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M-INJECTIVITY AND ASYMPTOTIC BEHAVIOUR

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ABSTRACT. Let R be a commutative Noetherian ring and M an R-module. In this paper we will consider the asymptotic behaviour of ideals relative to an R- module E which is M-injective.

1. Introduction

Throughout this paper R will denote a commutative Noetherian ring (with a non-zero identity). We shall follow Macdonald's terminology (see [5]) concerning secondary representation. So whenever an R-module L has a secondary representation, then the set of attached primes of L, which is uniquely determined, is denoted by $\mathrm{Att}_R(L)$.

In [2], H. Ansari-Toroghy and R. Y. Sharp showed that if M and E are respectively a finitely generated and an injective R modules, then $\mathrm{Hom}_R(M, E)$ has a secondary representation. Also they described $\mathrm{Att}_R(\mathrm{Hom}_R(M, E))$ in terms of $\mathrm{Ass}_R(M)$ and a certain set which is uniquely determined by E. In fact, this was the main key for studying the stability of some sequence of sets. Their method is much more dependent to the injective property of the module E such as the exactness of the functor $\mathrm{Hom}_R(-, E)$, and Matlis theorems concerning the injective modules (see [6]).

In [7], L. Melkerson and P. Schenzel, in a different method, obtained the above mentioned results in the case that M and E are respectively a Noetherian, and an injective modules over a commutative ring.

In this paper we will show that the above arguments are still true under a weaker condition when M is an R-module with the property that its zero submodule has a primary decomposition and E an R-module which is injective relative to M. In this case, the functor $\mathrm{Hom}_R(-, E)$ is not exact in general. We recall that E is injective relative to M (or E is

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M-injective) if and only if for any submodule N of M (up to embedding), the homomorphism $\mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(N, E)$ is epic (see [1]).

2. Auxiliary results

Let M be an R-module. A prime ideal P of R is said to be an associated prime ideal of M if there exists an element $x \in M$ such that $P = (0 :_R Rx)$ (see [3]). The set of associated primes of M is denoted by $\mathrm{Ass}_R(M)$.

The concept of coassociated prime ideals was introduced by L. Chambless, H. Zoschinger, and S. Yassemi in different ways. However, these concepts are equivalent (see [9, (1.6)] and [9, (1.7)]). In [9], the concept of coassociated prime ideals is introduced in terms of cocyclic modules: an R module L is cocyclic if $L \subseteq E(R/P)$ for some maximal ideal P of R (for an R-module X, we will use $E(X)$ to denote the injective envelope of X). Also a prime ideal P of R is said to be a coassociated prime of M if there exists a cocyclic homomorphic image L of M such that $(0 :_R L) = P$. The set of coassociated primes of M is denoted by $\mathrm{Coass}_R(M)$.

Remark 2.1 ([1, (16.8) and (16.13)]). Let M be an R-module. We have the following.

(a) If E is an R-module which is M-injective and if X is a submodule or a homomorphic image of M, then E is X-injective.

(b) If $(M_i)_{i\in I}$ is a family of R-modules and E is injective relative to M_i for each $i \in I$, then E is injective relative to $\oplus_{i\in I} M_i$.

(c) If E is M-injective and

$$0 \to K \to M \to L \to 0$$

is an exact sequence of R-modules and R-homomorphisms with middle term M, then

$$0 \to \mathrm{Hom}_R(L, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(K, E) \to 0$$

is also an exact sequence.

Remark 2.2.

(a) Let M and E be respectively a finitely generated and an injective R-module. Then $\mathrm{Hom}_R(M, E)$ has a secondary representation and we have

$$\mathrm{Att}_R(\mathrm{Hom}_R(M,E)) = \{P \in \mathrm{Ass}_R(M) : P \subseteq Q \quad \text{for some} \quad Q \in \mathrm{Ass}_R(E)\}.$$

(See [2, (2.1)] and [7, Lemma 1].)

(b) Let M be an R-module and let the zero submodule of M have a primary decomposition and let $0 = M_1 \cap M_2 \cap ... \cap M_n$ be a minimal primary decomposition of 0 where M_i is P_i-primary submodule of M for $i = 1, 2, ..., n$. Then for an injective R-module E,

$$\mathrm{Coass}_R(\mathrm{Hom}_R(M,E)) = \{P \in \mathrm{Ass}_R(M) : P \subseteq Q \quad \text{for some} \quad Q \in \mathrm{Ass}_R(E)\}.$$

(See [10, (3.6)].)

(c) Let M, L be R-modules. If $\mathrm{Hom}_R(M, L) \neq 0$, then there exists $P \in \mathrm{Ass}_R(M)$ such that $P \subseteq Q$ for some $Q \in \mathrm{Ass}_R(L)$. (See [9, (3.7)].)

LEMMA 2.3. *Suppose that E is an R-module which is injective relative to M. Then we have the following.*

(i) $\mathrm{Hom}_R(M,E) \neq 0$ *if only if there exists $P \in \mathrm{Ass}_R(M)$ such that $P \subseteq Q$ for some $Q \in \mathrm{Ass}_R(E)$.*

(ii) *If P is a prime ideal of R and M is a P-coprimary module and* $\mathrm{Hom}_R(M,E) \neq 0$, *then* $\mathrm{Hom}_R(M,E)$ *is a P-secondary module.*

Proof. (i) Let $P \in \mathrm{Ass}_R(M)$ with $P \subseteq Q$ for some $Q \in \mathrm{Ass}_R(E)$. Then we have the exact sequence

$$0 \to R/P \to M.$$

Since E is M-injective,

$$\mathrm{Hom}_R(M,E) \to \mathrm{Hom}_R(R/P,E) \to 0$$

is also an exact sequence by Remark 2.1 (c). Now $\mathrm{Hom}_R(R/Q,E) \neq 0$ because $\mathrm{Ass}_R(\mathrm{Hom}_R(R/Q,E)) = \mathrm{Supp}_R(R/Q) \cap \mathrm{Ass}_R(E)$ by [3, Chapter 4, Section 1, Proposition 10]. On the other hand since

$$R/P \to R/Q \to 0$$

is an exact sequence and E is R/P-injective,

$$0 \to \mathrm{Hom}_R(R/Q,E) \to \mathrm{Hom}_R(R/P,E)$$

is an exact sequence by Remark 2.1 (c). It implies that $\mathrm{Hom}_R(R/P,E) \neq 0$. Hence by the above arguments, $\mathrm{Hom}_R(M,E) \neq 0$. The reverse implication follows by Remark 2.2 (c).

(ii) Let $r \in R$. Then $M \xrightarrow{r} M$ is nilpotent or injective. Since E is M-injective, $\mathrm{Hom}_R(M, E) \xrightarrow{r} \mathrm{Hom}_R(M, E)$ is either nilpotent or surjective by Remark 2.1 (c). Hence $\mathrm{Hom}_R(M, E)$ is a P- secondary module and the proof is complete. □

3. Asymptotic behaviour

Throughout this section N will denote the set of positive integers.

In [4], Brodmann showed that if M is a Noetherian module over a commutative ring A, then the sequences of sets

$$\mathrm{Ass}_A(M/I^nM), \quad \text{resp.} \quad \mathrm{Ass}_A(I^{n-1}M/I^nM), \quad n \in N,$$

are ultimately constant. We will denote the ultimate constant values of the above sequences respectively by $As^\star(I, M)$ and $Bs^\star(I, M)$.

THEOREM 3.1. *Let M be an R-module with the property that its zero submodule has a primary decomposition, and suppose that E is an R-module which is injective relative to M. Then $\mathrm{Hom}(M, E)$ has a secondary representation and we have*

$$\begin{aligned} &\mathrm{Att}_R(\mathrm{Hom}_R(M, E)) \\ &= \{P \subseteq \mathrm{Ass}_R(M) : P \subseteq Q \quad \text{for some} \quad Q \in \mathrm{Ass}_R(E)\}. \end{aligned}$$

Proof. Let $0 = \cap_{i=i}^n M_i$ be a minimal primary decomposition of 0 where each M_i is P_i-primary. Let $\phi_i : M \to M/M_i$ $(1 \leq i \leq n)$ be the natural homomorphisim and let $T = \mathrm{Hom}(-, E)$. Then by Remark (2.1) (c), T is an exact functor over the category of all modules L which have the property that E is L-injective. On the other hand, during the proof, as you will see, we are facing only the exact sequences of R-modules with terms M, a submodule of M, a homomorphic image or a direct sum of the homomorphic images of M. Hence, by Remark (2.1) (a), and (2.1) (b) and the above arguments, we may assume $T = \mathrm{Hom}_R(-, E)$ is an exact functor. Now for each $i = 1, 2, ..., n$, set $S_i = T(\phi_i)T(M/M_i)$. Then each S_i is a submodule of $T(M)$ and it is isomorphic to $T(M/M_i)$. So it is either zero or P_i-secondary by Lemma 2.3. Now suppose that for $i = 1, 2, ..., r$, there exists $Q_i \in \mathrm{Ass}_R(E)$ such that $P_i \subseteq Q_i$, while this does not hold for $i = r + 1, ..., n$. Then by applying the functor T to the exact sequence of $0 \to M \to \oplus_{i=1}^n M/M_i$ and using Lemma 2.3, we have

$$T(M) = \mathrm{Hom}_R(M, E) = \sum_{i=1}^r S_i,$$

where each S_i is P_i-secondary for $i = 1, 2, ..., r$. Hence $T(M)$ has a secondary representation. We claim this is a minimal secondary representation. To see this, set for an integer j with $1 \leq j \leq r$, $K_j = \cap_{\substack{i=1 \\ i \neq j}}^{n} M_i$, and $Y_j = \oplus_{\substack{i=1 \\ i \neq j}}^{n} M/M_i$. Then from the exact sequence

$$0 \to K_j \to M \to Y_j \to 0,$$

we get the exact sequence

$$0 \to \mathrm{Hom}_R(Y_j, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(K_j, E) \to 0.$$

Hence we have $\mathrm{Hom}_R(M, E) = \sum_{\substack{i=1 \\ i \neq j}}^{r} S_i$ if and only if $\mathrm{Hom}_R(K_j, E) = 0$. But

$$K_j \cong K_j / K_j \cap M_j \cong (K_j + M_j)/K_j \subseteq M/K_j.$$

It implies that $\mathrm{Ass}_R(K_j) = \{P_j\}$ so that

$$0 \to A/P_j \to K_j,$$

is an exact sequence. Therefore we have the exact sequence

$$\mathrm{Hom}_R(K_j, E) \to \mathrm{Hom}_R(A/P_j, E) \to 0.$$

(Note that E is K_j-injective because $K_j \subseteq M$.) Now we have $\mathrm{Hom}_R(A/P_j, E) \neq 0$ by Lemma 2.3. It implies that $\mathrm{Hom}_R(K_j, E) \neq 0$. This completes the proof. □

THEOREM 3.2. *Let M be a finitely generated R-module, and suppose that E is an R-module which is injective relative to M. Further suppose that I be an ideal of R. Then the sequences of sets*

$$\mathrm{Att}_R((0 :_{\mathrm{Hom}_R(M,E)} I^n)), \ n \in N,$$

and

$$\mathrm{Att}_R((0 :_{\mathrm{Hom}_R(M,E)} I^n)/(0 :_{\mathrm{Hom}_R(M,E)} I^{n-1})), \ n \in N,$$

are ultimately constant.

Proof. Set $T = \mathrm{Hom}_R(-, E)$. Since E is M-injective, from the exact sequence

$$0 \to I^{n-1}M/I^nM \to M/I^nM \to M/I^{n-1}M \to 0,$$

we get the exact sequence

$$0 \to T(M/I^{n-1}M) \to T(M/I^nM) \to T(I^{n-1}M/I^nM) \to 0.$$

(See Remark 2.1 (c).) Also,

$$(0 :_{T(M)} I^n) \cong T(M/I^nM).$$

So we have

$$T(I^{n-1}M/I^nM) \cong (0 :_{T(M)} I^n)/(0 :_{T(M)} I^{n-1}).$$

Now the results follows from Theorem 3.1 and the fact that the sequences of sets of

$$\mathrm{Ass}_R(M/I^nM), \text{ resp. } \mathrm{Ass}_R(I^{n-1}M/I^nM), \ n \in N,$$

are ultimately constant. □

COROLLARY 3.3. *Let the situation be as in Theorem 3.2 and let denote the ultimate constant values of the sequences*

$$\mathrm{Att}_R((0 :_{\mathrm{Hom}_R(M,E)} I^n)), \ n \in N,$$

and

$$\mathrm{Att}_R((0 :_{\mathrm{Hom}_R(M,E)} I^n)/(0 :_{\mathrm{Hom}_R(M,E)} I^{n-1})), \ n \in N,$$

respectively by C and D. Then we have the following.

(i) $C = \{P \in As^\star(I,M) : P \subseteq Q$ *for some* $Q \in \mathrm{Ass}_R(E)\}$.
(ii) $D = \{P \in Bs^\star(I,M) : P \subseteq Q$ *for some* $Q \in \mathrm{Ass}_R(E)\}$.
(iii) $C - D \subseteq \mathrm{Att}_R(\mathrm{Hom}_R(M,E)) \cap V(I)$.

Proof. The result follows from the proof of Theorem 3.2 and the fact that

$$As^\star(I,M) - Bs^\star(I,M) \subseteq \mathrm{Ass}_R(M)$$

by [8]. □

COROLLARY 3.4. *Let M be an R-module, and suppose that E is an R-module which is injective relative to M. Then we have*

$$\{P \in \mathrm{Ass}_R(M) : P \subseteq Q \quad \text{for some} \quad Q \in \mathrm{Ass}_R(E)\}$$
$$\subseteq \mathrm{Coass}_R(\mathrm{Hom}_R(M,E)).$$

Proof. Let $P \in \mathrm{Ass}_R(M)$ and let $P \subseteq Q$ for some $Q \in \mathrm{Ass}_R(E)$. Then by Theorem 3.1, $P \in \mathrm{Att}_R(\mathrm{Hom}_R(R/P,E))$. Since E is M-injective, from the exact sequence

$$0 \to R/P \to M,$$

we get the exact sequence

$$\mathrm{Hom}_R(M,E) \to \mathrm{Hom}_R(R/P,E) \to 0$$

by using Remark 2.1 (c). Hence

$$\begin{aligned}&\mathrm{Att}_R(\mathrm{Hom}_R(R/P, E))\\ &= \mathrm{Coass}_R(\mathrm{Hom}_R(R/P, E))\\ &\subseteq \mathrm{Coass}_R(\mathrm{Hom}_R(M, E))\end{aligned}$$

by [9, (1.14) and (1.10)]. It implies that $P \in \mathrm{Coass}_R(\mathrm{Hom}_R(M, E))$ and the proof is complete. □

Remark 3.5. Let the situation be as in Corollary 3.4. Then the equality does not hold in general because it is not true in the case that our module E is an injective R-module (see [9, Example after (1.8)]).

Remark 3.6. Theorem 3.1 (resp. Theorem 3.2) extends [7, Theorem 1] (resp. [7, Theorem 2]).

Remark 3.7. In [10] S. Yassemi by using 2.2 (a), proved 2.2 (b). Theorem 3.1 extends this result and Also gives some further information in this case.

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