n+k-2}yx + \cdots + yx^{2n+k-1}) \\
& \cdot [f(x),x]x^{n-k-1} \Big\} \\
& + x^{3n-1}([f(x),x] + 2[F(x,y),x]) \\
& + (x^{3n-2}y + x^{3n-3}yx + \cdots + yx^{3n-1})[f(x),x]
\end{aligned}
$$

for all $x, y \in A$. Taking x^2 instead of y in (3.11), we obtain

$$
\begin{aligned}
0 = (n+2)\Big\{ & x^{2n}[f(x),x]x^{n-1} + x^{2n+1}[f(x),x]x^{n-2} \\
& + \cdots + x^{3n-1}[f(x),x]\Big\}x \\
+ (2n+2)x\Big\{ & x^{2n}[f(x),x]x^{n-1} + x^{2n+1}[f(x),x]x^{n-2} + \cdots \\
& + x^{3n-1}[f(x),x]\Big\} + nx^{3n}[f(x),x]
\end{aligned}
$$

for all $x \in A$. Thus, comparing the last relation with the relation (3.10), we have $x^{3n}[f(x),x] = 0$ for all $x \in A$. By Lemma 2.3, $D = 0$. In other words, $D_P = 0$ on A/P and we see that $D(A) \subseteq P$ and so the result follows since P is any primitive ideal. The proof of the theorem is complete. □

The following corollaries are due to Theorem 3.2.

COROLLARY 3.3. *Let D be a continuous linear Jordan derivation on a Banach algebra A such that $[[D(x),x],x] \in \mathrm{rad}(A)$ for a positive integer n and for all $x \in A$. Then $D(A) \subseteq \mathrm{rad}(A)$.*

COROLLARY 3.4. *Let D be a continuous linear Jordan derivation on a Banach algebra A such that $[D(x^n),x^n] \in \mathrm{rad}(A)$ for a positive integer n and for all $x \in A$. Then $D(A) \subseteq \mathrm{rad}(A)$.*

COROLLARY 3.5. *Let D be a continuous linear Jordan derivation on a Banach algebra A such that $[[D(x),x],x^n] \in \mathrm{rad}(A)$ for a positive integer n and for all $x \in A$. Then $D(A) \subseteq \mathrm{rad}(A)$.*

COROLLARY 3.6. *Let D be a continuous linear Jordan derivation on a Banach algebra A such that $[D(x), x^n] \in \mathrm{rad}(A)$ for a positive integer n and for all $x \in A$. Then $D(A) \subseteq \mathrm{rad}(A)$.*

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