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RIEMANNIAN SUBMANIFOLDS IN LORENTZIAN MANIFOLDS WITH THE SAME CONSTANT CURVATURES

JOONSANG PARK

ABSTRACT. We study nondegenerate immersions of Riemannian manifolds of constant sectional curvatures into Lorentzian manifolds of the same constant sectional curvatures with flat normal bundles. We also give a method to produce such immersions using the so-called Grassmannian system..

1. Introduction

The study of isometric immersions of the space forms $N^n(c)$ with constant sectional curvature c into the space forms $N^{n+k}(c')$ has been a classical problem in differential geometry. For example, nonexistence of an isometric immersion of the hyperbolic space form $\mathbb{H}^2 = N^2(-1)$ into $\mathbb{R}^3 = N^3(0)$ by Hilbert [3], existence of local isometric immersions of $N^{2n-1}(c)$ in $N^{2n}(c+1)$ and nonexistence of local immersions of $N^{2n-2}(c)$ in $N^{2n-1}(c+1)$ by Cartan [2], and generalizations of Cartan's work by Tenenblat and Terng [6], [7], [8] are known, and many other results have been obtained in [9] and [1], too.

On the other hand, recently the soliton theory in integrable systems has been developed extensively so that it can be applied to geometric problems. Notice that the sine-Gordon and the sinh-Gordon equations are special kind of soliton equations, which are related to local immersions of $N^2(c)$ into $N^3(c+1)$ for $c = -1$ and 0. In this vein, the so-called n-dimensional system or G/K system on a symmetric space developed by Terng [9] has succeeded in explaining some geometry of submanifold $N^n(c)$ in $N^{n+k}(c')$ in [9] and [1].

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In this paper, we study how a Riemannian space form $N^n(c)$ can be locally immersed into the Lorentzian space form $N^{n+k,1}(c)$, specifically, study suitable geometric conditions on such immersions and a method to produce them.

2. Submanifolds in Lorentzian space

First, we introduce basic knowledge and notations about Lorentzian geometry. For details, see [4] and [5]. Denote by $\mathbb{R}^{m,r}$ the vector space $\mathbb{R}^{m+r}$ with the nondegenerate metric of index r, $\langle x,y\rangle = \sum_{i=1}^{m} x_i y_i - \sum_{i=m+1}^{m+r} x_i y_i$. A basis $\{e_1, \cdots, e_{m+r}\}$ of $\mathbb{R}^{m,r}$ is called orthonormal if $\langle e_i, e_j\rangle = \epsilon_i \delta_{ij}$, where $\epsilon_i = 1$ for $i \le m$ and $\epsilon_i = -1$ for $i > m$. A pseudo-Riemannian manifold N which has a metric of index 1 is called a Lorentzian manifold. It is well-known (cf. [4]) that the complete connected $(m+1)$-dimensional Lorentzian manifold $N^{m,1}(c)$ of the constant sectional curvature $c = 0, 1, -1$ is the Lorentzian space $\mathbb{R}^{m,1}$, the Lorentzian sphere $\mathbb{S}^{m,1}$ or the Lorentzian hyperbolic space $\mathbb{H}^{m,1}$, respectively, where

$$\mathbb{S}^{m,1} = \{x \in \mathbb{R}^{m+1,1} \mid \langle x,x\rangle = 1\},$$
$$\mathbb{H}^{m,1} = \{x \in \mathbb{R}^{m,2} \mid \langle x,x\rangle = -1\}.$$

The usual differential d on $\mathbb{R}^{m,1}$, $\mathbb{R}^{m+1,1}$ or $\mathbb{R}^{m,2}$ induces the Levi-Civita connection $\bar\nabla$ on $N^{m,1}(c)$ by taking the orthogonal projection $\bar\nabla V$ of dV to the tangent space $TN^{m,1}(c)$ for a vector field V on $N^{m,1}(c)$.

Suppose $X : M^n \to N^{n+k,1}(c)$ is an isometric immersion of a Riemannian manifold M. Let $\{e_1, \cdots, e_{n+k+1}\}$ be a local orthonormal frame field such that $e_1, \cdots, e_n$ are tangent to M. From now on, we shall use the following index convention:

$$1 \le A, B, C \le n+k+1, \quad 1 \le i, j, k \le n, \quad n+1 \le \alpha, \beta, \gamma \le n+k+1.$$

Let $\{\omega_A\}$ be the coframe field dual to $\{e_A\}$, that is, $\omega_A(e_B) = \epsilon_A \delta_{AB}$, where, $\epsilon_A = \langle e_A, e_A\rangle$ so that $\epsilon_A = 1$ for $A \le n+k$ and $\epsilon_{n+k+1} = -1$. The first fundamental form on M is then given by $I = \sum_i \omega_i \otimes \omega_i$, which is a positive-definite metric. Let ω_{AB} be the connection 1-form corresponding to the canonical connection $\bar\nabla$,

$$\bar\nabla e_A = \sum_B e_B \otimes \omega_{BA}.$$

This induces the structure equations, Gauss, Codazzi and Ricci equations on M:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \tag{2.1}$$

$$\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = -\sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j} + c\,\omega_i \wedge \omega_j, \tag{2.2}$$

$$d\omega_{i\alpha} = -\sum_k \omega_{ik} \wedge \omega_{k\alpha} - \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha}, \tag{2.3}$$

$$d\omega_{\alpha\beta} + \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} = -\sum_i \omega_{\alpha i} \wedge \omega_{i\beta}. \tag{2.4}$$

From (2.2) and (2.4), we obtain the curvature 2-form Ω on M and the normal curvature 2-form Ω^ν as

$$\Omega_{ij} = \sum_\alpha \epsilon_\alpha \omega_{i\alpha} \wedge \omega_{j\alpha} + c\,\omega_i \wedge \omega_j, \tag{2.5}$$

$$\Omega^\nu_{\alpha\beta} = \sum_i \epsilon_\alpha \omega_{i\alpha} \wedge \omega_{i\beta}. \tag{2.6}$$

The shape operator A_v in the normal direction $v \in \nu M$ and the second fundamental form II are defined by

$$A = \sum_{j,\alpha} \epsilon_\alpha \omega_{j\alpha} \otimes \omega_\alpha \otimes e_j, \tag{2.7}$$

$$II = \sum_{j,\alpha} \omega_{j\alpha} \otimes \omega_j \otimes e_\alpha. \tag{2.8}$$

It is an elementary fact that M has constant sectional curvature c if and only if $\Omega_{ij} = c\,\omega_i \wedge \omega_j$. Thus by (2.5),

$$\sum_\alpha \epsilon_\alpha \omega_{i\alpha} \wedge \omega_{j\alpha} = 0 \quad \text{for} \quad i \neq j. \tag{2.9}$$

We may assume that when $c = 0, 1$ or -1, M is the Euclidean space $\mathbb{R}^n$, the unit sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$, respectively, as far as local immersions are concerned.

Now, suppose also that the normal bundle νM of M is flat, i.e., $\Omega^\nu = 0$. Then there exists a parallel normal frame $\{e_\alpha\}$ and it is easy to see that all the shape operators $\{A_v \mid v \in \nu_p M\}$ commute by (2.6), and thus they are simultaneously diagonalizable.

DEFINITION 2.1. Suppose $k+1 \geq n$. A Riemannian submanifold M^n in $N^{n+k,1}(c)$ is called *nondegenerate* if $(\operatorname{Im} II)_p = \{II(X,Y) \mid X, Y \in T_pM\}$ has dimension n for any $p \in M^n$ and the inner product on $\operatorname{Im} II$ induced by $\langle\ ,\ \rangle$ is nondegenerate.

A nondegenerate Riemannian submanifold M^n with a flat normal bundle has a strong geometric property, the existence of a curvature coordinate system. To see this, let $T_pM = E_1 \oplus \cdots \oplus E_r$ be the common eigen-decomposition for $\{A_v \mid v \in \nu_p M\}$. Then

$$A|_{E_i} = \lambda_i \otimes Id_{E_i} \tag{2.10}$$

for some $\lambda_1, \cdots, \lambda_r \in (\nu_p M)^*$. *The curvature normals* $v_1, \cdots, v_r$ in $\nu_p M$ are defined as the dual to λ_i, that is, $\lambda_i(v) = \langle v, v_i\rangle$. The following lemma holds as does in the case of Riemannian immersions into space forms ([9], [1]):

LEMMA 2.2. *Suppose M^n is nondegenerate and has a flat normal bundle. Then $r = n$ and the curvature normals $v_1, \cdots, v_n$ are linearly independent.*

Proof. Since $\langle II(X,Y), v\rangle = \langle A_v(X), Y\rangle$ for $v \in \nu_p M$ and $X, Y \in T_pM$, we have $(\operatorname{Im} II)^\perp = \operatorname{Ker}(A : \nu M \to T^*M \otimes TM)$. From $A_v|_{E_i} = \langle v, v_i\rangle Id_{E_i}$ for any i, $v \in \operatorname{Ker} A$ if and only if $v \in \operatorname{Span}\{v_i\}^\perp$. Hence $\operatorname{Span}\{v_i\} = (\operatorname{Ker} A)^\perp = \operatorname{Im} II$. Since M^n is nondegenerate, $r = n$ and $\{v_1, \cdots, v_n\}$ should be a basis of $\operatorname{Im} II$. □

According to the above lemma, we can also see that $\dim E_i = 1$ for each i and thus there exist a unique orthonormal tangent frame $\{e_i\}$ which diagonalize the shape operators simultaneously, up to signs and permutations, and they are smooth.

Using the frame $\{e_i\}$ and $\{e_\alpha\}$, the curvature normals can be expressed as

$$v_i = \sum_\alpha \epsilon_\alpha \langle v_i, e_\alpha \rangle e_\alpha = \sum_\alpha \epsilon_\alpha \lambda_{i\alpha} e_\alpha, \tag{2.11}$$

where $\lambda_{i\alpha} = \lambda_i(e_\alpha)$ and

$$\omega_{i\alpha} = \lambda_{i\alpha}\omega_i. \tag{2.12}$$

Thus $\langle v_i, v_j \rangle = \sum_\alpha \epsilon_\alpha \lambda_{i\alpha}\lambda_{j\alpha} = 0$ for $i \neq j$ by (2.9). Hence $v_1, \cdots,$ v_n are mutually orthogonal and not null vectors.

If all $v_1, \cdots, v_n$ are space-like, we say that the curvature normals are *space-like.* If not, then only one of them is time-like. We may assume v_n is a time-like vector in this case and say that the curvature normals are *Lorentzian.*

PROPOSITION 2.3. *Suppose M^n is a nondegenerate Riemannian submanifold of $N^{n+k,1}(c)$ with constant sectional curvature c and a parallel normal frame e_α and $k+1 \geq n$. Then there exist a coordinate system $(x_1, \cdots, x_n)$ and a map $b = (b_1, \cdots, b_n)^t$ such that $e_i = \frac{1}{b_i}\frac{\partial}{\partial x_i}$ are a principal tangent frame.*

Proof. From the Codazzi equations (2.3), using (2.12) and $\omega_{\alpha\beta} = 0$, we obtain

$$d\lambda_{i\alpha}(e_j) + (\lambda_{i\alpha} - \lambda_{j\alpha})\gamma_{iji} = 0 \quad \text{for } i \neq j, \tag{2.13}$$

and

$$(\lambda_{i\alpha} - \lambda_{k\alpha})\gamma_{ikj} = (\lambda_{i\alpha} - \lambda_{j\alpha})\gamma_{ijk} \quad \text{for distinct } i, j, k, \tag{2.14}$$

where $\omega_{ij} = \sum_k \gamma_{ijk}\omega_k$.

Take $b_i = |\langle v_i, v_i \rangle|^{-\frac{1}{2}}$. Multiplying $\epsilon_\alpha\lambda_{i\alpha}$ to (2.13) and $\epsilon_\alpha\lambda_{k\alpha}$ to (2.14), and summing up over α, we obtain

$$\gamma_{iji} = \frac{db_i(e_j)}{b_i} \quad \text{for} \quad i \neq j \quad \text{and} \quad \langle v_k, v_k \rangle \gamma_{ikj} = 0 \quad \text{for distinct } i, j, k.$$

Therefore,

$$\omega_{ij} = \frac{db_i(e_j)}{b_i}\omega_i - \frac{db_j(e_i)}{b_j}\omega_j. \tag{2.15}$$

By (2.15), we see that $\nabla_{e_i} e_j = \frac{db_i(e_j)}{b_i} e_i$. It is a direct calculation that

$$\left[b_i e_i, b_j e_j\right] = \nabla_{b_i e_i} b_j e_j - \nabla_{b_j e_j} b_i e_i = 0. \qquad \square$$

We now conclude the local geometry of the above submanifold as follows. Here, we denote by I_n the $n \times n$ identity matrix and $J_p = \mathrm{diag}(1, \cdots, 1, -1)$ is a $p \times p$ diagonal matrix.

THEOREM 2.4. *Let $X : M^n \longrightarrow N^{n+k,1}(c)$ be a nondegenerate isometric immersion of a Riemannian manifold M^n of constant sectional curvature c with a flat normal bundle, and assume $k+1 \geq n$. Then, for a local parallel normal frame e_α, there exist a curvature coordinate system $(x_1, \cdots, x_n)$, a map $b = (b_1, \cdots, b_n)^t$ and an $n \times (k+1)$ matrix-valued $B_1 = (b_{ij})$ such that $B_1 J_{k+1} B_1^t = I_n$ or $B_1 J_{k+1} B_1^t = J_n$ and the first and second fundamental forms are given by*

$$I = \sum_{i=1}^{n} b_i^2 dx_i^2, \qquad II = \sum_{i=1}^{n} \sum_{j=1}^{k+1} b_{ij} b_i dx_i^2 \otimes e_{n+j}.$$

The curvature normals are space-like when $B_1 J_{k+1} B_1^t = I_n$, and Lorentzian when $B_1 J_{k+1} B_1^t = J_n$.

Proof. Let $e_i = \frac{1}{b_i} \frac{\partial}{\partial x_i}$. Then e_i are an orthonormal tangent frame with its dual coframe $\omega_i = b_i dx_i$.

Define $b_{ij} = \lambda_{i,n+j} b_i$. Then by (2.12),

$$\omega_{i,n+j} = \lambda_{i,n+j} \omega_i = b_{ij} dx_i.$$

Hence, the second fundamental form is given as above. Orthonormality of the columns of B_1 follows from the fact that the curvature normals $v_i = \sum_\alpha \lambda_{i\alpha} e_\alpha$ are orthogonal and $b_i = |\langle v_i, v_i \rangle|^{-\frac{1}{2}}$. $\square$

3. Grassmannian system

To obtain Riemannian submanifolds in $N^{n+k,1}(c)$ described in Section 2, we will use a special partial differential equation called Grassmannian system. G/K systems are introduced by Terng in [9], and we mention some results from [9], which will be used in our case.

Let G/K be a rank n symmetric space with the involution $\sigma : \mathcal{G} \to \mathcal{G}$ on the Lie algebra $\mathcal{G}$ of G, $\mathcal{G} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition, and

$\mathcal{A} \subset \mathcal{P}$ a maximal abelian subalgebra with a basis $\{a_1, \ldots, a_n\}$. Let $\mathcal{A}^{\perp}$ denote the orthogonal complement of $\mathcal{A}$ in $\mathcal{G}$ with respect to the Killing form. G/K *system* for $v : R^n \to \mathcal{P} \cap \mathcal{A}^{\perp}$ is

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = \big[[a_i, v], [a_j, v]\big], \quad 1 \leq i \neq j \leq n, \tag{3.1}$$

where $v_{x_i} = \frac{\partial v}{\partial x_i}$. This is a Cauchy problem which can be solved for any generic data decaying rapidly along $(x_1, 0, \ldots, 0) \in \mathbb{R}^n$.

It is known that v is a solution of (3.1) if and only if the $\mathcal{G} \otimes \mathbb{C}$-valued connection 1-form on the trivial principal bundle $\mathbb{R}^n \times \mathcal{G}$ on $\mathbb{R}^n$

$$\theta_\lambda = \sum_{i=1}^{n} (a_i \lambda + [a_i, v]) dx_i \tag{3.2}$$

is flat for any $\lambda \in \mathbb{C}$.

Let $g : \mathbb{R}^n \to G$ and θ be a $\mathcal{G}$-valued connection 1-form. We call $g * \theta = g\theta g^{-1} - dgg^{-1}$ the gauge transformation of θ by g. It is obvious that if θ is flat, then $g * \theta$ is also flat.

To apply the theory of G/K system to our case, we take the Lorentzian Grassmannian system $G/K = O(n+m, r)/(O(n) \times O(m, r))$ related to the isometry group G of $N^{n+k,1}(c)$.

Let $\mathcal{M}_{p \times q}$ be the set of $p \times q$ matrices. For $c = 0, 1, -1$, the isometry groups G are given by

$$\mathrm{Isom}(\mathbb{R}^{n+k,1}) = \left\{ \begin{pmatrix} 1 & 0 \\ \xi & A \end{pmatrix} \;\middle|\; A \in O(n+k, 1),\ \xi^t \in \mathbb{R}^{n+k,1} \right\},$$

$$O(n+k+1, 1) = \left\{ A \in GL(n+k+2, \mathbb{R}) \;\middle|\; A^t J_1 A = J_1 \right\},$$

$$O(n+k, 2) = \left\{ A \in GL(n+k+2, \mathbb{R}) \;\middle|\; A^t J_{-1} A = J_{-1} \right\},$$

respectively. Here $J_1 = \mathrm{diag}\ (1, \cdots, 1, -1)$ and $J_{-1} = \mathrm{diag}\ (-1, 1, \cdots, 1, -1)$, and to fit into our purposes, we slightly modify the inner product on $\mathbb{R}^{n+k,2}$ by

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{n+k+1} x_i y_i - x_{n+k+2} y_{n+k+2},$$

and we identify $\mathbb{R}^{n+k,1}$ with $\{1\} \times \mathbb{R}^{n+k,1} \subset \mathbb{R}^{n+k,2}$ by $X \leftrightarrow (1, X)$.

The Lie algebras of the isometry groups G of $N^{n+k}(c)$ for $c = 0, 1, -1$ can be expressed in one way as the Lie algebra

$$\mathcal{G} = \left\{ \begin{pmatrix} 0 & -c\xi^t J \\ \xi & Y \end{pmatrix} \,\middle|\, Y \in o(n+k,1),\ \xi^t \in \mathbb{R}^{n+k+1} \right\},$$

where $J = \mathrm{diag}(1, \cdots, 1, -1) \in \mathcal{M}_{(n+k+1)\times(n+k+1)}$.

From now on, we abuse notation $J = \mathrm{diag}(1, \cdots, 1, -1)$ whatever the size is, and assume $k+1 \geq n$.

Define an involution σ on $\mathcal{G}$ by

$$\sigma(X) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & I_k \end{pmatrix} X \begin{pmatrix} -1 & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & I_k \end{pmatrix}, \ X \in \mathcal{G},$$

where I_p is the $p \times p$ identity matrix. Then the Cartan decomposition $\mathcal{G} = \mathcal{K} + \mathcal{P}$ is given by

$$\mathcal{K} = \left\{ \begin{pmatrix} 0 & -c\xi_1^t & 0 \\ \xi_1 & A & 0 \\ 0 & 0 & B \end{pmatrix} \,\middle|\, A \in o(n),\ B \in o(k,1),\ \xi_1^t \in \mathbb{R}^n \right\},$$

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & 0 & -c\xi_2^t J \\ 0 & 0 & -C^t J \\ \xi_2 & C & 0 \end{pmatrix} \,\middle|\, C \in \mathcal{M}_{(k+1)\times n},\ \xi_2^t \in \mathbb{R}^{k+1} \right\}.$$

Let $\mathcal{A}_I$ be an abelian subalgebra of $\mathcal{P}$ spanned by

$$a_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & D_i & 0 \\ 0 & -D_i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad 1 \leq i \leq n,$$

where the matrices are partitioned into blocks with sizes $(1, n, n, k+1-n)$ and $D_i = \mathrm{diag}(0, \cdots, 1, \cdots, 0)$, of which the only nonzero entry 1 occurs at the i-th entry. Then the solution v_I of the G/K system is of the form

$$v_I = \begin{pmatrix} 0 & 0 & -cb^t & 0 \\ 0 & 0 & -F^t & -G^t J \\ b & F & 0 & 0 \\ 0 & G & 0 & 0 \end{pmatrix}, \tag{3.3}$$

where $b^t \in \mathbb{R}^n$, $F \in \mathcal{M}_{n\times n}$ with $f_{ii} = 0$ and $G \in \mathcal{M}_{(k+1-n)\times n}$.

Put $\delta = \mathrm{diag}(dx_1, \cdots, dx_n)$, then the flat connection 1-form θ^I_λ in (3.2) becomes

$$\theta^I_\lambda = \begin{pmatrix} 0 & -cb^t\delta & 0 & 0 \\ \delta b & \delta F - F^t\delta & \lambda\delta & 0 \\ 0 & -\lambda\delta & \delta F^t - F\delta & \delta G^t J \\ 0 & 0 & -G\delta & 0 \end{pmatrix}. \tag{3.4}$$

We also take another abelian subalgebra $\mathcal{A}_{II}$ of $\mathcal{P}$ spanned by

$$a'_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_i \\ 0 & 0 & 0 & 0 \\ 0 & -JD_i & 0 & 0 \end{pmatrix}, \qquad 1 \le i \le n.$$

Then the solution and the flat connection 1-form become

$$v_{II} = \begin{pmatrix} 0 & 0 & 0 & -cb^tJ \\ 0 & 0 & -G^t & -F^tJ \\ 0 & G & 0 & 0 \\ b & F & 0 & 0 \end{pmatrix} \tag{3.5}$$

and

$$\theta^{II}_\lambda = \begin{pmatrix} 0 & -cb^t\delta & 0 & 0 \\ \delta b & \delta F - F^t\delta & 0 & \lambda\delta \\ 0 & 0 & 0 & -G\delta \\ 0 & -\lambda J\delta & J\delta G^t & J\delta F^tJ - F\delta \end{pmatrix}, \text{respectively.} \tag{3.6}$$

In this case, matrices are partitioned with sizes $(1, n, k+1-n, n)$.

4. Main Theorems

We now investigate on how to associate nondegenerate Riemannian submanifolds in $N^{n+k,1}(c)$ which has constant curvature c and a flat normal bundle to the solutions of the Lorentzian Grassmannian system.

THEOREM 4.1. *Suppose X is a nondegenerate local isometric immersion of a Riemannian manifold M^n into $N^{n+k}(c)$ of constant sectional curvature c with a flat normal bundle as in Theorem 2.4, where $k+1 \geq n$.*

If M has space-like curvature normals, then there exists v_I of the form (3.3), a solution of the system associated to $\mathcal{G}$ such that

$$F = \left(\frac{(b_i)_{x_j}}{b_j} \right), \quad (\omega_{ij}) = \delta F - F^t \delta, \quad \text{and} \quad B_1 dJB_1^t = \delta F^t - F\delta.$$

If M has Lorentzian curvature normals, then there exists a solution v_{II} of the form (3.5) such that

$$F = \left(\frac{(b_i)_{x_j}}{b_j} \right), \quad (\omega_{ij}) = \delta F - F^t \delta, \quad \text{and} \quad B_1 dJB_1^t J = J\delta F^t J - F\delta.$$

Proof. Suppose M has space-like curvature normals. Choose a parallel normal frame e_α and a tangent frame e_i as in Theorem 2.4. Then $\omega_i = b_i dx_i$. Put $b = (b_1, \cdots, b_n)^t$ and $\omega = (\omega_{ij})$. The structure equations, Gauss, Codazzi and Ricci equations for X are equivalent to saying that

$$\tilde{\theta}_1 = \begin{pmatrix} 0 & -cb^t\delta & 0 \\ \delta b & \omega & \delta B_1 \\ 0 & -JB_1^t\delta & 0 \end{pmatrix}$$

is flat. Also, it is easy to see that

$$\tilde{\theta}_\lambda = \begin{pmatrix} 0 & -cb^t\delta & 0 \\ \delta b & \omega & \lambda\delta B_1 \\ 0 & -\lambda JB_1^t\delta & 0 \end{pmatrix} \tag{4.1}$$

is flat for any $\lambda \in \mathbb{C}$. Let $F = (f_{ij}) \in \mathcal{M}_{n\times n}$, where $f_{ij} = \frac{(b_i)_{x_j}}{b_j}$ for $i \neq j$ and $f_{ii} = 0$. Since the connection 1-form ω on M satisfies

$$\omega_{ij} = \frac{(b_i)_{x_j}}{b_j} dx_i - \frac{(b_j)_{x_i}}{b_i} dx_j \quad \text{for} \quad i \neq j$$

by (2.15), we obtain

$$\omega = (\omega_{ij}) = \delta F - F^t \delta.$$

On the other hand, from the flatness of $\tilde{\theta}_\lambda$,

$$dJB_1^t \wedge \delta = -JB_1^t\delta \wedge \omega = -JB_1^t\delta \wedge (\delta F - F^t\delta) = JB_1^t(\delta F^t - F\delta) \wedge \delta$$

and thus

$$dJB_1^t = JB_1^t(\delta F^t - F\delta) + C\delta \tag{4.2}$$

for some $C \in \mathcal{M}_{(k+1)\times n}$. Extend B_1 to $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in O(k,1)$. Multiplying B_1 and B_2 on (4.2), we obtain

$$B_1\, dJB_1^t = (\delta F^t - F\delta) + B_1 C\delta \tag{4.3}$$

and

$$B_2\, dJB_1^t = B_2 C\delta. \tag{4.4}$$

Since $B_1\, dJB_1^t$ and $\delta F^t - F\delta$ are skew-symmetric, we have

$$B_1\, dJB_1^t = \delta F^t - F\delta. \tag{4.5}$$

By the same arguments as in [1], using $B_2\, dJB_1^t = B_2 C\delta$, it is easy to prove that $B_2\, dJB_2^t J$ is flat. Thus

(4.6)
$$B\, dB^{-1} = \begin{pmatrix} B_1\, dJB_1^t & B_1\, dJB_2^t J \\ B_2\, dJB_1^t & B_2\, dJB_2^t J \end{pmatrix} = \begin{pmatrix} \delta F^t - F\delta & -\delta C^t B_2^t J \\ B_2 C\delta & h^{-1}\, dh \end{pmatrix}$$

for some $h \in O(k-n,1)$.

Set $G = hB_2C$. Now, take a gauge transformation on $\tilde{\theta}_\lambda$ by

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & h \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & B \end{pmatrix},$$

then the resulting flat connection 1-form is $g * \tilde{\theta}_\lambda = \theta_\lambda^I$ defined as (3.4). Therefore, we can find a desired solution v_I of the system associated to $\mathcal{G}$.

The proof for the immersion which has Lorentzian curvature normals are similar only except taking $B = \begin{pmatrix} B_2 \\ B_1 \end{pmatrix} \in O(k,1)$ as an extension of B_1 to obtain $g * \tilde{\theta}_\lambda = \theta_\lambda^{II}$. □

Conversely, from a solution of the system associated to $\mathcal{G}$, we can produce the above kind of an immersion.

THEOREM 4.2. *Suppose $k+1 \geq n$. If v_I is a solution of the system associated to $\mathcal{G}$ defined as (3.3), then there exists a nondegenerate isometric immersion X of a Riemannian manifold M^n of constant sectional curvature c with a flat normal bundle into $N^{n+k}(c)$, which has space-like curvature normals, a parallel normal frame $\{e_\alpha\}$, a coordinate system $(x_1, \cdots, x_n)$, and an $\mathcal{M}_{n\times(k+1)}$-valued map B_1 with $B_1 J B_1^t = I$ such that the first and second fundamental forms are given by*

$$I = \sum_{i=1}^{n} b_i^2 dx_i^2, \qquad II = \sum_{i=1}^{n}\sum_{j=1}^{k+1} b_{ij} b_i dx_i^2 \otimes e_{n+j}.$$

If v_{II} is a solution defined as (3.5), then there exists a nondegenerate immersion $X : M^n \to N^{n+k}(c)$ of a Riemannian manifold M of sectional curvature c with a flat normal bundle, which has Lorentzian curvature normals, and a map $B_1 \in \mathcal{M}_{n\times(k+1)}$ with $B_1 J B_1^t = J$ such that the first and second fundamental forms are given as the same as above.

Proof. We will prove only the first case.

Consider the flat connection θ_λ^I as (3.4). Since $\begin{pmatrix} \delta F^t - F\delta & \delta G^t J \\ -G\delta & 0 \end{pmatrix}$ is flat, $B\,dB^{-1} = \begin{pmatrix} \delta F^t - F\delta & \delta G^t J \\ -G\delta & 0 \end{pmatrix}$ for some $B \in O(k,1)$. Put $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, where $B_1 \in \mathcal{M}_{n\times(k+1)}$. Taking a gauge transformation on θ_λ^I by

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & B^{-1} \end{pmatrix}$$

gives $g * \theta_\lambda^I = \tilde{\theta}_\lambda$, where $\tilde{\theta}_\lambda$ is of the form (4.1).

Now, let E be a G-valued map such that $E^{-1}dE = \tilde{\theta}_1$. Denote by X, e_i, e_α the columns of E. Then from

$$d(X, e_i, e_\alpha) = (X, e_i, e_\alpha)\,\tilde{\theta}_1,$$

we obtain

$$dX = \sum b_i dx_i \otimes e_i, \quad de_{n+j} = \sum_i \epsilon_{n+j} b_{ij} dx_i \otimes e_i.$$

Hence e_α are a parallel normal frame, and I and II are given as above. Flatness of the connection $\tilde{\theta}_1$ gives exactly the structure equations, the

Gauss and the Codazzi equations of the immersion X of a Riemannian manifold M^n of sectional curvature c. The fact that curvature normals are space-like comes from the orthonormality of the rows of B_1. Therefore, X gives a desired immersion. $\square$

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Department of Mathematics, Dongguk University, Seoul 100-715, Korea
E-mail: jpark@dgu.edu