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TOPOLOGICAL CONJUGACY OF DISJOINT FLOWS ON THE CIRCLE

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Abstract. Let $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ and $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ be disjoint flows defined on the unit circle $\mathbb{S}^1$, that is such flows that each their element either is the identity mapping or has no fixed point ($(V,\ +)$ is a 2-divisible nontrivial abelian group). The aim of this paper is to give a necessary and sufficient condition for topological conjugacy of disjoint flows i.e., the existence of a homeomorphism $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ satisfying

$$\Gamma \circ F^v = G^v \circ \Gamma, \qquad v \in V.$$

Moreover, under some further restrictions, we determine all such homeomorphisms.

1. Introduction

Let X be a topological space and $(V,\ +)$ be a 2-divisible nontrivial (i.e., $\mathrm{card}V > 1$) abelian group.

Recall that a family $\{F^v : X \longrightarrow X,\ v \in V\}$ of homeomorphisms such that

$$F^{v_1} \circ F^{v_2} = F^{v_1+v_2}, \qquad v_1,\ v_2 \in V$$

is called an *iteration group* or a *flow* (on X). A flow $\{F^v : X \longrightarrow X,\ v \in V\}$ is said to be *disjoint* if every its element either is the identity mapping or has no fixed point. Some results concerning disjoint flows on the unit circle $\mathbb{S}^1$ can be found in [3], [6], [7] and [9]. A flow $\{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in \mathbb{R}\}$ is called *continuous* if for every $z \in \mathbb{S}^1$ the mapping $\mathbb{R} \ni v \longmapsto F^v(z) \in \mathbb{S}^1$ is continuous. A continuous flow $\{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in \mathbb{R}\}$ is said to be *positively equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $z,\ w \in \mathbb{S}^1$ and $v \geq 0$, $d(z,\ w) < \delta$ implies

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$d(F^v(z),\ F^v(w)) < \epsilon$, where $d(z,\ w) := |x - y|$ for some $x,\ y \in \mathbb{R}$ with $|x - y| \leq \frac{1}{2}$ such that $e^{2\pi ix} = z$ and $e^{2\pi iy} = w$. Continuous flows have been studied in [14], and positively equicontinuous flows in [2].

Flows $\mathcal{F} = \{F^v : X \longrightarrow X,\ v \in V\}$ and $\mathcal{G} = \{G^v : X \longrightarrow X,\ v \in V\}$ are said to be *topologically conjugate* if there exists a homeomorphism $\Gamma : X \longrightarrow X$ for which

$$\Gamma \circ F^v = G^v \circ \Gamma, \qquad v \in V. \tag{1}$$

The problem of topological conjugacy of disjoint flows defined on open real intervals in the case where $V = \mathbb{R}$ has been examined by M. C. Zdun in [15] (see also [4] for the case when $V = \mathbb{Q}$). The aim of this paper is to give a necessary and sufficient condition for topological conjugacy of disjoint flows on $\mathbb{S}^1$. Moreover, under some further restrictions, we determine all homeomorphisms $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ satisfying (1). Our results generalize those obtained by the author in [7] (another generalization one can find in [10]), where a special case of disjoint flows under the assumptions that $V = \mathbb{R}$ and Γ preserves orientation has been studied.

2. Preliminaries

We begin by recalling the basic definitions and introducing some notation.

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and A^{d} stands for the set of all cluster points of A.

For any $v,\ w,\ z \in \mathbb{S}^1$ there exist unique $t_1,\ t_2 \in [0,\ 1)$ such that $we^{2\pi it_1} = z$ and $we^{2\pi it_2} = v$, so we can put

$$v \prec w \prec z \quad \text{if and only if} \quad 0 < t_1 < t_2$$

and

$$v \preceq w \preceq z \quad \text{if and only if} \quad t_1 \leq t_2 \ \text{ or } \ t_2 = 0$$

(see [3]). The properties of these relations can be found in [6] and [7]. It is easily seen that we also have

Remark 1. For any $v,\ w,\ z,\ c \in \mathbb{S}^1$, $v \prec w \prec z$ implies $c \cdot v \prec c \cdot w \prec c \cdot z$.

A set $A \subset \mathbb{S}^1$ is said to be an *open arc* if there are distinct $v,\ z \in \mathbb{S}^1$ for which

$$A = \overrightarrow{(v,\ z)} := \{w \in \mathbb{S}^1 : v \prec w \prec z\} = \{e^{2\pi it},\ t \in (t_v,\ t_z)\},$$

where $t_v,\ t_z \in \mathbb{R}$ are such that $e^{2\pi i t_v} = v,\ e^{2\pi i t_z} = z$ and $0 < t_z - t_v < 1$. Given a subset A of $\mathbb{S}^1$ with card$A \geq 3$ and a function F mapping A into $\mathbb{S}^1$ we say that F is *increasing* (respectively, *strictly increasing*) if for any $v,\ w,\ z \in A$ such that $v \prec w \prec z$ we have $F(v) \preceq F(w) \preceq F(z)$ (respectively, $F(v) \prec F(w) \prec F(z)$). For every homeomorphism $F : A \longrightarrow B$, where $A = \{e^{2\pi i t},\ t \in (a,\ b)\}$ and $B = \{e^{2\pi i t},\ t \in (c,\ d)\}$ are open arcs, there exists a unique homeomorphism $f : (a,\ b) \longrightarrow (c,\ d)$ with $F(e^{2\pi i x}) = e^{2\pi i f(x)}$ for $x \in (a,\ b)$ (see [3] and [6]).

It is well-known (see for instance [1], [5] and [13]) that for every continuous mapping $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ there is a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$, which is unique up to translation by an integer, and a unique integer k such that

$$F(e^{2\pi i x}) = e^{2\pi i f(x)} \quad \text{and} \quad f(x+1) = f(x) + k, \qquad x \in \mathbb{R}.$$

The function f is said to be the *lift* of F and the integer k is called the *degree* of F, and is denoted by $\deg F$. If $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a homeomorphism, then so is its lift. Furthermore, $|\deg F| = 1$. We say that a homeomorphism $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *preserves orientation* if $\deg F = 1$, which is clearly equivalent to the fact that the lift of F is increasing (recall that each element of a flow $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ preserves orientation). For every such homeomorphism F the number $\alpha(F) \in [0,\ 1)$ defined by

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} (\text{mod } 1), \qquad x \in \mathbb{R}$$

is called the *rotation number* of F. This number always exists and does not depend on x and f. Furthermore, $\alpha(F)$ is rational if and only if F has a periodic point. If $\alpha(F) \notin \mathbb{Q}$, then the non-empty set $L_F := \{F^n(z),\ n \in \mathbb{Z}\}^{\mathrm{d}}$ (the *limit set* of F) does not depend on $z \in \mathbb{S}^1$, is invariant with respect to F and either $L_F = \mathbb{S}^1$ or L_F is a perfect nowhere dense subset of $\mathbb{S}^1$ (see for instance [11] and [12]).

A flow $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is said to be *non-singular* if at least one its element has no periodic point, otherwise $\mathcal{F}$ is called a *singular* flow. By the *limit set* of a non-singular flow $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ we mean the set $L_{\mathcal{F}} := L_{F^v}$, where $F^v \in \mathcal{F}$ is an arbitrary homeomorphism with irrational rotation number. By the *limit set* of a disjoint flow $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ we mean the set $L_{\mathcal{F}} := \{F^v(z),\ v \in V\}^{\mathrm{d}}$, where z is an arbitrary element of $\mathbb{S}^1$. A non-singular or disjoint flow $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is called: – *dense*, if $L_{\mathcal{F}} = \mathbb{S}^1$; – *non-dense*, if $\emptyset \neq L_{\mathcal{F}} \neq \mathbb{S}^1$; – *discrete*, if $L_{\mathcal{F}} = \emptyset$ (see [9] and also [7]). It is worth pointing out that every discrete flow is both disjoint and singular, and every dense flow is disjoint (see [9]).

3. Main results

The proof of our next remark is obvious.

REMARK 2. (see also [3]) If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ and $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ are topologically conjugate flows, then $\mathcal{F}$ is disjoint if and only if so is $\mathcal{G}$.

As an immediate consequence of Theorem 9 in [2] and Remark 2 we obtain

COROLLARY 1. *Every positively equicontinuous flow* $\{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in \mathbb{R}\}$ *is disjoint.*

REMARK 3. (see [12] and [10]) If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ and $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ are flows and a homeomorphism $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ satisfies (1), then

$$\alpha(G^v) = \alpha(F^v) \deg \Gamma (\mathrm{mod}\ 1), \qquad v \in V.$$

According to Remark 3 we get

COROLLARY 2. *If* $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ *and* $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ *are topologically conjugate flows, then* $\mathcal{F}$ *is non-singular (respectively, singular) if and only if so is* $\mathcal{G}$.

LEMMA 1. *If* $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ *and* $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ *are non-singular flows, then there exists a* $v_0 \in V$ *for which* $\alpha(F^{v_0}),\ \alpha(G^{v_0}) \notin \mathbb{Q}$.

Proof. Let $v_{\mathcal{F}}, v_{\mathcal{G}} \in V$ be such that $\alpha(F^{v_{\mathcal{F}}}), \alpha(G^{v_{\mathcal{G}}}) \notin \mathbb{Q}$. If $\alpha(F^{v_{\mathcal{G}}}) \notin \mathbb{Q}$, then we put $v_0 := v_{\mathcal{G}}$. If $\alpha(G^{v_{\mathcal{F}}}) \notin \mathbb{Q}$, then we set $v_0 := v_{\mathcal{F}}$. Finally, assume that $\alpha(F^{v_{\mathcal{G}}}),\ \alpha(G^{v_{\mathcal{F}}}) \in \mathbb{Q}$ and let $v_0 := v_{\mathcal{G}} + v_{\mathcal{F}}$. Then, by Theorem 1 in [8], we have

$$\alpha(F^{v_0}) = \alpha(F^{v_{\mathcal{G}}+v_{\mathcal{F}}}) = \alpha(F^{v_{\mathcal{G}}} \circ F^{v_{\mathcal{F}}}) = \alpha(F^{v_{\mathcal{G}}}) + \alpha(F^{v_{\mathcal{F}}}) (\mathrm{mod}\ 1),$$

and therefore $\alpha(F^{v_0}) \notin \mathbb{Q}$. Similarly, $\alpha(G^{v_0}) \notin \mathbb{Q}$. □

LEMMA 2. *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ and $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ are non-singular (respectively, singular and disjoint) flows and a homeomorphism $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ satisfies (1), then $\Gamma[L_{\mathcal{F}}] = L_{\mathcal{G}}$.*

Proof. Fix a $z \in \mathbb{S}^1$. In the case where $\mathcal{F}$ and $\mathcal{G}$ are non-singular, take also a $v_0 \in V$ for which $\alpha(F^{v_0}),\ \alpha(G^{v_0}) \notin \mathbb{Q}$ (the existence of such a v_0 is guaranteed by Lemma 1). If the flows are non-singular, then using (1) together with the fact that Γ is a homeomorphism we get

$$\Gamma[L_{\mathcal{F}}] = \Gamma[\{F^{nv_0}(z),\ n \in \mathbb{Z}\}^{\mathrm{d}}] = \{G^{nv_0}(\Gamma(z)),\ n \in \mathbb{Z}\}^{\mathrm{d}} = L_{\mathcal{G}}.$$

Applying the same arguments to the case when $\mathcal{F}$ and $\mathcal{G}$ are both singular and disjoint we see that

$$\Gamma[L_{\mathcal{F}}] = \Gamma[\{F^v(z),\ v \in V\}^{\mathrm{d}}] = \{G^v(\Gamma(z)),\ v \in V\}^{\mathrm{d}} = L_{\mathcal{G}}. \qquad \square$$

The following fact follows immediately from Lemma 2.

COROLLARY 3. *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ and $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, v \in V\}$ are topologically conjugate non-singular (respectively, singular and disjoint) flows, then they are simultaneously either dense or non-dense or discrete.*

According to Corollary 3, we will consider the above three classes of flows separately.

3.1. Discrete flows

First, we shall deal with discrete flows. To do this, recall the following

PROPOSITION 1. (see [9]) *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a discrete flow, then $H_{\mathcal{F}} := \{\alpha(F^v),\ v \in V\} = \{\frac{k}{n},\ k = 0, \dots, n-1\}$ for a positive integer n and there exists a mapping $m : V \longrightarrow \{0, \dots, n-1\}$ such that $F^v = G^{m(v)}$ for all $v \in V$ and a homeomorphism $G \in \mathcal{F}$ with $\alpha(G) = \frac{1}{n} (\mathrm{mod}\ 1)$.*

THEOREM 1. *Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ and $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ are discrete flows and let $n_1,\ n_2 \in \mathbb{N}$ be such that $H_{\mathcal{F}} = \{\frac{k}{n_1},\ k = 0, \dots, n_1-1\}$ and $H_{\mathcal{G}} = \{\frac{k}{n_2},\ k = 0, \dots, n_2-1\}$. Suppose also that mappings $m_1 : V \longrightarrow \{0, \dots, n_1 - 1\}$, $m_2 : V \longrightarrow \{0, \dots, n_2 - 1\}$ and homeomorphisms $H_1 \in \mathcal{F}$, $H_2 \in \mathcal{G}$ with $\alpha(H_1) = \frac{1}{n_1} (\mathrm{mod}\ 1)$, $\alpha(H_2) = \frac{1}{n_2} (\mathrm{mod}\ 1)$ are such that $F^v = H_1^{m_1(v)}$, $G^v =$*

$H_2^{m_2(v)}$ *for* $v \in V$. *Then a homeomorphism* $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *of degree* 1 *satisfies* (1) *if and only if*

$$\Gamma \circ H_1 = H_2 \circ \Gamma \tag{2}$$

and $m_1 = m_2$, *whereas a homeomorphism* Γ *of degree* -1 *satisfies* (1) *if and only if*

$$\Gamma \circ H_1^{n_2-1} = H_2 \circ \Gamma \tag{3}$$

and

$$m_1(v) = \begin{cases} n_2 - m_2(v), & m_2(v) \neq 0, \\ 0, & m_2(v) = 0, \end{cases} \qquad v \in V. \tag{4}$$

Proof. Let $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a homeomorphism of degree 1. First, assume that Γ satisfies (1). Then, by Remark 3, we have

$$\alpha(H_1^{m_1(v)}) = \alpha(F^v) = \alpha(G^v) = \alpha(H_2^{m_2(v)}), \qquad v \in V$$

and $n_1 = n_2 =: n$. Theorem 1 in [8] and the facts that $\alpha(H_1) = \frac{1}{n}(\text{mod } 1)$, $\alpha(H_2) = \frac{1}{n}(\text{mod } 1)$ and $m_1(v)$, $m_2(v) \in \{0, ..., n-1\}$ for $v \in V$ now lead to

$$\frac{m_1(v)}{n} = \alpha(H_1^{m_1(v)}) = \alpha(H_2^{m_2(v)}) = \frac{m_2(v)}{n}, \qquad v \in V,$$

and therefore $m_1 = m_2$. Since the only element of $\mathcal{F}$ (respectively, $\mathcal{G}$) with the rotation number $\frac{1}{n}(\text{mod } 1)$ is H_1 (respectively, H_2), (2) also holds true.

If $m_1 = m_2 =: m$ and Γ fulfils (2), then

$$\Gamma \circ F^v = \Gamma \circ H_1^{m(v)} = H_2^{m(v)} \circ \Gamma = G^v \circ \Gamma, \qquad v \in V,$$

and (1) is proved.

Now, let $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a homeomorphism of degree -1 and assume that (1) holds true. Then from Remark 3 we conclude that

$$\alpha(F^v) = \begin{cases} 1 - \alpha(G^v), & \alpha(G^v) \neq 0, \\ 0, & \alpha(G^v) = 0, \end{cases} \qquad v \in V, \tag{5}$$

hence that

$$\{\frac{k}{n_1},\ k = 0, ..., n_1 - 1\} = H_{\mathcal{F}} = \{\frac{n_2 - k}{n_2},\ k = 1, ..., n_2\},$$

and finally that $n_1 = n_2 =: n$. Fix a $v \in V$. If $m_2(v) = 0$, then $\alpha(G^v) = 0$, and (5) implies $\alpha(F^v) = 0$. Therefore $m_1(v) = 0$. If $m_2(v) \neq 0$, then $\alpha(G^v) \neq 0$, and (5), Theorem 1 in [8] and the facts that $\alpha(H_1) =$

$\frac{1}{n}(\text{mod } 1)$, $\alpha(H_2) = \frac{1}{n}(\text{mod } 1)$ and $m_1(v)$, $m_2(v) \in \{0, ..., n-1\}$ for $v \in V$ show that

$$\begin{aligned} \frac{m_1(v)}{n} &= \alpha(H_1^{m_1(v)}) = \alpha(F^v) = 1-\alpha(G^v) \\ &= 1-\alpha(H_2^{m_2(v)}) = 1-\frac{m_2(v)}{n}. \end{aligned}$$

We thus get (4). From (5) it follows that H_2 is topologically conjugate with a homeomorphism with the rotation number $\frac{n-1}{n}$. Since the only such an element in $\mathcal{F}$ is H_1^{n-1}, (3) holds true.

Finally, if (4) is satisfied and Γ fulfils (3), then

$$\begin{aligned} G^v \circ \Gamma &= H_2^{m_2(v)} \circ \Gamma = \Gamma \circ H_1^{m_2(v)(n_2-1)} \\ &= \Gamma \circ H_1^{m_1(v)} = \Gamma \circ F^v, \qquad v \in V, \end{aligned}$$

and the proof is complete. □

3.2. Dense flows

Mappings $\varphi_{\mathcal{F}}$ and $c_{\mathcal{F}}$ guaranteed by our next proposition enable us to give a necessary and sufficient condition for topological conjugacy of dense (respectively, disjoint and non-dense) flows.

PROPOSITION 2. (see [9]) *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a dense or non-dense flow, then there exists a unique pair $(\varphi_{\mathcal{F}}, c_{\mathcal{F}})$ such that $\varphi_{\mathcal{F}} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a continuous function of degree 1 with $\varphi_{\mathcal{F}}(1) = 1$ and $c_{\mathcal{F}} : V \longrightarrow \mathbb{S}^1$ for which*

$$\varphi_{\mathcal{F}}(F^v(z)) = c_{\mathcal{F}}(v)\varphi_{\mathcal{F}}(z), \qquad z \in \mathbb{S}^1,\ v \in V.$$

The mapping $c_{\mathcal{F}}$ is given by $c_{\mathcal{F}}(v) = e^{2\pi i \alpha(F^v)}$ for $v \in V$ and fulfils the equation

$$c_{\mathcal{F}}(v_1 + v_2) = c_{\mathcal{F}}(v_1)c_{\mathcal{F}}(v_2), \qquad v_1,\ v_2 \in V. \tag{6}$$

The function $\varphi_{\mathcal{F}}$ is increasing and $\varphi_{\mathcal{F}}[L_{\mathcal{F}}] = \mathbb{S}^1$. Moreover, $\varphi_{\mathcal{F}}$ is a homeomorphism if and only if the flow $\mathcal{F}$ is dense.

THEOREM 2. *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ and $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ are non-singular (respectively, singular) dense flows and $l \in \{-1, 1\}$, then there is a homeomorphism $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ of degree l satisfying (1) if and only if $c_{\mathcal{F}} = c_{\mathcal{G}}^l$.*

Proof. If a homeomorphism $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ of degree l satisfies (1), then the equality $c_{\mathcal{F}} = c^l_{\mathcal{G}}$ follows immediately from Remark 3.

Now, assume that $c_{\mathcal{F}} = c^l_{\mathcal{G}}$ and note that from Proposition 2 and Remark 5 in [6] it follows that the functions $\varphi_{\mathcal{F}},\ \varphi_{\mathcal{G}} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ are orientation-preserving homeomorphisms such that $F^v(z) = \varphi^{-1}_{\mathcal{F}}\ (c_{\mathcal{F}}(v)\varphi_{\mathcal{F}}(z))$, $G^v(z) = \varphi^{-1}_{\mathcal{G}}(c_{\mathcal{G}}(v)\varphi_{\mathcal{G}}(z))$ for $v \in V,\ z \in \mathbb{S}^1$. Therefore, by Lemma 5 in [6], $\Gamma := \varphi^{-1}_{\mathcal{G}} \circ \varphi^l_{\mathcal{F}}$ is a homeomorphism of degree l which, as is easy to check, satisfies (1). □

3.3. Non-dense flows

Finally, we turn to non-dense flows. If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is such a flow, then its limit set is a non-empty perfect and nowhere dense subset of $\mathbb{S}^1$, and therefore

$$\mathbb{S}^1 \setminus L_{\mathcal{F}} = \bigcup_{q \in \mathbb{Q}} I(\mathcal{F})_q, \tag{7}$$

where $I(\mathcal{F})_q$ for $q \in \mathbb{Q}$ are open pairwise disjoint arcs.

LEMMA 3. (see [9]) *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-dense flow, then*

(i) *for every $q \in \mathbb{Q}$ the mapping $\varphi_{\mathcal{F}}$ is constant on $I(\mathcal{F})_q$,*
(ii) *if $A \subset \mathbb{S}^1$ is an open arc and $\varphi_{\mathcal{F}}$ is constant on A, then $A \subset I(\mathcal{F})_q$ for a $q \in \mathbb{Q}$,*
(iii) *for any distinct $p,\ q \in \mathbb{Q}$, $\varphi_{\mathcal{F}}[I(\mathcal{F})_p] \cap \varphi_{\mathcal{F}}[I(\mathcal{F})_q] = \emptyset$,*
(iv) *for any $q \in \mathbb{Q},\ v \in V$ there exists a $p \in \mathbb{Q}$ with $F^v[I(\mathcal{F})_q] = I(\mathcal{F})_p$,*
(v) *the sets* $\mathrm{Im}c_{\mathcal{F}}$ *and* $K_{\mathcal{F}} := \varphi_{\mathcal{F}}[\mathbb{S}^1 \setminus L_{\mathcal{F}}]$ *are countable,*
(vi) $K_{\mathcal{F}} \cdot \mathrm{Im}c_{\mathcal{F}} = K_{\mathcal{F}}$,
(vii) *the sets* $\mathrm{Im}c_{\mathcal{F}}$ *and* $K_{\mathcal{F}}$ *are dense in* $\mathbb{S}^1$.

According to Lemma 3 we can correctly define the bijection $\Phi_{\mathcal{F}} : \mathbb{Q} \longrightarrow K_{\mathcal{F}}$ and the mapping $T_{\mathcal{F}} : \mathbb{Q} \times V \longrightarrow \mathbb{Q}$ putting

$$\{\Phi_{\mathcal{F}}(q)\} := \varphi_{\mathcal{F}}[I(\mathcal{F})_q], \qquad q \in \mathbb{Q}, \tag{8}$$

$$T_{\mathcal{F}}(q,\ v) := \Phi^{-1}_{\mathcal{F}}(\Phi_{\mathcal{F}}(q)c_{\mathcal{F}}(v)), \qquad q \in \mathbb{Q},\ v \in V. \tag{9}$$

LEMMA 4. (see [9]) *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-dense flow, then*

(i) $T_{\mathcal{F}}(T_{\mathcal{F}}(q,\ v_1),\ v_2) = T_{\mathcal{F}}(q,\ v_1 + v_2), \quad q \in \mathbb{Q},\ v_1,\ v_2 \in V,$

(ii) $T_{\mathcal{F}}(q,\, 0) = q, \quad q \in \mathbb{Q}$,
(iii) $F^v[I(\mathcal{F})_q] = I(\mathcal{F})_{T_{\mathcal{F}}(q,\, v)}, \quad q \in \mathbb{Q},\ v \in V$.

We are now in a position to show a necessary condition for topological conjugacy of non-dense flows.

THEOREM 3. *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ and $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ are non-singular (respectively, singular) non-dense flows and a homeomorphism $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ satisfies (1), then $c_{\mathcal{F}} = c_{\mathcal{G}}^{\deg \Gamma}$ and $K_{\mathcal{G}} = d \cdot (K_{\mathcal{F}})^{\deg \Gamma}$ for a $d \in \mathbb{S}^1$.*

Proof. Since the equality $c_{\mathcal{F}} = c_{\mathcal{G}}^{\deg \Gamma}$ is an immediate consequence of Remark 3, it remains to prove that there is a $d \in \mathbb{S}^1$ for which $K_{\mathcal{G}} = d \cdot (K_{\mathcal{F}})^{\deg \Gamma}$. To do this let us first note that from Lemma 2 and (7) we have

$$\bigcup_{q \in \mathbb{Q}} \Gamma[I(\mathcal{F})_q] = \bigcup_{q \in \mathbb{Q}} I(\mathcal{G})_q,$$

and therefore there exists a bijection $\Phi : \mathbb{Q} \longrightarrow \mathbb{Q}$ such that $\Gamma[I(\mathcal{F})_q] = I(\mathcal{G})_{\Phi(q)}$ for $q \in \mathbb{Q}$. This together with Lemma 4(iii) and (1) gives

$$\Phi(T_{\mathcal{F}}(q,\, v)) = T_{\mathcal{G}}(\Phi(q),\, v), \quad q \in \mathbb{Q},\ v \in V$$

and, by (9),

$$\Phi(\Phi_{\mathcal{F}}^{-1}(\Phi_{\mathcal{F}}(q) c_{\mathcal{F}}(v))) = \Phi_{\mathcal{G}}^{-1}(\Phi_{\mathcal{G}}(\Phi(q)) c_{\mathcal{G}}(v)), \quad q \in \mathbb{Q},\ v \in V.$$

Now, putting $q := \Phi_{\mathcal{F}}^{-1}(z)$ for $z \in K_{\mathcal{F}}$, $\delta := \Phi_{\mathcal{G}} \circ \Phi \circ \Phi_{\mathcal{F}}^{-1}$ we see from $c_{\mathcal{F}} = c_{\mathcal{G}}^{\deg \Gamma}$ that

$$\delta(z c_{\mathcal{F}}(v)) = \delta(z)(c_{\mathcal{F}}(v))^{\deg \Gamma}, \quad z \in K_{\mathcal{F}},\ v \in V. \tag{10}$$

It is obvious that the mapping $\delta : K_{\mathcal{F}} \longrightarrow K_{\mathcal{G}}$ is a bijection. Analysis similar to that in the proof of Theorem 2 in [7] shows that if $\deg \Gamma = 1$ (respectively, $\deg \Gamma = -1$), it is also strictly increasing (respectively, decreasing). Since from Lemma 3(vii) it follows that the sets $K_{\mathcal{F}}$ and $K_{\mathcal{G}}$ are dense in $\mathbb{S}^1$, Corollary 1 in [7] shows that δ can be extended to a continuous function $\hat{\delta} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. Using (10), the density of the sets $K_{\mathcal{F}}$ and $\mathrm{Im} c_{\mathcal{F}}$ in $\mathbb{S}^1$ (see Lemma 3(vii)) and the continuity of the mapping $\hat{\delta}$ we get $\hat{\delta}(zw) = \hat{\delta}(z) w^{\deg \Gamma}$ for $z,\ w \in \mathbb{S}^1$, and consequently

$$K_{\mathcal{G}} = \delta[K_{\mathcal{F}}] = \hat{\delta}[K_{\mathcal{F}}] = \hat{\delta}(1) \cdot (K_{\mathcal{F}})^{\deg \Gamma}. \qquad \square$$

Before we state our next result let us observe that since $\delta(w) = \hat{\delta}(1)w^{\deg\Gamma}$ for $w \in K_{\mathcal{F}}$, the definition of the mapping δ gives

$$\Phi(q) = \Phi_{\mathcal{G}}^{-1}((\Phi_{\mathcal{F}}(q))^{\deg\Gamma}\hat{\delta}(1)), \qquad q \in \mathbb{Q}.$$

THEOREM 4. *Assume that* $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ *and* $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ *are non-singular (respectively, singular), non-dense and disjoint flows and let* $l \in \{-1, 1\}$. *Suppose also that* $c_{\mathcal{F}} = c_{\mathcal{G}}^l$ *and* $K_{\mathcal{G}} = d\cdot(K_{\mathcal{F}})^l$ *for a* $d \in \mathbb{S}^1$. *Then the following construction determines all homeomorphisms* $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *of degree* l *satisfying* (1).

1°. *Take a* $d \in \mathbb{S}^1$ *for which* $K_{\mathcal{G}} = d \cdot (K_{\mathcal{F}})^l$ *and define the mapping* $\Psi : \mathbb{Q} \longrightarrow \mathbb{Q}$ *by*

$$\Psi(q) := \Phi_{\mathcal{G}}^{-1}((\Phi_{\mathcal{F}}(q))^l d), \qquad q \in \mathbb{Q}.$$

2°. *Introduce the following equivalence relation on* $\mathbb{Q}$

$$p\,\mathcal{R}\,q \quad \text{if and only if there is a } v \in V \text{ such that } p = T_{\mathcal{F}}(q, v).$$

3°. *Take an* $E \subset \mathbb{Q}$ *having exactly one point in common with each equivalence class with respect to the relation* $\mathcal{R}$ *and define*

$$\{A(q)\} := [q]_{\mathcal{R}} \cap E, \qquad q \in \mathbb{Q}.$$

4°. *Choose an arbitrary function* $W : \mathbb{Q} \longrightarrow V$ *with*

$$T_{\mathcal{F}}(A(q), W(q)) = q, \qquad q \in \mathbb{Q}. \tag{11}$$

5°. *If* $l = 1$ *(respectively,* $l = -1$*), then take strictly increasing (respectively, decreasing) homeomorphisms*

$$\Gamma_e : I(\mathcal{F})_e \longrightarrow I(\mathcal{G})_{\Psi(e)}, \qquad e \in E. \tag{12}$$

6°. *Define the strictly increasing (respectively, decreasing) mapping* $\Gamma_0 : \bigcup_{q\in\mathbb{Q}} I(\mathcal{F})_q \longrightarrow \bigcup_{q\in\mathbb{Q}} I(\mathcal{G})_q$ *putting*

$$\Gamma_0(z) := (G^{W(q)} \circ \Gamma_{A(q)} \circ F^{-W(q)})(z), \qquad z \in I(\mathcal{F})_q,\ q \in \mathbb{Q}. \tag{13}$$

7°. *Extend the function* Γ_0 *to a strictly increasing (respectively, decreasing) and continuous mapping* $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$.

Proof. We first show that constructed in the above way Γ is a homeomorphism of the circle with $\deg\Gamma = l$ satisfying (1).

It is easily seen that $\Psi : \mathbb{Q} \longrightarrow \mathbb{Q}$ is a bijection such that $(\Phi_{\mathcal{G}} \circ \Psi \circ \Phi_{\mathcal{F}}^{-1})(z) = z^l d$ for $z \in K_{\mathcal{F}}$. This together with Lemma 3(vi) and (9) gives

$$\Phi_{\mathcal{G}}(\Psi(T_{\mathcal{F}}(q, v))) = \Phi_{\mathcal{G}}(\Psi(q))(c_{\mathcal{F}}(v))^l, \qquad q \in \mathbb{Q},\ v \in V.$$

On the other hand, from (9) it also follows that

$$\Phi_{\mathcal{G}}(T_{\mathcal{G}}(\Psi(q),\ v)) = \Phi_{\mathcal{G}}(\Psi(q))c_{\mathcal{G}}(v), \qquad q \in \mathbb{Q},\ v \in V,$$

so, in consequence,

$$\Psi(T_{\mathcal{F}}(q,\ v)) = T_{\mathcal{G}}(\Psi(q),\ v), \qquad q \in \mathbb{Q},\ v \in V. \tag{14}$$

An easy computation shows that $\mathcal{R}$ is an equivalence relation. The existence of the mapping $W : \mathbb{Q} \longrightarrow V$ for which (11) holds true follows immediately from the definition of $\mathcal{R}$. Moreover, according to (11) and (9), we have $T_{\mathcal{F}}(q,\ -W(q)) = A(q)$ for $q \in \mathbb{Q}$. From this, (13), Lemma 4(iii), (12), (14) and (11) it may be concluded that

$$\Gamma_0[I(\mathcal{F})_q] = I(\mathcal{G})_{\Psi(q)}, \qquad q \in \mathbb{Q}. \tag{15}$$

Our next goal is to prove that if $l = 1$ (respectively, $l = -1$), then the mapping $\Gamma_0 : \bigcup_{q\in\mathbb{Q}} I(\mathcal{F})_q \longrightarrow \bigcup_{q\in\mathbb{Q}} I(\mathcal{G})_q$ is strictly increasing (respectively, decreasing). To do this, fix $x,\ w,\ z \in \bigcup_{q\in\mathbb{Q}} I(\mathcal{F})_q$ for which $x \prec w \prec z$. We shall show that $\Gamma_0(x) \underset{(\succ)}{\prec} \Gamma_0(w) \underset{(\succ)}{\prec} \Gamma_0(z)$. In order to get this relation, it is convenient to consider three cases.

(i) $\{x,\ w,\ z\} \subset I(\mathcal{F})_q$ for a $q \in \mathbb{Q}$.
Since G^v and F^v for $v \in V$ are orientation-preserving homeomorphisms of the circle, our assertion follows from (13) and Lemmas 11 and 10 in [7].

(ii) $\mathrm{card}(\{x,\ w,\ z\} \cap I(\mathcal{F})_q) = 2$ for a $q \in \mathbb{Q}$.
By Lemma 2 in [6] we may assume that $x,\ w \in I(\mathcal{F})_q$. Fixing a $u \in I(\mathcal{F})_q$ such that $w \in \overrightarrow{(x,\ u)} \subset I(\mathcal{F})_q$ we conclude from (i) and (15) that $\Gamma_0(w) \in \overrightarrow{(\Gamma_0(x),\ \Gamma_0(u))} \subset I(\mathcal{G})_{\Psi(q)}$ (respectively, $\Gamma_0(w) \in \overrightarrow{(\Gamma_0(u),\ \Gamma_0(x))} \subset I(\mathcal{G})_{\Psi(q)}$). As $z \notin I(\mathcal{F})_q$ and Ψ is a bijection we also have $\Gamma_0(z) \notin I(\mathcal{G})_{\Psi(q)}$, and therefore $\Gamma_0(w) \in \overrightarrow{(\Gamma_0(x),\ \Gamma_0(z))}$ (respectively, $\Gamma_0(w) \in \overrightarrow{(\Gamma_0(z),\ \Gamma_0(x))}$).

(iii) $\mathrm{card}(\{x,\ w,\ z\} \cap I(\mathcal{F})_q) \leq 1$ for a $q \in \mathbb{Q}$.
Let $p,\ q,\ r \in \mathbb{Q}$ be pairwise distinct numbers for which $x \in I(\mathcal{F})_q$, $w \in I(\mathcal{F})_p$, $z \in I(\mathcal{F})_r$. Then $I(\mathcal{F})_q \prec I(\mathcal{F})_p \prec I(\mathcal{F})_r$, and the fact that the function $\varphi_{\mathcal{F}}$ is increasing together with Lemma 3, (8) and Remark 1 gives

$$\Phi_{\mathcal{F}}(q)d \prec \Phi_{\mathcal{F}}(p)d \prec \Phi_{\mathcal{F}}(r)d \quad \text{and} \quad \frac{1}{\Phi_{\mathcal{F}}(q)}d \succ \frac{1}{\Phi_{\mathcal{F}}(p)}d \succ \frac{1}{\Phi_{\mathcal{F}}(r)}d.$$

Consequently, using the definition of Ψ, (8), Lemma 3 and the facts that the mapping $\varphi_{\mathcal{G}}$ is increasing and $I(\mathcal{G})_q$ for $q \in \mathbb{Q}$ are pairwise disjoint open arcs, we get $I(\mathcal{G})_{\Psi(q)} \underset{(\succ)}{\prec} I(\mathcal{G})_{\Psi(p)} \underset{(\succ)}{\prec} I(\mathcal{G})_{\Psi(r)}$, and (15) now leads to $\Gamma_0(x) \underset{(\succ)}{\prec} \Gamma_0(w) \underset{(\succ)}{\prec} \Gamma_0(z)$.

Next, observe that (7), (15) and the fact that Ψ is a bijection give $\Gamma_0[\mathbb{S}^1 \setminus L_{\mathcal{F}}] = \mathbb{S}^1 \setminus L_{\mathcal{G}}$. Since the sets $\mathbb{S}^1 \setminus L_{\mathcal{F}}$ and $\mathbb{S}^1 \setminus L_{\mathcal{G}}$ are dense in $\mathbb{S}^1$, from Lemmas 12 and 13 in [7] it follows that Γ_0 can be extended to a strictly increasing (respectively, decreasing) and continuous function $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. Clearly, $\Gamma[\mathbb{S}^1] = \mathbb{S}^1$ and, by Remark 3 in [7], Γ is an injection. Thus Γ is a homeomorphism and, in view of Lemma 11 in [7], $\deg \Gamma = l$.

We now show that

$$(P) \qquad \begin{array}{l} \text{if } T_{\mathcal{F}}(p,\, u) = T_{\mathcal{F}}(p,\, v) \text{ for some } p \in \mathbb{Q},\ u,\ v \in V, \\ \text{then } F^u = F^v \text{ and } G^u = G^v. \end{array}$$

Fix $p \in \mathbb{Q}$, u, $v \in V$ with $T_{\mathcal{F}}(p,\, u) = T_{\mathcal{F}}(p,\, v)$. By (9) we obtain $c_{\mathcal{F}}(u) = c_{\mathcal{F}}(v)$, which together with (6) yields $c_{\mathcal{F}}(u-v) = 1 = c_{\mathcal{G}}(u-v)$, and (9) now shows that

$$(16) \qquad T_{\mathcal{F}}(p,\, u-v) = p \quad \text{and} \quad T_{\mathcal{G}}(p,\, u-v) = p.$$

Let $I(\mathcal{F})_p = \overrightarrow{(a_p,\, b_p)}$, $I(\mathcal{G})_p = \overrightarrow{(a'_p,\, b'_p)}$. Since from Lemma 4(iii) and (16) we have $F^{u-v}[I(\mathcal{F})_p] = I(\mathcal{F})_p$ and $G^{u-v}[I(\mathcal{G})_p] = I(\mathcal{G})_p$, the fact that F^{u-v} and G^{u-v} are strictly increasing gives $F^{u-v}(a_p) = a_p$ and $G^{u-v}(a'_p) = a'_p$. But the flows $\mathcal{F}$ and $\mathcal{G}$ are disjoint, and so $F^u = F^v$ and $G^u = G^v$.

Fix $q \in \mathbb{Q}$, $v \in V$. By (11), Lemma 4 and the fact that $A(q) = A(T_{\mathcal{F}}(q,\, v))$ we get

$$T_{\mathcal{F}}(q,\, v) = T_{\mathcal{F}}(A(T_{\mathcal{F}}(q,\, v)),\, W(q) + v).$$

Putting $p := T_{\mathcal{F}}(q,\, v)$ we see that $p = T_{\mathcal{F}}(A(p),\, W(q) + v)$, which together with (11) implies $T_{\mathcal{F}}(A(p),\, W(p)) = T_{\mathcal{F}}(A(p),\, W(q) + v)$. (P) and the definition of p now give

$$(17) \qquad \begin{array}{l} F^{W(q)+v} = F^{W(p)} = F^{W(T_{\mathcal{F}}(q,\, v))} \quad \text{and} \\ G^{W(q)+v} = G^{W(p)} = G^{W(T_{\mathcal{F}}(q,\, v))}. \end{array}$$

Take $v \in V$, $z_0 \in \mathbb{S}^1 \setminus L_{\mathcal{F}}$ and let $q \in \mathbb{Q}$ be such that $z_0 \in I(\mathcal{F})_q$. By Lemma 4(iii) we have $F^v(z_0) \in I(\mathcal{F})_{T_{\mathcal{F}}(q,\, v)}$, and from the equality $\Gamma \mid \mathbb{S}^1 \setminus L_{\mathcal{F}} = \Gamma_0$, (13), (18) and the fact that $A(q) = A(T_{\mathcal{F}}(q,\, v))$ it may be concluded that $G^v(\Gamma(z_0)) = \Gamma(F^v(z_0))$. The density of $\mathbb{S}^1 \setminus L_{\mathcal{F}}$ in $\mathbb{S}^1$ and the continuity of G^v, F^v and Γ finally give (1).

Now, let $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a homeomorphism of degree l satisfying (1). Putting

$$\Gamma_0 := \Gamma \mid \bigcup_{q\in\mathbb{Q}} I(\mathcal{F})_q \quad \text{and} \quad \Gamma_e := \Gamma \mid I(\mathcal{F})_e, \quad e \in E$$

we see that if $l = 1$ (respectively, $l = -1$), then $\Gamma_0 : \bigcup_{q\in\mathbb{Q}} I(\mathcal{F})_q \longrightarrow \bigcup_{q\in\mathbb{Q}} I(\mathcal{G})_q$ and $\Gamma_e : I(\mathcal{F})_e \longrightarrow I(\mathcal{G})_{\Phi(e)}$ are strictly increasing (respectively, decreasing) homeomorphisms for which (13) holds true.

Thus, the above construction determines all homeomorphisms $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ of degree l satisfying (1). □

As an immediate consequence of Theorems 3 and 4 we obtain

THEOREM 5. *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ and $\mathcal{G} = \{G^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ are non-singular (respectively, singular), non-dense and disjoint flows and $l \in \{-1, 1\}$, then there is a homeomorphism $\Gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ of degree l satisfying* (1) *if and only if $c_{\mathcal{F}} = c_{\mathcal{G}}^l$ and $K_{\mathcal{G}} = d \cdot (K_{\mathcal{F}})^l$ for a $d \in \mathbb{S}^1$.*

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