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A REMARK OF EISENSTEIN SERIES AND THETA SERIES

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ABSTRACT. As a by-product of [5], we produce algebraic integers of certain values of quotients of Eisenstein series. And we consider the relation of $\theta_3(0,\tau)$ and $\theta_3(0,\tau^n)$. That is, we show that $|\theta_3(0,\tau^n)| = |\theta_3(0,\tau)|$, $\Delta(0,\tau) = \Delta(0,\tau^n)$ and $J(\tau) = J(\tau^n)$ for some $\tau \in \mathfrak{h}$.

1. Introduction

Let k be an imaginary quadratic field, $\mathfrak{h}$ the complex upper half plane, $\tau \in \mathfrak{h} \cap k$, $p = e^{\pi i \tau}$ and $\Delta(\tau)$ the modular discriminant. The *Weierstrass* $\wp$*-function(relative to a lattice* Λ_τ) is defined by the series

$$\wp(z) = \wp(z;\Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\} \quad ([10]).$$

In [5], [6], [7] and [8], we dealt with certain algebraic integers as values of elliptic functions constructed from Weierstrass $\wp$-function by using infinite products.

Kim [9] has obtained the leading finite-size corrections to the spectra of the asymmetric XXZ chain and the related six-vertex model in statistical physics. He established this at zero vertical field, and at zero horizontal field. The results are related by a 90° rotation. Kim's results are related to theta series. Baxter [1] had a direct proof of Kim's identities.

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In [4], we dealt with certain algebraic integers as values of theta functions and derived analogues of Burndt-Chan-Zhang's results ([2]), which would be a generalization for the case of m even.

In Section 2, we shall consider the following problem:

PROBLEM. Find $\tau \in \mathfrak{h}$ which satisfies $|\theta_3(0, z^n)| = |\theta_3(0, z)|$.

We also consider the same problem for the modular discriminant $\Delta(z)$ and $J(z)$-invariant. Using these relations, we find the algebraic integers derived from infinite products.

Throughout the article we adopt the following notations:

- $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$
- $\theta_3(0, \tau) := \prod_{n=1}^{\infty}(1 - p^{2n})(1 + p^{2n-1})^2$
- $G_k(\Lambda_\tau) := G_k(\tau) = \sum_{\omega \in \Lambda_\tau - \{0\}} \frac{1}{\omega^k}$ Eisenstein series with weight k
- $g_2(\tau) = 60G_4(\tau)$
- $g_3(\tau) = 140G_6(\tau)$
- $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = (2\pi)^{12}\eta(\tau)^{24}$
- $j(\tau) = 1728\dfrac{g_2(\tau)^3}{\Delta(\tau)}$

2. Some relations of modular functions

We know from [3] and [4] that

$$\wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2 \prod_{n=1}^{\infty}(1 - p^{2n})^4(1 + p^{(2n-1)})^8,$$

$$\wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2 \prod_{n=1}^{\infty}(1 - p^{2n})^4(1 - p^{(2n-1)})^8,$$

$$\wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{\tau}{2}\right) = 16\pi^2 p \prod_{n=1}^{\infty}(1 - p^{2n})^4(1 + p^{2n})^8.$$

Thus we get

$$(K-1) \qquad \wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2\theta_3(0, \tau)^4.$$

Let $f(X) = X^2 + uX + v \in \mathbb{Z}[X]$. If $\tau \in \mathfrak{h}$ and $\tau^2 + u\tau + v = 0$, then $\tau = \frac{-u+\sqrt{u^2-4v}}{2} = r(\cos\alpha + i\sin\alpha)$ where $r = \sqrt{v}$, $\cos\alpha = \frac{-u}{2\sqrt{v}}$, and $\sin\alpha = \frac{\sqrt{4v-u^2}}{2\sqrt{v}}$.

THEOREM 2.1. *If $0 < \alpha < \frac{\pi}{n}$ and u is odd, then the followings are satisfied:*

(1) *If v is even then*

$$\theta_3(0,\tau)^4 = \theta_3(0,\tau^k)^4,$$

where $k = 1, 2, \cdots, n$.

In particular, if $|\tau| = 1$ then there does not exist $u, v \in \mathbb{Z}$ satisfying $\theta_3(0,\tau)^4 = \theta_3(0,\tau^2)^4$.

(2) *If v is odd then*

$$\begin{cases} \theta_3(0,\tau^2)^4 = \theta_3(0,\tau+1)^4 & \text{if } n \geq 2, \\ \theta_3(0,\tau^4)^4 = \theta_3(0,\tau)^4 & \text{if } n \geq 3. \end{cases}$$

In particular, if $|\tau| = 1$ then $\tau = \dfrac{1+\sqrt{-3}}{2}$, $u = -1$, and $v = 1$.

Proof. Since the Weierstrass $\wp$-function is an even elliptic function, $\wp(\tau) = \wp(-\tau)$ and $\wp(\tau + 1) = \wp(\tau)$. By $(K-1)$,

$$\begin{aligned} -\pi^2\theta_3(0,\tau^2)^4 &= \wp\left(\frac{\tau^2}{2}\right) - \wp\left(\frac{1}{2}\right) \\ &= \wp\left(\frac{-u\tau - v}{2}\right) - \wp\left(\frac{1}{2}\right) \\ &= \wp\left(\frac{\tau + \epsilon(v)}{2}\right) - \wp\left(\frac{1}{2}\right) \\ &= -\pi^2\theta_3(0,\tau+\epsilon(v))^4, \end{aligned}$$

where

$$\epsilon(v) = \begin{cases} 0 & \text{if } v \text{ is even }, \\ 1 & \text{if } v \text{ is odd }. \end{cases}$$

Thus we consider the following two cases.

(1) Since $\tau^2 = -u\tau - v = u_2\tau + v_2$, $\tau^3 = \tau\tau^2 = \tau(u_2\tau + v_2) = u_2\tau^2 + v_2\tau = u_2(u_2\tau + v_2) + v_2\tau = (u_2^2 + v_2)\tau + u_2v_2 = u_3\tau + v_3$, $\tau^4 = \tau\tau^3 = \tau(-u_3\tau - v_3) = u_4\tau + v_4$, $\cdots$, and $\tau^n = \tau\tau^{n-1} = \tau(-u_{n-1}\tau - v_{n-1}) = u_n\tau + v_n$ where u_i is odd and v_i even with $i = 2, \cdots, n$, we obtain $\theta_3(0,\tau)^4 = \theta_3(0,\tau^2)^4 = \theta_3(0,\tau^3)^4 = \cdots = \theta_3(0,\tau^n)^4$.

(2) Since $\tau^2 = -u\tau - v$ and $\tau^4 = (-u\tau - v)^2 = u'\tau + v'$ where u, v, u' are odd and v' even, we derive $\theta_3(0,\tau+1)^4 = \theta_3(0,\tau^2)^4$ and $\theta_3(0,\tau^4)^4 = \theta_3(0,\tau)^4$.

Furthermore, if $\tau^2 + u\tau + v = 0$ and $\tau = \dfrac{-u \pm \sqrt{u^2 - 4v}}{2}$ then $1 = |\tau| = \dfrac{u^2 + 4v - u^2}{4} = v$. Thus case (2) is possible, while case (1) is impossible. Since $u^2 - 4v < 0$ and $\tau^2 \in \mathfrak{h}$, we get $u = -1$. Consequently, $\tau = \dfrac{1 + \sqrt{-3}}{2}$. We have the theorem. $\square$

For an example of Theorem 2.1, we let

$$\begin{aligned} S_v &= \{\tau \in \mathfrak{h} : \theta_3(0, \tau^2)^4 = \theta_3(0, \tau)^4,\ |\tau|^2 = v, \tau^2 + u\tau + v = 0 \\ &\qquad \text{with } u \text{ odd}, v \text{ even}\}, \\ S'_v &= \{\tau \in \mathfrak{h} : \theta_3(0, \tau^2)^4 = \theta_3(0, \tau + 1)^4,\ |\tau|^2 = v, \tau^2 + u\tau + v = 0 \\ &\qquad \text{with } u, v \text{ odd}\}. \end{aligned}$$

By Theorem 2.1, we have:

$$\begin{aligned} &S_1 = S_{2v+1} = \emptyset \text{ for all } v \in \mathbb{Z}^+, \\ &S_2 = \left\{\frac{1+\sqrt{-7}}{2}\right\}, \\ &S_4 = \left\{\frac{1+\sqrt{-15}}{2}, \frac{3+\sqrt{-7}}{2}\right\}, \\ &S_6 = \left\{\frac{1+\sqrt{-23}}{2}, \frac{3+\sqrt{-19}}{2}\right\}, \\ &S_8 = \left\{\frac{1+\sqrt{-31}}{2}, \frac{3+\sqrt{-23}}{2}, \frac{5+\sqrt{-7}}{2}\right\}, \\ &S_{10} = \left\{\frac{1+\sqrt{-39}}{2}, \frac{3+\sqrt{-31}}{2}, \frac{5+\sqrt{-15}}{2}\right\}, \\ &S_{12} = \left\{\frac{1+\sqrt{-47}}{2}, \frac{3+\sqrt{-39}}{2}, \frac{5+\sqrt{-23}}{2}\right\}, \cdots, \\ &S_{54} = \left\{\frac{1+\sqrt{-215}}{2}, \frac{3+\sqrt{-207}}{2}, \frac{5+\sqrt{-191}}{2}, \frac{7+\sqrt{-167}}{2}, \right. \\ &\qquad \left. \frac{9+\sqrt{-135}}{2}, \frac{11+\sqrt{-95}}{2}, \frac{13+\sqrt{-47}}{2}\right\}, \cdots, \end{aligned}$$

$$S'_1 = \left\{ \frac{1+\sqrt{-3}}{2} \right\},$$
$$S'_2 = S'_{2v} = \emptyset \text{ for all } v \in \mathbb{Z}^+,$$
$$S'_3 = \left\{ \frac{1+\sqrt{-11}}{2}, \frac{3+\sqrt{-3}}{2} \right\},$$
$$S'_5 = \left\{ \frac{1+\sqrt{-19}}{2}, \frac{3+\sqrt{-11}}{2} \right\},$$
$$S'_7 = \left\{ \frac{1+3\sqrt{-3}}{2}, \frac{3+\sqrt{-19}}{2}, \frac{5+\sqrt{-3}}{2} \right\},$$
$$S'_9 = \left\{ \frac{1+\sqrt{-35}}{2}, \frac{3+5i}{2}, \frac{5+\sqrt{-11}}{2} \right\},$$
$$S'_{11} = \left\{ \frac{1+\sqrt{-43}}{2}, \frac{3+\sqrt{-35}}{2}, \frac{5+\sqrt{-19}}{2} \right\}, \dots .$$

By Theorem 2.1.(1), we get the pair (u, v) of the coefficients of the polynomial $f_1(X) = X^2 + uX + v$ with $f_1(\tau) = 0$ and τ, u, v as in Theorem 2.1.(1). If k is a positive integer less than or equal to n then $\theta_3(0, \tau^k)^4 = \theta_3(0, \tau)^4$.

In this way, we put $\tau^{2k} = u_k\tau + v_k$. Then, (u_k, v_k) is the pair of the coefficients of the polynomial $f_k(X) = X^k - u_kX - v_k$ with $f_k(\tau) = 0$ and τ as in Theorem 2.1.(1). If we choose (u_2, v_2) with $\tau \in S_n$ and $0 < \alpha < \frac{\pi}{n}$, then we can find (u_k, v_k) by the following relations

$(u_2, v_2) = (-u, -v)$,

$(u_k, v_k) = (u_2u_{k-1} + v_{k-1}, u_{k-1}v_2)$ for $3 \le k \le n$.

EXAMPLE 2.2. If $u_1 = -5$, $v_1 = 8$, $\tau = \dfrac{5+\sqrt{-7}}{2}$, then we have

$$(u_1, v_1) = (-5, 8),$$
$$(u_2, v_2) = (5, -8),$$
$$(u_3, v_3) = (17, -40),$$
$$(u_4, v_4) = (45, -136),$$
$$(u_5, v_5) = (361, -360),$$
$$(u_6, v_6) = (1445, -2888),$$
$$(u_7, v_7) = (10125, -34696),$$
$$(u_8, v_8) = (15929, -81000).$$

From Theorem 2.1, we get the following corollary.

COROLLARY 2.3. *If* $0 < \alpha < \frac{\pi}{2}$ *and* $u\tau^2 + v\tau + 1 = 0$ *where* u, v *integer, then the followings hold.*

(1) *If* u *is odd and* v *even, then* $\theta_3(0, \frac{-1}{\tau})^4 = \theta_3(0, \tau)^4$.
(2) *If* u *is odd and* v *odd, then* $\theta_3(0, \frac{-1}{\tau})^4 = \theta_3(0, \tau + 1)^4$.

Proof. Note that $\theta_3(0, \frac{-1}{\tau})^4 = \theta_3(0, \frac{u\tau^2+v\tau}{\tau})^4 = \theta_3(0, u\tau + v)^4$. By Theorem 2.1, the conclusion is immediate. □

THEOREM 2.4. *Let* u *be an odd integer and* v *any integer. If* $0 < \alpha < \frac{\pi}{n}$ *and* $\tau^n + u\tau + v = 0$ *where* n *is an integer* $(\neq 0, 1)$, *then*

$$\Delta(\tau^n) = \Delta(\tau).$$

In particular, if $n = 2$, *then there does not exist* $u, v \in \mathbb{Z}$ *such that* $|\tau| = 1$.

Proof. It follows from [3, p. 69] that
$(K-2)$
$$\Delta(\tau) = 16\pi^{12}\theta_3(0,\tau)^8\theta_3(0,\tau+1)^8\left(\theta_3(0,\tau)^4 - \theta_3(0,\tau+1)^4\right)^2.$$

Let v be even. By $(K-2)$ and Theorem 2.1, we obtain that

$$\begin{aligned}\Delta(\tau^n) &= 16\pi^{12}\theta_3(0,\tau^n)^8\theta_3(0,\tau^n+1)^8\left(\theta_3(0,\tau^n)^4 - \theta_3(0,\tau^n+1)^4\right)^2\\ &= 16\pi^{12}\theta_3(0,\tau)^8\theta_3(0,\tau+1)^8\left(\theta_3(0,\tau)^4 - \theta_3(0,\tau+1)^4\right)^2\\ &= \Delta(\tau).\end{aligned}$$

When v is odd,

$$\begin{aligned}\Delta(\tau^n) &= 16\pi^{12}\theta_3(0,\tau^n)^8\theta_3(0,\tau^n+1)^8\left(\theta_3(0,\tau^n)^4 - \theta_3(0,\tau^n+1)^4\right)^2\\ &= 16\pi^{12}\theta_3(0,\tau^n)^8\theta_3(0,\tau)^8\left(\theta_3(0,\tau^n)^4 - \theta_3(0,\tau)^4\right)^2\\ &= 16\pi^{12}\theta_3(0,\tau+1)^8\theta_3(0,\tau)^8\left(\theta_3(0,\tau+1)^4 - \theta_3(0,\tau)^4\right)^2\\ &= \Delta(\tau).\end{aligned}$$

Finally, if $n = 2$ and $|\tau| = 1$, then we check the following two cases.

Case 1. If v is odd then $\theta_3(0, \tau^2) = \theta_3(0, \tau)$. By Theorem 2.1, it cannot happen.

Case 2. If v is even then $\theta_3(0, \tau^2 + 1) = \theta_3(0, \tau)$. We see by Theorem 2.1 that it is impossible.

Consequently, if $\tau^2 + u\tau + v = 0$ and $|\tau| = 1$, then $\Delta(\tau^2) \neq \Delta(\tau)$. □

COROLLARY 2.5. *Let a, u be odd integers and b, v be any integers. If $0 < \alpha < \frac{\pi}{\max\{n^l, m^t\}}$ and $\tau^{n^l} + u\tau + v = \tau^{m^t} + a\tau + b = 0$, where $n, m \neq 0, 1$ integers then $\Delta(\tau^{n^l}) = \Delta(\tau^{m^t}) = \Delta(\tau)$. That is,*

$$e^{2\pi i\tau^{n^l}} \prod_{k=1}^{\infty}(1 - e^{2\pi ik\tau^{n^l}})^{24} = e^{2\pi i\tau} \prod_{k=1}^{\infty}(1 - e^{2\pi ik\tau})^{24}.$$

Proof. It is clear from Theorem 2.4. □

COROLLARY 2.6. *Let u be odd integer and v be any integer. If $0 < \alpha < \frac{\pi}{n}$ and $\tau^n + u\tau + v = 0$ where $(n \neq 0, 1)$ is an integer, then*

$$J(\tau^n) = J(\tau).$$

In particular, if $n = 2$, then there does not exist $u, v \in \mathbb{Z}$ such that $|\tau| = 1$.

Proof. We see from [3] that

$$J(\tau) = \frac{1}{54} \frac{\left(\theta_3(0,\tau)^8 + \theta_3(0,\tau+1)^8 + \left(\theta_3(0,\tau)^4 - \theta_3(0,\tau+1)^4\right)^2\right)^3}{\theta_3(0,\tau)^8\theta_3(0,\tau+1)^8\left(\theta_3(0,\tau)^4 - \theta_3(0,\tau+1)^4\right)^2}.$$

Then, in a similar way as in Theorem 2.4, we have the conclusion. □

In like manner, we also obtain analogous formulae for Eisenstein series.

COROLLARY 2.7. *Let u be odd integer and v be any integer. If $0 < \alpha < \frac{\pi}{n}$ and $\tau^n + u\tau + v = 0$ where $(n \neq 0, 1)$ is an integer, then*

$$g_2(\tau^n) = g_2(\tau) \text{ and } g_3(\tau^n) = g_3(\tau).$$

3. Eisenstein series

We adopt the following notation

$$M_{2k} = \{ \text{ modular forms of weight } 2k \text{ for } \Gamma(1) \},$$

which is a $\mathbb{C}$-vector space([11, 12]). As is well known [11], the set $\{G_4^b G_6^c : b, c \in \mathbb{Z}, 2b + 3c = k\}$ constitutes a basis for M_{2k}.

LEMMA 3.1. *For $k \geq 2$, $G_{2k} = \sum_{i=1}^{t} a_i G_4^{b_i} G_6^{c_i}$ where t=dim M_{2k}, $a_i \in \mathbb{Q}$ and $2b_i + 3c_i = k$.*

Proof. We see from [11, 12] that

$$(K-3) \qquad G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n\geq 1}\sigma_{2k-1}(n)p^n,$$

where $\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s}$ and $\sigma_k(n) = \sum_{d|n} d^k$ are the Riemann ζ-function and k^{th}-power divisor function, respectively.

We know from [11, 12] that $\zeta(2k)$ is a rational multiple of π^{2k}.

Since $G_{2k} \in M_{2k}$, we have

$$(K-4) \qquad G_{2k}(\tau) = \sum_{i=1}^{t} a_i G_4^{b_i} G_6^{c_i}$$

with t=dim M_{2k} and $a_i \in \mathbb{C}$.

From $(K-3)$ and $(K-4)$, we see that

$$\begin{aligned}(\pi)^{2k}(d_1 + d_2 p + d_3 p^2 + \cdots) = (\pi)^{2k}\{&a_1(e_{11} + e_{12}p + e_{13}p^2 + \cdots)\\ &+ a_2(e_{21} + e_{22}p + e_{23}p^2 + \cdots)\\ &+ \cdots\\ &+ a_t(e_{t1} + e_{t2}p + e_{t3}p^2 + \cdots)\},\end{aligned}$$

and hence

$$\begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1t}\\ e_{21} & e_{22} & \cdots & e_{2t}\\ \cdots & \cdots & \cdots & \cdots\\ e_{t1} & e_{t2} & \cdots & e_{tt}\end{pmatrix}\begin{pmatrix} a_1\\ a_2\\ \cdot\\ a_t\end{pmatrix} = \begin{pmatrix} d_1\\ d_2\\ \cdot\\ d_t\end{pmatrix}$$

with d_i, $e_{i,j} \in \mathbb{Q}$. Since $\{G_4^b G_6^c : a, b \in \mathbb{Z}, 2b + 3c = k\}$ is a basis for the space M_{2k}, $a_1, \cdots, a_t$ are uniquely determined in $\mathbb{C}$. Therefore $a_1, \cdots, a_t \in \mathbb{Q}$. □

Using Corollary 2.7 and Lemma 3.1, we get the following.

COROLLARY 3.2. *Let u be odd integer and v be any integer. If $0 < \alpha < \frac{\pi}{n}$ and $\tau^n + u\tau + v = 0$ where $(n \neq 0, 1)$ is an integer, then*

$$G_{2k}(\tau^n) = G_{2k}(\tau)$$

with $k \geq 2$.

PROPOSITION 3.3. ([5]) *If* $k \geq 2$, $\dfrac{G_{2k}(\tau)}{(\pi)^{2k}\eta(\tau)^{4k}}$ *is an algebraic number. And if* $k \geq 2$ *and* $G_{2k}(\tau) \neq 0$, *then* $\dfrac{G_{2k}(\tau)}{\eta(\tau)^{4k}}$ *is a transcendental number.*

Using Corollary 3.2 and Proposition 3.3, we derive

COROLLARY 3.4. *Let* u *be odd integer and* v *be any integer, and let* $0 < \alpha < \frac{\pi}{n}$ *and* $\tau^n + u\tau + v = 0$ *where* $(n \neq 0, 1)$ is an integer. If $k \geq 2$, $\dfrac{G_{2k}(\tau^n)}{(\pi)^{2k}\eta(\tau^n)^{4k}}$ is an algebraic number. And if $k \geq 2$ and $G_{2k}(\tau^n) \neq 0$, then $\dfrac{G_{2k}(\tau^n)}{\eta(\tau^n)^{4k}}$ is a transcendental number.

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