Bull. Korean Math. Soc. **39** (2002), No. 2, pp. 269–276

A NECESSARY AND SUFFICIENT CONDITION FOR THE CONVERGENCE OF THE MANN SEQUENCE FOR A CLASS OF NONLINEAR OPERATORS

C. E. Chidume and B. V. C. Nnoli

Abstract. Let E be a real Banach space. Let $T : E \to E$ be a map with $F(T) := \{x \in E : Tx = x\} \neq \emptyset$ and satisfying the accretive-type condition

$$\langle x - Tx, j(x - x^*)\rangle \geq \lambda \|x - Tx\|^2,$$

for all $x \in E$, $x^* \in F(T)$ and $\lambda > 0$. We prove some necessary and sufficient conditions for the convergence of the Mann iterative sequence to a fixed point of T.

1. Introduction

Let E be a real Banach space, E^* its dual and let $\langle .,. \rangle$ denote the generalized duality pairing between E and E^*. Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined for each $x \in E$ by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$

It is well known that if E is smooth then J is single-valued. In the sequel we shall denote the single-valued normalized duality map by j.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* in the terminology of Browder and Petryshyn [1] if there exists $\lambda > 0$ and for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \lambda \|x - y - (Tx - Ty)\|^2. \quad (1.1)$$

Received October 24, 2001.

2000 Mathematics Subject Classification: 47H05, 47H09, 47J05.

Key words and phrases: demicontractive, condition(A), Banach spaces.

The research of the second author was supported by the fellowship from the OEA of the Abdus Salam ICTP, Trieste, Italy for the Ph.D. programme in Mathematics for Sub-Saharan Africa, at the University of Nigeria, Nsukka, Nigeria.

Without loss in generality we may assume $\lambda \in (0,1)$. If I denotes the identity operator, then inequality (1.1) can be written in the form

$$(1.2) \qquad \langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \lambda \|(I-T)x-(I-T)y\|^2.$$

We observe that if $E = H$ (a Hilbert space), inequality (1.1) (and hence (1.2)) is equivalent to the following inequality

$$(1.3) \quad \|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \; \lambda = \frac{(1-k)}{2}$$

(see e.g., [9]). Let $F(T) := \{x^* \in D(T) : x^* = Tx^*\}$ denote the set of fixed points of T. If $F(T) \neq \emptyset$ and (1.2) holds for all $x \in D(T)$ and $x^* \in F(T)$ then we obtain the following inequality

$$(1.4) \qquad \langle x - Tx, j(x-x^*)\rangle \geq \lambda\|x-Tx\|^2$$

which is either called *demicontractive* (in the terminology of Hicks and Kubicek [5]) or is said to satisfy *condition(A)* (in the terminology of Maruster [7]). We adopt the former definition. If $F(T) \neq \emptyset$, (1.3) holds for $k = 0$, and for all $x \in D(T)$ and $x^* \in F(T)$ then T is *quasi-nonexpansive.* Thus the class of demicontractive maps contains the class of quasi-nonexpansive maps. It follows from (1.4) that

$$(1.5) \quad \|Tx-x^*\| \leq \frac{1+\lambda}{\lambda}\|x-x^*\| = L\|x-x^*\|, \quad \text{where} \quad L = \frac{1+\lambda}{\lambda}.$$

The class of demicontractive maps has been studied by several authors (see e.g., [1-3], [5], [9]). Let K be a closed convex subset of a Hilbert space H and $T : K \to K$ strictly pseudocontractive, Browder and Petryshyn [1] Proved the convergence of the Mann iterative sequence [6] to a fixed point of T under the assumption that T is demiclosed at 0 (a map T is demiclosed at 0, see [7], if $\{u_n\}$ is a sequence in K which converges weakly to u and $\{Tu_n\}$ converges strongly to 0 then $Tu = 0$). In 1977 Maruster [7] and Hicks and Kubicek [5] studied the results of Browder and Petryshyn [1] (though independently) when the map T is demicontractive. Chidume [2] extended the results of Maruster [7] and Hicks and Kubicek [5] to real Banach spaces which satisfy Linderstrauss smoothness condition and admit weakly sequentially continuous duality map. He then posed the following question:

Question. Can the requirement that E possesses a weakly continuous duality map be dispensed with?

It is our purpose in this paper to prove some necessary and sufficient conditions for the Mann iterative sequence (see e.g., [6]) to converge to fixed points of demicontractive maps. Our theorems provide an affirmative answer to the above question and also improve some important known results in [1-3], [5], [9].

2. Preliminaries

In the sequel we shall make use of the following lemma and remark.

LEMMA 2.1. (See [[10] Lemma 1, p. 303]). *Let* $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$\beta_{n+1} \le \beta_n + b_n, \quad n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ *then* $\lim_{n\to\infty} \beta_n$ *exists.*

REMARK 2.2. Let E be a real normed linear space. Then $\forall\, x,\ y \in E$ the following inequality holds

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{2.1}$$

for $j(x+y) \in J(x+y)$.

3. Main results

For the rest of this paper, we denote by $L > 0$ the constant appearing in inequality (1.5) and by $\lambda \in (0,1)$ the constant appearing in inequality (1.1).

LEMMA 3.1. *Let E be a real Banach space and $T : E \to E$ be a demicontractive map with $F(T) \neq \emptyset$. Let $\{a_n\}_{n=1}^{\infty} \subset [0,1]$ be a real sequence such that $\sum\limits_{n=0}^{\infty} a_n^2 < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in E$ by*

$$x_{n+1} = (1-a_n)x_n + a_n T x_n, \quad n \ge 1. \tag{3.1}$$

Then $\forall\, x^* \in F(T)$,

(a) *There exists $M > 0$ such that $\|x_n - x^*\| \le M$. Moreover $\lim\limits_{n\to\infty} \|x_n - x^*\|$ exists.*

(b) $\|x_{n+1} - x^*\| \le (1+a_n^2)\|x_n - x^*\| + \delta_n$, *where* $\delta_n = 2a_n^2(1+L)(3+L)M$.

(c) $\|x_{n+m} - x^*\| \le D\|x_n - x^*\| + D \sum\limits_{k=n}^{n+m-1} a_k^2$, *where* $D = e^{\sum_{k=n}^{n+m-1} a_k^2}$.

Proof of (a). From (1.4), (2.1) and (3.1) we get the following estimates.

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ \leq\ & (1-a_n)^2\|x_n - x^*\|^2 + 2a_n\langle Tx_n - x^*, j(x_{n+1} - x^*)\rangle \\ =\ & (1-a_n)^2\|x_n - x^*\|^2 + 2a_n\langle x_{n+1} - x^*, j(x_{n+1} - x^*)\rangle \\ & -2a_n\langle x_{n+1} - Tx_{n+1}, j(x_{n+1} - x^*)\rangle \\ & +2a_n\langle Tx_n - Tx_{n+1}, j(x_{n+1} - x^*)\rangle \\ \leq\ & (1-a_n)^2\|x_n - x^*\|^2 + 2a_n\|x_{n+1} - x^*\|^2 \\ & -2a_n\lambda\|x_{n+1} - Tx_{n+1}\|^2 \\ & +2a_n\|Tx_n - Tx_{n+1}\|\|x_{n+1} - x^*\|. \end{aligned} \tag{3.2}$$

Moreover,

$$\|x_{n+1} - x_n\| = a_n\|x_n - Tx_n\| \leq a_n(1+L)\|x_n - x^*\| \tag{3.3}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|x_{n+1} - x_n\| + \|x_n - x^*\| \\ &\leq [a_n(1+L)+1]\|x_n - x^*\|. \end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) in (3.2) yields

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ \leq\ & (1-a_n)^2\|x_n - x^*\|^2 + 2a_n[a_n(1+L)+1]^2\|x_n - x^*\|^2 \\ & +2a_nL[a_n(1+L)][a_n(1+L)+1]\|x_n - x^*\|^2 \\ \leq\ & [1+a_n^2]\|x_n - x^*\|^2 + 2a_n^2(1+L)(3+L)\|x_n - x^*\|^2 \qquad (3.5) \\ \leq\ & (1+\sigma_n)\|x_n - x^*\|^2, \qquad (3.6) \end{aligned}$$

where $\sigma_n := a_n^2[1+2(1+L)(3+L)]$. Observe that $\sum_{n=1}^{\infty}\sigma_n < \infty$. From (3.6) we get

$$\|x_{n+1} - x^*\|^2 \leq \prod_{i=1}^{n}(1+\sigma_i)\|x_1 - x^*\|^2 \leq e^{\sum_{i=1}^{\infty}\sigma_i}\|x_1 - x^*\|^2.$$

So that $\|x_n - x^*\| \leq M$ for some $M > 0$. If we set $\beta_n = \|x_n - x^*\|$ and $b_n = \sigma_n M$ then, by Lemma 1.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

Proof of (b). From (3.5) we get

$$\|x_{n+1} - x^*\|^2 \leq [1 + a_n^2 + \rho_n]\|x_n - x^*\|^2,$$

where $\rho_n := 2a_n^2(1+L)(3+L)$. Moreover

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq [1+a_n^2+\rho_n]^{\frac{1}{2}}\|x_n-x^*\| \\ &\leq [1+a_n^2+\rho_n]\|x_n-x^*\| \\ &\leq (1+a_n^2)\|x_n-x^*\|+\rho_n M \\ &= (1+a_n^2)\|x_n-x^*\|+\delta_n,\end{aligned}$$

where $\delta_n := \rho_n M = 2a_n^2(1+L)(3+L)M$.

Proof of (c). From (b), for all positive integers n, m we get the following estimates.

$$\begin{aligned}&\|x_{n+m}-x^*\| \\ &\leq (1+a_{n+m-1}^2)\|x_{n+m-1}-x^*\|+\delta_{n+m-1} \\ &\leq (1+a_{n+m-1}^2)(1+a_{n+m-2}^2)\|x_{n+m-2}-x^*\| \\ &\quad +(1+a_{n+m-1}^2)\delta_{n+m-2}+\delta_{n+m-1} \\ &\leq \cdots \leq \prod_{i=n}^{n+m-1}(1+a_i^2)\|x_n-x^*\|+\prod_{i=n}^{n+m-1}(1+a_i^2)\sum_{i=n}^{n+m-1}\delta_i \\ &\leq e^{\sum_{i=n}^{n+m-1}a_i^2}\|x_n-x^*\|+e^{\sum_{i=n}^{n+m-1}a_i^2}\sum_{i=n}^{n+m-1}\delta_i \\ &\leq D\|x_n-x^*\|+D\sum_{i=n}^{n+m-1}\delta_i,\end{aligned}$$

where $D = e^{\sum_{i=n}^{n+m-1}a_i^2}$. □

THEOREM 3.2. *Let E be a real Banach space and $T : E \to E$ be a demicontractive map with $F(T) \neq \emptyset$. Let $\{a_n\}_{n=1}^{\infty} \subset [0,1]$ be a real sequence such that $\sum_{n=0}^{\infty} a_n^2 < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in E$ by*

$$x_{n+1} = (1-a_n)x_n + a_n T x_n \quad n \geq 1.$$

Then $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof. From (b) of Lemma 3.1 we obtain

$$d(x_{n+1}, F(T)) \leq (1 + a_n^2)d(x_n, F(T)) + \delta_n.$$

Since $\liminf_{n\to\infty} d(x_n. F(T)) = 0$ we have from (a) of Lemma 2.1 that

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

It now suffices to show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. For this, let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{i=n}^{\infty} \delta_i < \infty$, there exists a positive integer N_1 such that $\forall\, n \geq N_1$

$$d(x_n, F(T)) \leq \frac{\epsilon}{3D} \quad \text{and} \quad \sum_{i=n}^{\infty} \delta_i \leq \frac{\epsilon}{6D}.$$

In particular there exists $x^* \in F(T)$ such that $d(x_{N_1}, x^*) \leq \frac{\epsilon}{3D}$. Now from Lemma 2.1 (c) we have, $\forall\, n \geq N_1$, that

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - x^*\| + \|x_n - x^*\| \\ &\leq D\|x_{N_1} - x^*\| + D \sum_{i=N_1}^{N_1+m-1} \delta_i \\ &\quad + D\|x_{N_1} - x^*\| + D \sum_{i=N_1}^{N_1+m-1} \delta_i \\ &\leq \epsilon.\end{aligned}$$

Hence $\lim_{n\to\infty} x_n$ exists (since E is complete). Suppose that $\lim_{n\to\infty} x_n = x^*$. We now show that $x^* \in F(T)$. But given any $\tilde{\epsilon} > 0$ there exists a positive integer $N_2 \geq N_1$ such that $\forall\, n \geq N_2$

$$\|x_n - x^*\| \leq \frac{\tilde{\epsilon}}{2(1+L)} \quad \text{and} \quad d(x_n, F(T)) \leq \frac{\tilde{\epsilon}}{2(1+3L)}. \tag{3.7}$$

Thus, there exists $y^* \in F(T)$ such that

$$\|x_{N_2} - y^*\| = d(x_{N_2}, y^*) \leq \frac{\tilde{\epsilon}}{2(1+3L)}.$$

From (3.7), we obtain the following estimates.

$$\begin{aligned} & \|Tx^* - x^*\| \\ \le\ & \|Tx^* - y^*\| + 2\|Tx_{N_2} - y^*\| + \|x_{N_2} - y^*\| + \|x_{N_2} - x^*\| \\ \le\ & L\|x^* - y^*\| + 2L\|x_{N_2} - y^*\| + \|x_{N_2} - y^*\| + \|x_{N_2} - x^*\| \\ \le\ & L\|x_{N_2} - x^*\| + L\|x_{N_2} - y^*\| + (1+2L)\|x_{N_2} - y^*\| + \|x_{N_2} - x^*\| \\ =\ & (1+L)\|x_{N_2} - x^*\| + \| + (1+3L)\|x_{N_2} - y^*\| \\ \le\ & \tilde{\epsilon}. \end{aligned}$$

Since $\tilde{\epsilon} > 0$ is arbitrary we have that $Tx^* = x^*$. This completes the proof. □

THEOREM 3.3. *Let E be a real Banach space and $T : E \to E$ be a demicontractive map with $F(T) \neq \emptyset$. Let $\{a_n\}_{n=1}^{\infty} \subset [0,1]$ be a real sequence such that $\sum_{n=0}^{\infty} a_n^2 < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in E$ by*

$$x_{n+1} = (1 - a_n)x_n + a_n Tx_n, \quad n \geq 1.$$

Then $\{x_n\}_{n=1}^{\infty}$ converges if and only if there exists some infinite subsequence of $\{x_n\}_{n=1}^{\infty}$ which converges to $x^* \in F(T)$.

Proof. Let $x^* \in F(T)$ and $\{x_{n_j}\}_{j=1}^{\infty}$ a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{j\to\infty} \|x_{n_j} - x^*\| = 0$. Since, by Lemma 2.1 (a), $\lim_{n\to\infty} \|x_n - x^*\|$ exists then $\lim_{n\to\infty} \|x_n - x^*\| = 0$. □

REMARK 3.4. It is well known that if a Banach space E possesses a weakly sequentially continuous duality map then E satisfies Opial's condition (see [4]). However, the $L_p (1 < p < \infty)$ spaces with $p \neq 2$ do not satisfy Opial's condition (see [8]). Our theorems apply, in particular, to L_p spaces, $1 < p < \infty$, thus providing an affirmative answer to the above mentioned question of Chidume [2]. Furthermore, our theorems improve the results in [1-3], [5], [7] and [9] in various ways.

References

[1] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math Anal. Appl. **20** (1967), 297–228.

[2] C. E. Chidume, *The solution by iteration of nonlinear equations in certain Banach spaces*, J. Nig. Math. Soc. **3** (1984), 57–62.

[3] M. K. Ghosh and L. Debnath, *Convergence of Ishikawa iterates of quasi-nonexpansive mappings*, J. Math Anal. Appl. **207** (1997), 96–103.

[4] J. P. Gossez and E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific. J. Math. **40** (1972), 565–573.

[5] T. L. Hicks and J. R. Kubicek, *On the Mann iteration process in Hilbert space*, J. Math Anal. Appl. **59** (1977), 498–504.

[6] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[7] S. Maruster, *The solution by iteration of nonlinear equations*, Proc. Amer. Math. Soc. **66** (1977), 69–73.

[8] Z. Opial, *Weak convergence of the sequence of successive approximation for non-expansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[9] M. O. Osilike and A. Udomene, *Demiclosedness principle and convergence theorems for stictly pseudocontractive mappings of the Browder-Petryshyn type*, J. Math Anal. Appl. **256** (2001), no. 2, 431–445.

[10] Tan and Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math Anal. Appl. **178** (1993), 301–308.

C. E. Chidume, The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy
E-mail: chidume@ictp.trieste.it

B. V. C. Nnoli, Department of Mathematics, University of Jos, Jos, Plateau State, Nigeria
E-mail: nnolib@unijos.edu.ng