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EXCHANGE RINGS SATISFYING STABLE RANGE CONDITIONS

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ABSTRACT. In this paper, we establish necessary and sufficient conditions for an exchange ring R to satisfy the n-stable range condition. It is shown that an exchange ring R satisfies the n-stable range condition if and only if for any regular $a \in R^n$, there exists a unimodular $u \in^n R$ such that $au \in R$ is a group member, and if and only if whenever $a \overset{\sim}{-}_n b$ with $a \in R, b \in M_n(R)$, there exist $u \in R^n, v \in^n R$ such that $a = ubv$ with $uv = 1$. As an application, we observe that exchange rings satisfying the n-stable range condition can be characterized by Drazin inverses. These also give nontrivial generalizations of [7, Theorem 10], [13, Theorem 10], [15, Theorem] and [16, Theorem 2A].

1. Introduction

Let R be an associative ring with identity. R is said to satisfy the n-stable range condition if for every split epimorphism $M = (\oplus_{i=1}^n R) \oplus H \to R$ there is a splitting $\psi : R \to M$ such that $M = \psi(R) \oplus L \oplus H$ with $L \subseteq \oplus_{i=1}^n R$. It is well known that, if A is a right R-module and $\mathrm{End}_R A$ satisfies the n-stable range condition, then $A^{n+1} \oplus B \cong A \oplus C$ implies $A^n \oplus B \cong C$ for any right R-modules B and C. The n-stable range condition plays an important role in algebraic K-theory. For example, if R satisfies the n-stable range condition then the central extension $1 \to \mathrm{K}_2(m, R) \to \mathrm{St}(m, R) \to \mathrm{E}(m, R) \to 1$ is a universal central extension for all $m \geq \max(n+3, 5)$ and $\mathrm{K}_2(m, R) \to \mathrm{K}_2(m+1, R)$ are surjective for all $m \geq n + 2$. Furthermore, one sees that $\mathrm{K}_{2,m}(R) \cong \mathrm{K}_2(R)$ provided that $m \geq n + 2$. Also we have $\mathrm{K}_1(R) \cong \mathrm{U}(R)/\mathrm{W}(R)$ provided that R satisfies the 1-stable range condition. Many authors have studied rings satisfying the n-stable range condition (see [1], [3-6] and [10-14]).

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Recall that R is an exchange ring if, for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\bigoplus_{i\in I} A_i')$. The class of exchange rings is very large. It includes all regular rings, all π-regular rings, all strongly π-regular rings, all semiperfect rings, all left or right continuous rings, all clean rings, all unit C^*-algebras of real rank zero and all right semi-artinian rings(cf. [2-4], [9-12] and [14] and [17]).

In this paper, we establish necessary and sufficient conditions for an exchange ring R to satisfy the n-stable range condition. It is shown that an exchange ring R satisfies the n-stable range condition if and only if for any regular $a \in R^n$, there exists a unimodular $u \in^n R$ such that $au \in R$ is a group member. In addition, we introduce n-pseudo similarity for exchange rings. Using this new concept, we show that an exchange ring R satisfies the n-stable range condition is equivalent to whenever $a \widetilde{\sim}_n b$ with $a \in R, b \in \mathrm{M}_n(R)$, there exist $u \in R^n, v \in^n R$ such that $a = ubv$ with $uv = 1$. As an application, we observe that exchange rings satisfying the n-stable range condition can be characterizated by Drazin inverses. These also give nontrivial generalizations of [7, Theorem 10], [13, Theorem 10], [15, Theorem] and [16, Theorem 2A].

Throughout this paper, all rings are associative rings with identity and all modules are right unitary modules. The symbol $M \lesssim^{\oplus} N$ means that M is isomorphic to a direct summand of a module N. The symbol mR denotes the $M_m(R)$-R-bimodule $\{(x_1, \ldots, x_m)^T \mid x_1, \ldots, x_m \in R\}$, where scalar multiplications are products of matrices over the ring R. $R^n = \{(a_1, \cdots, a_n) \mid a_1, \cdots, a_n \in R\}, {}^n R = \{(a_1, \cdots, a_n)^T \mid a_1, \cdots, a_n \in R\}$. Call $x \in R^n (x \in^n R)$ is unimodular provided that $xy = 1 (yx = 1)$ for some $y \in^n R (y \in R^n)$. $x \in R^n (x \in^n R)$ is said to be regular provided that $x = xyx$ for some $y \in^n R (y \in R^n)$.

2. Group members

Recall that $g \in R$ is a group member of R provided that there exists some $g^{\#} \in R$ such that $g = gg^{\#}g$, $g^{\#} = g^{\#}gg^{\#}$ and $gg^{\#} = g^{\#}g$. That is, $g \in R$ has group inverse $g^{\#} \in R$. The author proved that a regular ring R is one-sided unit-regular if and only if for any regular $a \in R$, there exists some one-sided unit $u \in R$ such that $au \in R$ is a group member. By virtue of group members, we now extend [7, Theorem 10] and characterize exchange rings satisfying the n-stable range condition, which also extend [16, Theorem 2A].

LEMMA 2.1. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R satisfies the n-stable range condition.*
(2) *Whenever $a(^nR) = eR$ with $a \in R^n, e = e^2 \in R$, there exists a unimodular $u \in^n R$ such that $au = e$.*
(3) *Whenever $a(^nR) = eR$ with $a \in R^n, e = e^2 \in R$, there exists a unimodular $u \in R^n$ such that $a = eu$.*

Proof. (1)$\Rightarrow$(2) Given $a(^nR) = eR$ with $a \in R^n, e = e^2 \in R$, then $ax = e$ for some $x \in^n R$. Furthermore, we have $r_1, \cdots, r_n \in R$ such that $a(1, 0, \cdots, 0)^T = er_1, a(0, 1, \cdots, 0)^T = er_2, \cdots, a(0, 0, \cdots, 1)^T = er_n$. Hence $a = a\mathrm{diag}(1, \cdots, 1) = e(r_1, r_2, \cdots, r_n)$. Set $y = (r_1, r_2, \cdots, r_n)$. Then we claim that $ax = e$ and $a = ey$ for $x \in^n R, y \in R^n$. Since $yx + (1 - yx) = 1$, we can find a $z \in R^n$ such that $y + (1 - yx)z = u \in R^n$ is unimodular. So we have $uv = 1$ for a $v \in^n R$, and then $(y + (1 - yx)z)v = 1$. It is easy to verify that $eyx = ax = e$. Therefore $e = e(y + (1 - yx)z)v = eyv = av$, as desired.

(2) $\Rightarrow$ (1) Given any regular $x \in R^n$, there exists a $y \in^n R$ such that $x = xyx$. We easily check that $x(^nR) = (xy)R$. Hence we can find a unimodular $u \in^n R$ such that $xu = xyx$. Therefore $x = xyx = xux$. It follows by [18, Theorem 9] that R satisfies the n-stable range condition.

(1)$\Rightarrow$(3) is analogous to the consideration above.

(3)$\Rightarrow$(1) For any regular $a \in R^n$, there exists $b \in^n R$ such that $a = aba$. Set $e = ab$. Then $a(^nR) = eR$ with $e = e^2 \in R$. Hence $a = eu$ for a unimodular $u \in R^n$. It follows by [12, Theorem 4.2] that R satisfies the n-stable range condition. □

THEOREM 2.2. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R satisfies the n-stable range condition.*
(2) *Whenever $a(^nR) = gR$ with $a \in R^n$ and group member $g \in R$, there exists a unimodular $u \in^n R$ such that $au = g$.*
(3) *Whenever $a(^nR) = gR$ with $a \in R^n$ and group member $g \in R$, there exists a unimodular $u \in^n R$ such that $a = gu$.*

Proof. (2)$\Rightarrow$(1) is clear by Lemma 2.1.

(1)$\Rightarrow$(2) Suppose that $a(^nR) = gR$ with $a \in R^n$ and group member $g \in R$. Because $g \in R$ has a group inverse $g^\#$ in R, we see that $g = g(g^\# + 1 - g^\# g)g$. Clearly, $(g^\# + 1 - g^\# g)^{-1} = g + 1 - g^\# g \in \mathrm{U}(R)$. Set $g(g^\# + 1 - g^\# g) = e$. Then $a(^nR) = eR$. In view of Lemma 2.1, we can find a unimodular $u \in^n R$ such that $au = e = g(g^\# + 1 - g^\# g)$. Hence

$au(g^{\#}+1-g^{\#}g)^{-1}=g$. One easily checks that $u(g^{\#}+1-g^{\#}g)^{-1}\in^n R$ is unimodular, as required.

(3)⇒(1) Since every idempotent has group inverse, the implication is obvious by Lemma 2.1.

(1)⇒(3) Suppose that $a(^nR)=gR$ with $a\in R^n$ and group member $g\in R$. Similarly to the consideration above, we have $a(^nR)=eR$ with $e=g(g^{\#}+1-g^{\#}g)$. Using Lemma 2.1 again, we can find a unimodular $u\in^n R$ such that $a=eu=g(g^{\#}+1-g^{\#}g)u$. Clearly, $(g^{\#}+1-g^{\#}g)u\in R^n$ is also unimodular. Therefore we complete the proof. □

In general, R satisfies the n-stable range condition is not equivalent to for any regular $a\in R^n$, there exists a unimodular $u\in^n R$ such that $au\in R$ is a group member if R is not an exchange ring. For example, $\mathbb{Z}$ satisfies the condition (2) above for $n=1$, while it has no stable range one. But we note that the class of exchange rings satisfying the finite stable ranges is very large. We refer the reader to [3], [11-12] and [18]. In order to have more examples of exchange rings satisfying the n-stable range condition, we raise the question of when a trivial extension is an exchange ring satisfying the n-stable range condition.

PROPOSITION 2.3. *Let R be a ring, M an R-R-bimodule. If R is an exchange ring satisfying the n-stable range condition, then so is the trivial extension $R\ltimes M$.*

Proof. We easily check that $\mathrm{U}(R\ltimes M)=\{(u,m)\mid u\in\mathrm{U}(R),m\in M\}$. Hence $\mathrm{J}(R\ltimes M)=\{(r,m)\mid r\in\mathrm{J}(R),m\in M\}$. Therefore $R\ltimes M/\mathrm{J}(R\ltimes M)\cong R/\mathrm{J}(R)$. Since R is an exchange ring satisfies the n-stable range condition, so is $R\ltimes M/\mathrm{J}(R\ltimes M)$. Hence $R\ltimes M$ satisfies the n-stable range condition. Clearly, $\mathrm{J}(R\ltimes M)$ is an exchange ring and idempotents in $R\ltimes M/\mathrm{J}(R\ltimes M)$ can be lifted modulo $\mathrm{J}(R\ltimes M)$. In view of [2, Theorem 2.2], we claim that $R\ltimes M$ is an exchange ring, as asserted. □

3. n-Pseudo similarity

So as to get more characterizations of exchange rings satisfying the n-stable range condition, we now introduce the concept of the n-pseudo similarity over exchange rings.

Let $a\in R, b\in \mathrm{M}_n(R)$. Then we say that a is n-pseudo similar to b in R provided that there exist $x\in^n R, y,z\in R^n$ such that $xay=b, zbx=a$

and $xyx = xzx = x$. We denote it by $a \overset{\sim}{\approx}_n b$. Clearly, pseudo similarity on a ring R is equivalent to 1-pseudo similarity. It is well known that R satisfies cancellation if and only if pseudo similarity coincides with pseudo similarity (see [15, Theorem]). Unfortunately, Guralnick and Lanski's technique can not be applied to exchange rings satisfying the n-stable range condition. We need to develop some new methods.

LEMMA 3.1. *Let $e = e^2 \in R$ and $f = f^2 \in M_n(R)$. Then*

(1) *$eR \lesssim^{\oplus} f(nR)$ if and only if there exist $a \in R^n$ and $b \in^n R$ such that $e = ab$, $a = eaf$ and $b = fbe$.*

(2) *$eR \cong f(nR)$ if and only if there exist $a \in R^n$ and $b \in^n R$ such that $e = ab$, $f = ba$, $a = eaf$ and $b = fbe$.*

Proof. (1) Suppose that $eR \lesssim^{\oplus} f(nR)$. We have a projection $p : f(nR) \to eR$. Let $\{\eta_1, \dots, \eta_t\}$ be the standard basis of nR. Then we have $x \in nR$ such that $e = p(fx)$, so

$$\begin{aligned} e &= e^2 = ep(fx)e \\ &= e\big(p(\textstyle\sum_{j=1}^{n} f\eta_j\eta_j^T fx)\big)e \\ &= \big(e(\textstyle\sum_{j=1}^{t} p(f\eta_j)\eta_j^T)f\big)(fx)e. \end{aligned}$$

Let $a = e(\sum_{j=1}^{n} p(f\eta_j)\eta_j^T)f$ and $b = fxe$. Then $a = eaf$, $b = fbe$, and $e = ab$, as desired.

Conversely, suppose $e = ab$ with $a \in R^n$ and $b \in^n R$ such that $a = eaf$ and $b = fbe$. We construct an R-homomorphism $\psi : f(nR) \to eR$ by $\psi(fx) = afx$ for $x \in R$. For any $er \in eR$ with $r \in R$, there exists $x = br \in nR$ such that $\psi(fx) = afx = afbr = abr = er$. So ψ is an epimorphism. Because eR is a projective right R-module, we know that the exact sequence $0 \to \mathrm{Ker}\psi \to f(nR) \to eR \to 0$ is split. Hence, $eR \oplus \mathrm{Ker}\psi \cong f(nR)$. Therefore, $eR \lesssim^{\oplus} f(nR)$.

(2) Analogously the consideration above, we easily complete the proof. □

LEMMA 3.2. *Let R be a ring with $a \in R, b \in M_n(R)$. Then the following are equivalent:*

(1) *$a \overset{\sim}{\approx}_n b$.*

(2) *There exist $x \in^n R, y \in R^n$ such that $a = ybx, b = xay, x = xyx$ and $y = yxy$.*

Proof. (2)⇒(1) is trivial.

(1)⇒(2) Since $a \overset{\sim}{\approx} b$, there are $x \in^n R, y, z \in R^n$ such that $xay = b, zbx = a$ and $xyx = xzx = x$ By replacing y with yxy and z with

zxz, we can assume $y = yxy$ and $z = zxz$. We directly check that $xazxy = xzbxzxy = xzbxy = xay = b, zxybx = zxyxayx = zxayx = zbx = a, zxy = zxyxzxy$ and $x = xzxyx$, as required. $\square$

In [15], Guralnick and Lanski observed that the definition of pseudo-similarity in [15] differs from that introduced by Hartwig and Putcha (Linear Algebra Appl., 39, **1981**, 125–132). In fact, we have proved that the two kind of concepts coincide with each other in the lemma above. Now we characterize exchange rings satisfying the n-stable range condition by virtue of n-pseudo similarity. We also note that if R is an exchange ring satisfying the n-stable range condition then so is $M_n(R)$. Thus n-pseudo similarity also give new relations for matrices over exchange rings.

THEOREM 3.3. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R satisfies the n-stable range condition.*
(2) *For any $a \in R, b \in M_n(R)$, $a \overset{\sim}{\approx}_n b$ implies that there exist $u \in R^n, v \in^n R$ such that $a = ubv$ with $uv = 1$.*

Proof. (1)⇒(2) Assume that $a \overset{\sim}{\approx}_n b$. By Lemma 3.2, we have $x \in^n R, y \in R^n$ such that $a = ybx, b = xay, x = xyx$ and $y = yxy$. Since R satisfies the n-stable range condition, by [18, Theorem 9], we have a unimodular $s \in R^n$ such that $x = xsx$. Assume that $st = 1$. Set $w = x + (I_n - xs)t(1 - sx)$. Then we verify that $x = wsx$ ant $sw = 1$. Let $u = (1-yx-sx)s(I_n-xy-xs), v = (I_n-xy-xs)w(1-yx-sx)$. One easily checks that $(I_n - xy - xs)^2 = I_n, (1 - yx - sx)^2 = 1$ and $uv = 1$. Furthermore, we show that $ubv = (1-yx-sx)s(I_n-xy-xs)b(I_n-xy-xs)w(1-yx-sx) = (1-yx-sx)s(I_n-xy-xs)bx = yxsbx = ybx = a$, as desired.

(2)⇒(1) Given $eR \cong f(nR)$ with $e = e^2 \in R$, $f = f^2 \in M_n(R)$, in view of Lemma 3.2, there exist $a \in R^n$ and $b \in^n R$ such that $e = ab$, $f = ba$, $a = eaf$, and $b = fbe$. Clearly, we have $e = afb$, $f = bea$, $a = aba$. So $e \overset{\sim}{\approx}_n f$. Thus we can find $u \in R^n$ and $v \in^n R$ such that $e = ufv$ with $uv = 1$. It is easy to check that $1 - e = u(I_n - f)v = ((1-e)u(I_n-f))((I_n-f)v(1-e))$. Using Lemma 3.2 again, we conclude that $(1 - e)R \lesssim^{\oplus} (I_n - f)(nR)$.

Given R-module decompositions $R = A_1 \oplus B_1$ and $nR = A_2 \oplus B_2$ with $A_1 \cong A_2$, there are $e = e^2 \in R$ and $f = f^2 \in M_n(R)$ such that $A_1 = eR$ and $A_2 = f(nR)$. Thus, $eR \cong f(nR)$, whence $B_1 \cong R/A_1 \cong$

$(1-e)R \lesssim^{\oplus} (I_n - f)(nR) \cong nR/A_2 \cong B_2$. According to [18, Theorem 13], we complete the proof. □

Recall that an exchange ring R is strongly separative if the following condition holds: For all finitely generated projective right R-modules A, B and C, $2C \oplus A \cong C \oplus B \Longrightarrow C \oplus A \cong B$(cf. [3]).

COROLLARY 3.4. *Let R be a strongly separative exchange ring, $n \geq 2$ a positive integer. For any $a \in R, b \in M_n(R)$, $a \overset{\sim}{\approx}_n b$, there exist $u \in R^n, v \in^n R$ such that $a = ubv$ with $uv = 1$.*

Proof. Since R is a strongly separative exchange ring, it satisfies the 2-stable range condition. So we complete the proof by Theorem 3.3. □

PROPOSITION 3.5. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R satisfies the n-stable range condition.*
(2) *For any regular $a \in R^n$, there exists a unimodular $u \in R^n, v \in^n R$ such that $aa^- = ua^-av$ with $uv = 1$.*
(3) *For any regular $a \in R^n$, there exists a unimodular $u \in^n R$ such that $a^-au = uaa^-$.*

Proof. (2)⇒(1) Assume that $eR \cong f(nR)$ with $e = e^2 \in R, f = f^2 \in \mathrm{M}_n(R)$. By Lemma 3.1, we have $a \in R^n$ and $b \in^n R$ such that $e = ab, f = ba, a = eaf$ and $b = fbe$. Thus we claim that $a = ea = aba$. So we may choose $a^- = b$. Hence $ab = u(ba)v$ and $uv = 1$ for some $u \in^n R, v \in^n R$. Therefore $1 - e = u(I_n - f)v = ((1-e)u(I_n - f))((I_n - f)v(1-e))$. Consequently, we show that $(1-e)R \lesssim^{\oplus} (I_n - f)(nR)$, as needed.

(1)⇒(2) Given any regular $a \in R^n$, there exists $a^- \in^n R$ such that $a = aa^-a$. We construct a map $\psi : aa^-R = a(^nR) \to a^-a(^nR)$ given by $\psi(ax) = a^-ax$ for any $x \in^n R$. Obviously, we claim that $\psi : aa^-R \cong a^-a(^nR)$. According to Lemma 3.1 and Lemma 3.2, we see that $aa^- \overset{\sim}{\approx}_n a^-a$. Hence the result follows by Theorem 3.3.

(1)⇔(3) is analogous to the discussion above. □

Recall that $a \in R$ is said to be strongly π-regular provided that there exist $n \geq 1$ and $x \in R$ such that $a^n = a^{n+1}x, ax = xa$ and $x = xax$. By an argument of P. Ara, we know that every strongly π-regular element of an exchange ring is unit-regular. Clearly, the solution $x \in R$ is unique, and we say that x is the Drazin inverse a^d of a. As an application of Theorem 3.3, we now give a new necessary and sufficient condition under which an exchange ring R satisfies the n-stable range condition, which also shows that the n-stable range condition over exchange rings can be determined by Drazin inverses.

THEOREM 3.6. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R satisfies the n-stable range condition.*
(2) *Whenever $ab \in R, ba \in M_n(R)$ are strongly π-regular with $a \in R^n, b \in^n R$, there exists a unimodular $u \in R^n, v \in^n R$ such that $(ab)^d = u(ba)^d v$ with $uv = 1$.*
(3) *Whenever $ab \in R, ba \in M_n(R)$ are strongly π-regular with $a \in R^n, b \in^n R$, there exists a unimodular $u \in R^n$ such that $(ab)(ab)^d = u(ba)(ba)^d v$ with $uv = 1$.*

Proof. (1)⇒(2) Since $ab \in R$ are strongly π-regular, we have $k \geq 1$ such that $(ab)^k = (ab)^{k+1}(ab)^d$, $(ab)(ab)^d = (ab)^d(ab)$ and $(ab)^d = (ab)^d(ab)(ab)^d$. Likewise, we have $l \geq 1$ such that $(ba)^l = (ba)^{l+1}(ba)^d$, $(ba)(ba)^d = (ba)^d(ba)$ and $(ba)^d = (ba)^d(ba)(ba)^d$. So we verify that

$$\begin{aligned}
&(ab)^{k+l+2}a(ba)^d(ba)^d b\\
&= (ab)a(ba)^{k+l+1}(ba)^d(ba)^d b\\
&= (ab)a(ba)^{k+l}(ba)^d b\\
&= a(ba)^{k+l+1}(ba)^d b\\
&= a(ba)^{k+l}b = (ab)^{k+l+1},
\end{aligned}$$

$$\begin{aligned}
&(ab)(a(ba)^d(ba)^d b)\\
&= a(ba)^d(ba)^d(ba)b\\
&= a(ba)^d b\\
&= (a(ba)^d(ba)^d b)(ab),\\
&(a(ba)^d(ba)^d b)(ab)(a(ba)^d(ba)^d b)\\
&= a(ba)^d(ba)^d b.
\end{aligned}$$

Hence $(ab)^d = a(ba)^d(ba)^d b$. By a similar route, we also have $(ba)^d = b(ab)^d(ab)^d a$. Furthermore, we check that

$$(ba)^d = (ba)^d(ba)(ba)^d = (ba)^d bab(ab)^d(ab)^d a = (ba)^d b(ab)^d a.$$

Clearly, $(ba)^d ba(ba)^d b = (ba)^d b$. Thus $(ab)^d \precsim_n (ba)^d$, as required.

(2) $\Rightarrow$ (1) Suppose that $eR \cong f(^nR)$ with idempotents $e \in R, f \in \mathrm{M}_n(R)$. Then we can find some $a \in R^n, b \in^n R$ such that $e = ab, f = ba, a = eaf$ and $b = fbe$. Obviously, e and f are both strongly π-regular, hence $(ab)^d \overset{\sim}{\approx}_n (ba)^d$. Inasmuch as $e^d = e$ and $f^d = f$, we claim that $e = ufv$ and $uv = 1$ for some $u \in R^n, v \in^n R$. Therefore R satisfies the n-stable range condition by Theorem 3.3.

(1) $\Rightarrow$ (3) Similarly to the consideration above, one easily checks that $(ab)(ab)^d \overset{\sim}{\approx}_n (ba)(ba)^d$, as desired.

(3) $\Rightarrow$ (1) Given $eR \cong f(^nR)$ with $e = e^2 \in R, f = f^2 \in \mathrm{M}_n(R)$, then there exist $a \in R^n, b \in^n R$ such that $e = ab, f = ba, a = eaf$ and $b = fbe$. Analogously to the consideration in (2) $\Rightarrow$ (1), we obtain the result. □

COROLLARY 3.7. *Let $\mathbb{F}$ be a field. For any $a \in \mathbb{F}^n, b \in^n \mathbb{F}$, there exists a unimodular $u \in \mathbb{F}^n, v \in^n \mathbb{F}$ such that $(ab)^d = u(ba)^d v$ with $uv = 1$.*

Proof. Since $\mathbb{F}$ is a field, we know that $\mathbb{F}$ and $\mathrm{M}_n(\mathbb{F})$ are both strongly π-regular rings. Hence $(ab)^d$ and $(ba)^d$ exist for any $a \in \mathbb{F}^n, b \in^n \mathbb{F}$. Thus the proof is complete by Theorem 3.6. □

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