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STABLE CLASS OF EQUIVARIANT ALGEBRAIC VECTOR BUNDLES OVER REPRESENTATIONS

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Dedicated to Professor Fuichi Uchida on his 60th birthday

ABSTRACT. Let G be a reductive algebraic group and let B, F be G-modules. We denote by $\mathrm{VEC}_G(B, F)$ the set of isomorphism classes in algebraic G-vector bundles over B with F as the fiber over the origin of B. Schwarz (or Kraft-Schwarz) shows that $\mathrm{VEC}_G(B, F)$ admits an abelian group structure when $\dim B/\!\!/G = 1$. In this paper, we introduce a stable functor $\mathrm{VEC}_G(B, F^\infty)$ and prove that it is an abelian group for *any* G-module B. We also show that this stable functor will have nice properties.

1. Introduction

Throughout this paper, we will work in the algebraic category over the field of complex numbers $\mathbb{C}$ and G will denote a reductive group unless otherwise stated. Finite groups, $\mathbb{C}^*$-tori (i.e., products of $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$) and semisimple groups are examples of reductive groups, and it is known that any reductive group is obtained as a group extension by these three types of groups (see [2] for example). One may also think of a reductive group as "complexification" of a compact Lie group (see [20] for example), e.g. the complexification of the circle group S^1 is $\mathbb{C}^*$.

The research of this paper is motivated by the following problem.

Equivariant Serre Problem. Is any G-vector bundle over a G-module B (= a G-representation space) trivial, i.e., isomorphic to a product bundle $\mathbf{F} := B \times F \to B$ for some G-module F?

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One can ask the same question in other categories. It is a classical result that the problem has an affirmative solution in the smooth category because the base space B is equivariantly contractible. Recently it has affirmatively been answered in the holomorphic category ([5]).

However, the situation is not so simple in the algebraic category. When G is trivial, the Equivariant Serre Problem is nothing but the famous Serre conjecture which was solved affirmatively by D. Quillen [18] and A. Suslin [21]. This result is extended to the case when G is abelian by Masuda-Moser-Petrie [14]. Another type of partial affirmative solution to the problem is as follows. The affine variety $B/\!\!/G$, whose coordinate ring is the ring $\mathcal{O}(B)^G$ of G-invariant polynomials on B, is called the algebraic quotient of B by the G-action. When $\dim B/\!\!/G = 0$, it follows from Luna slice theorem [10] that the Equivariant Serre Problem has an affirmative solution. G. Schwarz [19] (see also [8]) attacked the next case where $\dim B/\!\!/G = 1$, and surprisingly found counterexamples to the problem for many non-abelian groups G. After his breakthrough, more counterexamples have been found ([6], [13, 15], [16, 17]), where $\dim B/\!\!/G$ is not necessarily one. On the other hand, Bass and Haboush ([4]) proved (before the breakthrough by Schwarz) that every G-vector bundle over a G-module is *stably* trivial, i.e., it becomes trivial when added to a suitable trivial G-vector bundle, for any G. See [12] for more information on our subject.

For G-modules B and F we denote by $\mathrm{VEC}_G(B, F)$ the set of isomorphism classes in G-vector bundles over B whose fiber over the origin is isomorphic to F. We often abbreviate a G-vector bundle $\pi\colon E \to B$ as E, and denote its isomorphism class by $[E]$. Needless to say, $\mathrm{VEC}_G(B, F)$ contains the isomorphism class of the product bundle $\mathbf{F}$, and if $\mathrm{VEC}_G(B, F)$ contains an element different from $[\mathbf{F}]$, then it provides a counterexample to the Equivariant Serre Problem. Following [16, 17] we also consider a subset

$$\mathrm{VEC}_G(B, F; S) := \{[E] \in \mathrm{VEC}_G(B, F) \mid [E \oplus \mathbf{S}] = [\mathbf{F} \oplus \mathbf{S}]\}$$

for a G-module S. The result of Bass and Haboush mentioned above says that the union of $\mathrm{VEC}_G(B, F; S)$ over all G-modules S agrees with $\mathrm{VEC}_G(B, F)$.

Schwarz [19] (and Kraft-Schwarz [8]) proved that if $\dim B/\!\!/G = 1$, then $\mathrm{VEC}_G(B, F)$ admits an abelian group structure and is isomorphic to $\mathbb{C}^p$ for some non-negative integer p depending on B and F. They also established a formula to compute the dimension p in terms of invariant theory and found that p could be positive for many G, B and F.

The group structure on $\mathrm{VEC}_G(B, F)$ is as follows. When $\dim B/\!\!/G = 1$, they showed that the Whitney sum with $\mathbf{F}$ induces a bijective correspondence

$$(*) \qquad \mathrm{VEC}_G(B, F) \xrightarrow[\cong]{\oplus \mathbf{F}} \mathrm{VEC}_G(B, F \oplus F).$$

Therefore, given $[E_1]$ and $[E_2]$ in $\mathrm{VEC}_G(B, F)$, there is a unique element $[E_3]$ in $\mathrm{VEC}_G(B, F)$ such that $[E_1 \oplus E_2] = [E_3 \oplus \mathbf{F}]$, and the sum of $[E_1]$ and $[E_2]$ is defined to be $[E_3]$, giving the abelian group structure on $\mathrm{VEC}_G(B, F)$. The map $(*)$ above also induces a bijection between $\mathrm{VEC}_G(B, F; S)$ and $\mathrm{VEC}_G(B, F \oplus F; S)$ for any S, so that $\mathrm{VEC}_G(B, F; S)$ becomes a subgroup of $\mathrm{VEC}_G(B, F)$ when $\dim B/\!\!/G = 1$.

However, when $\dim B/\!\!/G \geq 2$, the map $(*)$ above is not known to be bijective, so we do not know whether $\mathrm{VEC}_G(B, F)$ admits an abelian group structure under Whitney sum. To get around this, we consider the following direct system

$$\xrightarrow{\oplus \mathbf{F}} \mathrm{VEC}_G(B, F^n) \xrightarrow{\oplus \mathbf{F}} \mathrm{VEC}_G(B, F^{n+1}) \xrightarrow{\oplus \mathbf{F}}$$

where F^n denotes the direct sum of n copies of F, and define

$$\mathrm{VEC}_G(B, F^\infty) := \varinjlim_n \mathrm{VEC}_G(B, F^n).$$

Similarly $\mathrm{VEC}_G(B, F^\infty; S)$ can be defined. $\mathrm{VEC}_G(B, F^\infty)$ and $\mathrm{VEC}_G(B$ $F^\infty; S)$ are apparently abelian monoids under Whitney sum, but it turns out

THEOREM 1.1. *$\mathrm{VEC}_G(B, F^\infty)$ is an abelian group and $\mathrm{VEC}_G(B, F^\infty; S)$ is its subgroup under Whitney sum for any G-modules B, F and S.*

REMARK. $\mathrm{VEC}_G(B, F^\infty)$ and $\mathrm{VEC}_G(B, F^\infty; S)$ are both trivial when $\dim B/\!\!/G = 0$, and isomorphic to $\mathrm{VEC}_G(B, F)$ and $\mathrm{VEC}_G(B, F; S)$ respectively when $\dim B/\!\!/G = 1$.

In the proof of the theorem above, we define a *surjective* homomorphism

$$\mathcal{V}\colon (R/I)^* \to \mathrm{VEC}_G(B, F^\infty; S),$$

where R is the ring of G-vector bundle endomorphisms of $\mathbf{S}$, I is a two sided ideal in R and $(R/I)^*$ is the group of units in R/I. Note that when S is the trivial one-dimensional module $\mathbb{C}$, R is isomorphic to $\mathcal{O}(B)^G$, in particular, commutative. The homomorphism $\mathcal{V}$ has a nontrivial kernel Γ^∞ in general. When $(R/I)^*$ is commutative (e.g.

$S = \mathbb{C}$), one can transfer the multiplicative group $(R/I)^*/\Gamma^\infty$ to an additive group isomorphically using a logarithmic map. It turns out that the additive group is a finitely generated $\mathcal{O}(B)^G$-module. Thus we have

THEOREM 1.2. *If R/I is commutative (e.g. $S = \mathbb{C}$), then* $\mathrm{VEC}_G(B, F^\infty; S)$ *is isomorphic to a finitely generated $\mathcal{O}(B)^G$-module, in particular, a complex vector space (of possibly countably infinite dimension) as groups.*

The author believes that the theorem above would hold without the commutativity assumption on R/I and even for $\mathrm{VEC}_G(B, F^\infty)$. In fact, when $\dim B/\!\!/G = 1$, $\mathrm{VEC}_G(B, F^\infty)$ is isomorphic to $\mathrm{VEC}_G(B, F)$ as remarked above and $\mathrm{VEC}_G(B, F)$ is isomorphic to a truncated polynomial ring $\mathbb{C}[t]/(t^p)$ in one variable t for some non-negative integer p by the result of Schwarz. The assumption that $\dim B/\!\!/G = 1$ is equivalent to $\mathcal{O}(B)^G$ being a polynomial ring in one variable, so $\mathbb{C}[t]$ can be identified with $\mathcal{O}(B)^G$ and then $\mathbb{C}[t]/(t^p)$ is certainly a finitely generated $\mathcal{O}(B)^G$-module in this case.

When $\dim B/\!\!/G = 1$, Schwarz proved more. He showed that there is a "universal" G-vector bundle $\mathcal{E} \in \mathrm{VEC}_G(B \oplus \mathbb{C}^p, F)$ such that mapping $c \in \mathbb{C}^p$ to $\mathcal{E}|_{B\times\{c\}} \in \mathrm{VEC}_G(B, F)$ is bijective. Let m be a non-negative integer. To any morphism (i.e., polynomial map) $f\colon \mathbb{C}^m \to \mathbb{C}^p = \mathrm{VEC}_G(B, F)$, we assign a bundle induced from $\mathcal{E}$ by a map $1 \oplus f\colon B \oplus \mathbb{C}^m \to B \oplus \mathbb{C}^p$. This produces a map

$$\mathrm{Mor}(\mathbb{C}^m, \mathrm{VEC}_G(B, F)) = \mathrm{VEC}_G(B, F) \otimes \mathcal{O}(\mathbb{C}^m) \to \mathrm{VEC}_G(B \oplus \mathbb{C}^m, F)$$

where $\mathrm{Mor}(X, Y)$ denotes the set of morphisms from X to Y and the tensor product is taken over $\mathbb{C}$. The universality of the bundle $\mathcal{E}$ implies that the above map is injective, and it is claimed in [11] that the map is actually bijective. The following result implies that there might exist the product formula above even when $\dim B/\!\!/G \geq 2$.

THEOREM 1.3. *If R/I is commutative (e.g. $S = \mathbb{C}$), then*

$$\mathrm{VEC}_G(B \oplus \mathbb{C}^m, F^\infty; S) \cong \mathrm{VEC}_G(B, F^\infty; S) \otimes \mathcal{O}(\mathbb{C}^m)$$

as groups.

This paper is organized as follows. In Section 2 we review the method introduced in [16, 17] to produce elements in $\mathrm{VEC}_G(B, F; S)$ and to distinguish them. It is the main tool used in this paper. We discuss its stable version in Section 3 and Theorem 1.1 is proved in Section 4. In Section 5 we consider a $\mathbb{C}^*$-action on B commuting with the

G-action. In Section 6 we study $(R/I)^*/\Gamma^\infty$, which is isomorphic to $\mathrm{VEC}_G(B, F^\infty; S)$, using the $\mathbb{C}^*$-action on B when R/I is commutative, and prove Theorem 1.2. Theorem 1.3 is proved in Section 7.

2. Subbundle method

In this section we review the method introduced in [16, 17]. Let $[E]$ be an element of $\mathrm{VEC}_G(B, F; S)$. Since $E \oplus \mathbf{S}$ is isomorphic to $\mathbf{F} \oplus \mathbf{S}$, there is a G-vector bundle surjective homomorphism $L : \mathbf{F} \oplus \mathbf{S} \to \mathbf{S}$ whose kernel $\ker L$ is isomorphic to E. Let $L' : \mathbf{F} \oplus \mathbf{S} \to \mathbf{S}$ be another surjective homomorphism. Then it is not difficult to see that $\ker L'$ is isomorphic to $\ker L$ if and only if there is a G-vector bundle automorphism A of $\mathbf{F} \oplus \mathbf{S}$ such that $L' = LA$. Therefore, the study of $\mathrm{VEC}_G(B, F; S)$ splits into two steps: one is the study of G-vector bundle surjective homomorphisms from $\mathbf{F} \oplus \mathbf{S}$ to $\mathbf{S}$ (in other words, construction of G-vector bundles) and the other is the study of G-vector bundle automorphisms of $\mathbf{F} \oplus \mathbf{S}$ (in other words, distinction of G-vector bundles). One can formulate this as follows. Let $\mathrm{sur}(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})^G$ be the set of G-vector bundle surjective homomorphisms from $\mathbf{F} \oplus \mathbf{S}$ to $\mathbf{S}$ and let $\mathrm{aut}(\mathbf{F} \oplus \mathbf{S})^G$ be the group of G-vector bundle automorphisms of $\mathbf{F} \oplus \mathbf{S}$. The group $\mathrm{aut}(\mathbf{F} \oplus \mathbf{S})^G$ acts on $\mathrm{sur}(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})^G$ as above. Then the fact mentioned above can be restated as follows.

THEOREM 2.1 ([16, 17]). *The map sending $L \in \mathrm{sur}(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})^G$ to $\ker L$ induces a bijection*

$$\mathrm{sur}(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})^G / \mathrm{aut}(\mathbf{F} \oplus \mathbf{S})^G \cong \mathrm{VEC}_G(B, F; S).$$

The following example will illustrate our method well.

EXAMPLE 2.2. Let $\mathrm{O}_2 = \mathbb{C}^* \rtimes \mathbb{Z}/2$. For a positive integer n we denote by V_n the 2-dimensional O_2-module with the actions of $g \in \mathbb{C}^*$ and of the nontrivial element in $\mathbb{Z}/2$ respectively given by

$$\begin{pmatrix} g^n & 0 \\ 0 & g^{-n} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then one easily checks that $\mathcal{O}(V_n)^{\mathrm{O}_2}$ is a polynomial ring in one variable and it is proved in [19] that $\mathrm{VEC}_{\mathrm{O}_2}(V_1, V_m) \cong \mathbb{C}^{m-1}$ and $\mathrm{VEC}_{\mathrm{O}_2}(V_1, V_m) = \mathrm{VEC}_{\mathrm{O}_2}(V_1, V_m; \mathbb{C})$. This provided the first counterexamples to the Equivariant Serre Problem.

Here is an explicit description of elements in $\mathrm{VEC}_{\mathrm{O}_2}(V_1, V_m)$ found in [16, 17]. To a polynomial $f(t)$ in one variable t with $f(0) = 1$, we associate

$$E_f := \{(a,b,x,y,z) \in V_1 \times (V_m \oplus \mathbb{C}) \mid b^m x + a^m y + f(ab)z = 0\},$$

where $(a,b) \in V_1, (x,y) \in V_m$ and $z \in \mathbb{C}$. Taking the projection on V_1, one sees that E_f defines an element of $\mathrm{VEC}_{\mathrm{O}_2}(V_1, V_m,;\mathbb{C})$. In fact, the 1×3 matrix $L_f := (b^m, a^m, f(ab))$ is of rank one at any point $(a,b) \in V_1$, so

$$L_f \colon V_1 \times (V_m \oplus \mathbb{C}) \to V_1 \times \mathbb{C}$$

is a surjective O_2-vector bundle homomorphism and $\ker L_f = E_f$.

On the other hand, it follows from the equivariance that an O_2-vector bundle automorphism A of the product bundle $V_1 \times (V_m \oplus \mathbb{C})$ is a 3×3 matrix of this form

$$A = \begin{pmatrix} p & a^{2m}q & a^m r \\ b^{2m}q & p & b^m r \\ b^m s & a^m s & w \end{pmatrix},$$

where p, q, r, s, w are polynomials in $ab = t$. An elementary computation shows that

$$\det A = (p - t^m q)(pw + t^m qw - 2t^m rs).$$

Since A is algebraic and invertible, $\det A$ must be a nonzero constant and hence so are the both factors above. It follows that

$$w \equiv \text{a nonzero constant} \pmod{t^m}.$$

Let $h(t)$ be another polynomial with $h(0) = 1$ and suppose that $[E_h] = [E_f]$ in $\mathrm{VEC}_{\mathrm{O}_2}(V_1, V_m;\mathbb{C})$. Then $L_h = L_f A$ for some automorphism A. Comparing the last entries in L_h and L_f and using the congruence on w above, one concludes that $h(t) \equiv f(t) \pmod{t^m}$. This shows that the correspondence : $\mathbb{C}^{m-1} \to \mathrm{VEC}_{\mathrm{O}_2}(V_1, V_m;\mathbb{C})$ given by $(c_1, \dots, c_{m-1}) \to [E_c]$, where $c(t) = 1 + c_1 t + \cdots + c_{m-1}t^{m-1}$, is injective. A more careful but elementary observation shows that this correspondence is bijective.

In this case, the universal bundle $\mathcal{E}$ mentioned in the introduction can be described as

$$\mathcal{E} = \{(a, b, c_1, \dots, c_{m-1}, x, y, z) \in (V_1 \oplus \mathbb{C}^{m-1}) \times (V_m \oplus \mathbb{C}) \\ \mid b^m x + a^m y + c(ab)z = 0\}$$

with the projection on $V_1 \times \mathbb{C}^{m-1}$.

The following general argument was developed keeping the above example in mind. We review the definition of the invariants which distinguish elements in $\mathrm{VEC}_G(B, F; S)$.

For G-vector bundles P and Q over the same base space B, we denote by $\mathrm{mor}(P,Q)^G$ the set of G-vector bundle homomorphisms from P to Q. We write an element L in $\mathrm{sur}(\mathbf{F}\oplus\mathbf{S},\mathbf{S})^G$ as $L=(L(F,S),L(S,S))$ where $L(F,S)\in\mathrm{mor}(\mathbf{F},\mathbf{S})^G$ and $L(S,S)\in\mathrm{mor}(\mathbf{S},\mathbf{S})^G=:R$. Since L is a surjective homomorphism and G is reductive, there is an element $M\in\mathrm{mor}(\mathbf{S},\mathbf{F}\oplus\mathbf{S})^G$ such that LM is the identity map on $\mathbf{S}$ (see [3]), i.e.,

$$L(S,S)M(S,S)+L(F,S)M(S,F)=1,$$

where $M(S,S)$ and $M(S,F)$ are defined similarly to $L(S,S)$ and $L(F,S)$. We denote by I the ideal in R generated by G-vector bundle endomorphisms of $\mathbf{S}$ which factor through $\mathbf{F}$, i.e., I is generated by composition of elements in $\mathrm{mor}(\mathbf{F},\mathbf{S})^G$ and $\mathrm{mor}(\mathbf{S},\mathbf{F})^G$. The identity above implies that $L(S,S)$ is in $(R/I)^*$, i.e., a unit in R/I.

Now let A be an element in $\mathrm{aut}(\mathbf{F}\oplus\mathbf{S})^G$. Then $\ker(LA)$ is isomorphic to $\ker L$ and we have

$$(LA)(S,S)=L(F,S)A(S,F)+L(S,S)A(S,S),$$

where $A(S,F)$ and $A(S,S)$ are defined similarly to $L(F,S)$ and $L(S,S)$. The first term at the right hand side above is an element of I and it is not difficult to see that $A(S,S)$ is a unit in R/I. Therefore, if we denote by Γ the subgroup of $(R/I)^*$ represented by elements $A(S,S)$ for $A\in\mathrm{aut}(\mathbf{F}\oplus\mathbf{S})^G$, then we have a well-defined map

$$\rho\colon\ \mathrm{VEC}_G(B,F;S)\to(R/I)^*/\Gamma$$

sending $[\ker L]$ to the equivalence class of $L(S,S)$. This is the invariant introduced in [16, 17] and used to distinguish elements in $\mathrm{VEC}_G(B,F;S)$ (see also [13, 15]). In Example 2.2, one can check that $R=\mathcal{O}(V_1)^{\mathrm{O}_2}=\mathbb{C}[t]$ $(t=ab)$, $I=(t^m)$ and $\Gamma=\mathbb{C}^*$; so $(R/I)^*/\Gamma$ bijectively corresponds to the set of truncated polynomials of degree at most $m-1$ and with constant term 1. Moreover, the map ρ is bijective in this case. There are many cases where ρ is bijective but it is not known whether ρ is always bijective. However we will see later that the map ρ^∞ induced from ρ on $\mathrm{VEC}_G(B,F^\infty;S)$ is bijective for any G-modules B,F and S.

3. Stabilization

First we make sure that $\mathrm{VEC}_G(B,F^\infty;S)$ is closed under Whitney sum. Suppose $[E_i]\in\mathrm{VEC}_G(B,F^{n_i};S)$ for $i=1,2$. Then, since $E_i\oplus\mathbf{S}\cong$

$\mathbf{F}^{n_i} \oplus \mathbf{S}$, we have

$$E_1 \oplus E_2 \oplus \mathbf{S} \cong E_1 \oplus \mathbf{F}^{n_2} \oplus \mathbf{S} \cong \mathbf{F}^{n_1} \oplus \mathbf{F}^{n_2} \oplus \mathbf{S} \cong \mathbf{F}^{n_1+n_2} \oplus \mathbf{S},$$

which shows that $[E_1 \oplus E_2]$ lies in $\mathrm{VEC}_G(B, F^{n_1+n_2}; S)$. It follows that $\mathrm{VEC}_G(B, F^\infty; S)$ is closed under Whitney sum.

$\mathrm{VEC}_G(B, F^\infty; S)$ can be described in terms of sur and aut as in Theorem 2.1. We think of $\mathrm{sur}(\mathbf{F}^n \oplus \mathbf{S}, \mathbf{S})^G$ (resp. $\mathrm{aut}(\mathbf{F}^n \oplus \mathbf{S})^G$) as a subset (resp. a subgroup) of $\mathrm{sur}(\mathbf{F}^{n+1} \oplus \mathbf{S}, \mathbf{S})^G$ (resp. $\mathrm{aut}(\mathbf{F}^{n+1} \oplus \mathbf{S})^G$) by defining to be zero (resp. the identity) on the added factor $\mathbf{F}$, and define $\mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S}, \mathbf{S})^G$ (resp. $\mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G$) to be the union of $\mathrm{sur}(\mathbf{F}^n \oplus \mathbf{S}, \mathbf{S})^G$ (resp. $\mathrm{aut}(\mathbf{F}^n \oplus \mathbf{S})^G$) over all positive integers n. The group $\mathrm{aut}(\mathbf{F}^n \oplus \mathbf{S})^G$ acts on $\mathrm{sur}(\mathbf{F}^n \oplus \mathbf{S}, \mathbf{S})^G$ and it follows from Theorem 2.1 that we have a bijection

$$\mathrm{sur}(\mathbf{F}^n \oplus \mathbf{S}. \mathbf{S})^G / \mathrm{aut}(\mathbf{F}^n \oplus \mathbf{S})^G \cong \mathrm{VEC}_G(B, F^n; S)$$

for each n. Therefore, the group $\mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G$ acts on $\mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S}, \mathbf{S})^G$ and we obtain a bijection

$$\mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S}, \mathbf{S})^G / \mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G \cong \mathrm{VEC}_G(B, F^\infty; S). \tag{3.1}$$

The map ρ applied to F^n instead of F produces a map

$$\rho^n \colon \mathrm{VEC}_G(B, F^n; S) \to (R/I)^*/\Gamma^n$$

for each positive integer n. Here Γ^n is a subgroup of $(R/I)^*$ defined for F^n, and since $\mathrm{aut}(\mathbf{F}^n \oplus \mathbf{S})^G$ is a subgroup of $\mathrm{aut}(\mathbf{F}^{n+1} \oplus \mathbf{S})^G$, Γ^n is a subgroup of Γ^{n+1}. We define Γ^∞ to be the union of Γ^n over all positive integers n. Then the maps ρ^n induce a map

$$\rho^\infty \colon \mathrm{VEC}_G(B, F^\infty; S) \to (R/I)^*/\Gamma^\infty.$$

We do not know whether ρ^n is bijective for each n, but we will prove the following in the next section.

THEOREM 3.1. *The map ρ^∞ is bijective (in fact, a group isomorphism) for any G-modules B, F and S.*

4. Proof of Theorem 1.1

As we did in the previous section for $\mathrm{sur}(\mathbf{F}^n \oplus \mathbf{S}, \mathbf{S})^G$, we think of $\mathrm{mor}(\mathbf{F}^n, \mathbf{S})^G$ (resp. $\mathrm{mor}(\mathbf{S}, \mathbf{F}^n)^G$) as a subset of $\mathrm{mor}(\mathbf{F}^{n+1}, \mathbf{S})^G$ (resp. $\mathrm{mor}(\mathbf{S}, \mathbf{F}^{n+1})^G$) by defining to be zero on the added factor and denote by $\mathrm{mor}(\mathbf{F}^\infty, \mathbf{S})^G$ (resp. $\mathrm{mor}(\mathbf{S}, \mathbf{F}^\infty)^G$) the union of $\mathrm{mor}(\mathbf{F}^n, \mathbf{S})^G$ (resp. $\mathrm{mor}(\mathbf{S}, \mathbf{F}^n)^G$) over all positive integers n. Let $\phi_1, \ldots, \phi_\ell$ be elements in $\mathrm{mor}(\mathbf{F}^\infty, \mathbf{S})^G$. Then each ϕ_i lies in $\mathrm{mor}(\mathbf{F}^{n_i}, \mathbf{S})^G$ for some positive integer n_i. We define

$$(\phi_1, \dots, \phi_\ell)(v) := \sum_{i=1}^{\ell} \phi_i(v) \quad \text{for } v \in \mathbf{F},$$

so that $(\phi_1, \dots, \phi_\ell)$ is an element in $\mathrm{mor}(\mathbf{F}^{\sum n_i}, \mathbf{S})^G$ and hence in $\mathrm{mor}(\mathbf{F}^\infty, \mathbf{S})^G$.

Since $\mathrm{mor}(\mathbf{F}, \mathbf{S})^G = \mathrm{Mor}(B, \mathrm{Hom}(F, S))^G$ and $\mathrm{Mor}(B, V)^G$ is finitely generated as an $\mathcal{O}(B)^G$-module for any G-module V as is well-known, $\mathrm{mor}(\mathbf{F}, \mathbf{S})^G$ is a finitely generated $\mathcal{O}(B)^G$-module. Let $\Phi_1, \Phi_2, \dots, \Phi_k$ be generators of $\mathrm{mor}(\mathbf{F}, \mathbf{S})^G$ as an $\mathcal{O}(B)^G$-module. We set

$$\Phi := (\Phi_1, \Phi_2, \dots, \Phi_k) \in \mathrm{mor}(\mathbf{F}^k, \mathbf{S})^G \subset \mathrm{mor}(\mathbf{F}^\infty, \mathbf{S})^G$$

and think of it as an element of $\mathrm{mor}(\mathbf{F}^\infty, \mathbf{S})^G$.

Lemma 4.1. *Any element in the ideal I is of the form $\Phi\Psi$ with some $\Psi \in \mathrm{mor}(\mathbf{S}, \mathbf{F}^\infty)^G$.*

Proof. By definition, the ideal I is generated by elements in $R = \mathrm{mor}(\mathbf{S}, \mathbf{S})^G$ which factors through $\mathbf{F}$. Therefore, any element α in I is of the form $\sum \phi_i\psi_i$ with some $\phi_i \in \mathrm{mor}(\mathbf{F}, \mathbf{S})^G$ and $\psi_i \in \mathrm{mor}(\mathbf{S}, \mathbf{F})^G$. Since Φ_j's are generators of $\mathrm{mor}(\mathbf{F}, \mathbf{S})^G$ as an $\mathcal{O}(B)^G$-module, each ϕ_i is a linear combination of $\Phi_1, \dots, \Phi_k$ over $\mathcal{O}(B)^G$. Therefore, $\alpha = \sum \phi_i\psi_i = \sum_{j=1}^k \Phi_j\Psi_j$ with some $\Psi_j \in \mathrm{mor}(\mathbf{S}, \mathbf{F})^G$ because $\mathrm{mor}(\mathbf{S}, \mathbf{F})^G$ is also an $\mathcal{O}(B)^G$-module. This means that if we set $\Psi = (\Psi_1, \Psi_2, \dots, \Psi_k) \in \mathrm{mor}(\mathbf{S}, \mathbf{F}^\infty)^G$, then $\alpha = \Phi\Psi$. □

If (ϕ, T) is an element of $\mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S}, \mathbf{S})^G$, where $\phi \in \mathrm{mor}(\mathbf{F}^\infty, \mathbf{S})^G$ and $T \in R = \mathrm{mor}(\mathbf{S}, \mathbf{S})^G$, then $[T]$ in R/I is a unit as is observed in Section 2. Conversely, if T is an element of R whose image $[T]$ in R/I is a unit, then there is an element Y in R such that $TY \equiv 1 \mod I$. It follows from Lemma 4.1 that there is $\Psi \in \mathrm{mor}(\mathbf{S}, \mathbf{F}^\infty)^G$ such that $\Phi\Psi + TY = 1$. This means that the pair (Φ, T) is an element of $\mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S}, \mathbf{S})^G$.

We denote $\ker(\phi, T)$ by $E_\phi(T)$, and by $\{E\}$ the element in $\mathrm{VEC}_G(B, F^\infty; S)$ determined by a G-vector bundle E. The argument above shows that if $\{E_\phi(T)\}$ is an element in $\mathrm{VEC}_G(B, F^\infty; S)$, then so is $\{E_\Phi(T)\}$. With this understood we have

Lemma 4.2. $\{E_\phi(T)\} = \{E_\Phi(T)\}$.

Proof. Since $(\phi, T) \in \mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S}, \mathbf{S})^G$, there are elements $\psi \in \mathrm{mor}(\mathbf{S}, \mathbf{F}^\infty)^G$ and $Y \in R$ such that $\phi\psi + TY = 1$. Hence we have

$$(\phi, \Phi, T)\begin{pmatrix} 1 & -\psi\Phi & 0 \\ 0 & 1 & 0 \\ 0 & -Y\Phi & 1 \end{pmatrix} = (\phi, 0, T),$$

where the square matrix above is in $\mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G$. This together with (3.1) shows that $\{E_{\phi\oplus\Phi}(T)\} = \{E_{\phi\oplus 0}(T)\}$. Here $\{E_{\phi\oplus 0}(T)\} = \{E_\phi(T)\}$ because $E_{\phi\oplus 0}(T)$ is isomorphic to Whitney sum of $E_\phi(T)$ and a certain number of $\mathbf{F}$. Therefore we have $\{E_{\phi\oplus\Phi}(T)\} = \{E_\phi(T)\}$. Changing the role of ϕ and Φ, we obtain $\{E_{\Phi\oplus\phi}(T)\} = \{E_\Phi(T)\}$. Thus, it suffices to prove that $\{E_{\phi\oplus\Phi}(T)\} = \{E_{\Phi\oplus\phi}(T)\}$, but this follows from the following identity and (3.1):

$$(\phi, \Phi, T)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\Phi, \phi, T),$$

where the square matrix above is in $\mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G$. □

As noted before Lemma 4.2, we have an element $\{E_\Phi(T)\} \in \mathrm{VEC}_G(B, F^\infty; S)$ for any $T \in R$ such that $[T] \in (R/I)^*$.

LEMMA 4.3. *If* $[T] = [T'] \in (R/I)^*$, *then* $\{E_\Phi(T)\} = \{E_\Phi(T')\}$.

Proof. Since $T \equiv T' \mod I$, there is $\Psi \in \mathrm{mor}(\mathbf{S}, \mathbf{F}^\infty)^G$ such that $T' = T + \Phi\Psi$ by Lemma 4.1. Then

$$(\Phi, T)\begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix} = (\Phi, T')$$

where the square matrix above is in $\mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G$. This together with (3.1) proves the lemma. □

Lemma 4.3 tells us that the correspondence $[T] \to \{E_\Phi(T)\}$ yields a well-defined map

$$\mathcal{V}\colon (R/I)^* \to \mathrm{VEC}_G(B, F^\infty; S),$$

and Lemma 4.2 tells us that $\mathcal{V}$ is independent of the choice of Φ and is surjective.

LEMMA 4.4. (1) $\mathcal{V}([1]) = \{\mathbf{F}\}$.
(2) $\mathcal{V}([T'][T]) = \mathcal{V}([T']) \oplus \mathcal{V}([T])$ *for any* $[T'], [T] \in (R/I)^*$.

Proof. (1) Since $(0,1) \in \mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S}, \mathbf{S})^G$, $\{E_0(1)\} = \{E_\Phi(1)\}$ by Lemma 4.2. Here $E_0(1)$ is nothing but $\mathbf{F}$, so statement (1) is proved.

(2) By definition

$$\begin{aligned}
&\mathcal{V}([T'][T]) = \mathcal{V}([T'T]) = \{E_\Phi(T'T)\},\\
&\mathcal{V}([T']) = \{E_\Phi(T')\},\\
&\mathcal{V}([T]) = \{E_\Phi(T)\}.
\end{aligned}$$

Since $E_\Phi(1) \cong \mathbf{F}$ by (1) above, it suffices to prove that

$$E_\Phi(T'T) \oplus E_\Phi(1) \cong E_\Phi(T') \oplus E_\Phi(T).$$

Here the left hand side is the kernel of

$$L := \begin{pmatrix} \Phi & 0 & T'T & 0 \\ 0 & \Phi & 0 & 1 \end{pmatrix} \in \mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S} \oplus \mathbf{S}, \mathbf{S} \oplus \mathbf{S})^G$$

while the right hand side is the kernel of

$$L' := \begin{pmatrix} \Phi & 0 & T' & 0 \\ 0 & \Phi & 0 & T \end{pmatrix} \in \mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S} \oplus \mathbf{S}, \mathbf{S} \oplus \mathbf{S})^G.$$

Since $[T] \in (R/I)^*$ and $(R/I)^*$ is a group, there is $Y \in R$ such that $TY \equiv YT \equiv 1 \mod I$. Set $P := 1 - YT$ and $Q := Y(Y-1)$. Then $P \equiv 0 \mod I$ and $TQ \equiv Y - 1 \mod I$. Observe that

$$L \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & Y - PQ & P \\ 0 & 0 & Y - 1 - (T+P)Q & T + P \end{pmatrix} = \begin{pmatrix} \Phi & 0 & T' + p_1 & p_2 \\ 0 & \Phi & p_3 & T + p_4 \end{pmatrix}$$

where $p_i \in I$, and that

$$\begin{pmatrix} \Phi & 0 & T' + p_1 & p_2 \\ 0 & \Phi & p_3 & T + p_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\Psi_1 & -\Psi_2 \\ 0 & 1 & -\Psi_3 & -\Psi_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = L',$$

where $\Psi_i \in \mathrm{mor}(\mathbf{S}, \mathbf{F}^\infty)^G$ such that $p_i = \Phi\Psi_i$ for each i (such Ψ_i exists by Lemma 4.1). One can check that the two square matrices above are both in $\mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S} \oplus \mathbf{S})^G$ by applying elementary operations. This shows that the kernels of L and L', which are respectively $E_\Phi(T'T) \oplus \mathbf{F}$ and $E_\Phi(T') \oplus E_\Phi(T)$, are isomorphic. □

Proof of Theorem 1.1. The map $\mathcal{V}\colon (R/I)^* \to \mathrm{VEC}_G(B, F^\infty; S)$ is surjective as noted before and $(R/I)^*$ is a group. So it follows from Lemma 4.4 that the abelian monoid $\mathrm{VEC}_G(B, F^\infty; S)$ is actually an abelian group, i.e., any element in $\mathrm{VEC}_G(B, F^\infty; S)$ has an inverse in it.

It follows from the result of Bass-Haboush mentioned in the introduction that the union of $\mathrm{VEC}_G(B, F^n; S)$ over all G-modules S agrees with $\mathrm{VEC}_G(B, F^n)$. Therefore the union of $\mathrm{VEC}_G(B, F^\infty; S)$ over all G-modules S agrees with $\mathrm{VEC}_G(B, F^\infty)$. Since $\mathrm{VEC}_G(B, F^\infty; S)$ is a group under Whitney sum, so is $\mathrm{VEC}_G(B, F^\infty)$. □

Proof of Theorem 3.1. Any element in Γ^∞ is represented by $[A(S, S)]$ for some $A \in \mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G$. Since $(A(F^\infty, S), A(S, S))A^{-1} = (0, 1)$, the element $(A(F^\infty, S), A(S, S))$ in $\mathrm{sur}(\mathbf{F}^\infty \oplus \mathbf{S}, \mathbf{S})^G$ produces the trivial element in $\mathrm{VEC}_G(B, F^\infty; S)$. This shows that $\ker \mathcal{V} \supset \Gamma^\infty$. On the other hand, the composition $\rho^\infty \mathcal{V}\colon (R/I)^* \to (R/I)^*/\Gamma^\infty$ is just the projection, so $\ker \mathcal{V} \subset \Gamma^\infty$. Thus $\ker \mathcal{V} = \Gamma^\infty$ and $\mathcal{V}$ induces an isomorphism $\tilde{\mathcal{V}}\colon (R/I)^*/\Gamma^\infty \to \mathrm{VEC}_G(B, F^\infty; S)$. Since $\rho^\infty \tilde{\mathcal{V}}$ is the identity and $\tilde{\mathcal{V}}$ is an isomorphism, ρ^∞ is also an isomorphism. □

5. $\mathbb{C}^*$-action and grading

Since B is a G-module, scalar multiplication gives a $\mathbb{C}^*$-action on B commuting with the G-action. Keeping this example in mind, we consider a general $\mathbb{C}^*$-action on B commuting with the G-action. The $\mathbb{C}^*$-action induces an action on $\mathrm{Mor}(B, V)^G$ and makes it a $\mathbb{C}^*$-module for any G-module V. In fact, we define $(cf)(x) := f(cx)$ for $c \in \mathbb{C}^*, f \in \mathrm{Mor}(B, V)^G$ and $x \in B$. Then $\mathrm{Mor}(B, V)^G$ decomposes into a direct sum of eigenspaces, i.e.,

$$\mathrm{Mor}(B, V)^G = \bigoplus_{k \in \mathbb{Z}} \mathrm{Mor}(B, V)^G_{(k)},$$

where $\mathbb{C}^*$ acts on $\mathrm{Mor}(B, V)_{(k)}$ as scalar multiplication by k-th power. Note that

$$\mathrm{Mor}(B, V)^G_{(0)} = \mathrm{Mor}(B, V)^{G \times \mathbb{C}^*} = \mathrm{Mor}(B /\!\!/ \mathbb{C}^*, V)^G.$$

For an element $P \in \mathrm{Mor}(B, V)^G$, we denote by $P_{(k)}$ the degree k homogeneous component of P. It is obvious that $\mathrm{sur}(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})^G$ and $\mathrm{aut}(\mathbf{F} \oplus \mathbf{S})^G$, which are respectively subsets of $\mathrm{Mor}(B, \mathrm{Hom}(F \oplus S, S))^G$ and $\mathrm{Mor}(B, \mathrm{Hom}(F \oplus S, F \oplus S))^G$, are invariant under the $\mathbb{C}^*$-actions, so both of them inherit gradings. Moreover, it is obvious that the map from $\mathrm{sur}(\mathbf{F} \oplus \mathbf{S}, \mathbf{S})^G$ and $\mathrm{aut}(\mathbf{F} \oplus \mathbf{S})^G$ to R defined by taking the (S, S)-component is $\mathbb{C}^*$-equivariant and hence so is the map $\rho\colon \mathrm{VEC}_G(B, F; S) \to (R/I)^*/\Gamma$.

The $\mathbb{C}^*$-action makes $\mathcal{O}(B)$ a $\mathbb{C}^*$-module as above. We say that $\mathcal{O}(B)$ is *positively graded* if $\mathcal{O}(B)_{(k)} = 0$ for all $k < 0$. The $\mathbb{C}^*$-actions we will use later are the ones obtained as scalar multiplication on B or on a factor of B when B is a direct sum of two G-modules, and $\mathcal{O}(B)$ is positively graded for these actions. The following lemma can easily be checked for them.

LEMMA 5.1 ([3]). *If $\mathcal{O}(B)$ is positively graded for the $\mathbb{C}^*$-action, then the algebraic quotient map $\pi\colon B \to B/\!\!/\mathbb{C}^*$ restricted to the $\mathbb{C}^*$-fixed point set $B^{\mathbb{C}^*}$ gives an isomorphism between $B^{\mathbb{C}^*}$ and $B/\!\!/\mathbb{C}^*$.*

We note that if the grading on $\mathcal{O}(B)$ is positive, then so is the grading on $\operatorname{Mor}(B, V)^G$.

LEMMA 5.2. *If $\mathcal{O}(B)$ is positively graded by the $\mathbb{C}^*$-action, then $L_{(0)} \in \operatorname{sur}(\mathbf{F}\oplus\mathbf{S},\mathbf{S})^{G\times\mathbb{C}^*}$ and $A_{(0)} \in \operatorname{aut}(\mathbf{F}\oplus\mathbf{S})^{G\times\mathbb{C}^*}$ for $L \in \operatorname{sur}(\mathbf{F}\oplus\mathbf{S},\mathbf{S})^G$ and $A \in \operatorname{aut}(\mathbf{F}\oplus\mathbf{S})^G$.*

Proof. As remarked in the previous section, there is an element $M \in \operatorname{mor}(\mathbf{S},\mathbf{F}\oplus\mathbf{S})^G$ such that LM is the identity. Since $(LM)_{(0)} = L_{(0)}M_{(0)}$ (where we use the assumption that our grading is positive) and the identity is of degree zero, it follows that $L_{(0)}M_{(0)}$ is the identity. This shows that $L_{(0)}\colon \mathbf{F}\oplus\mathbf{S} \to \mathbf{S}$ is also surjective. A similar argument shows that $A_{(0)}$ is again an automorphism of $\mathbf{F}\oplus\mathbf{S}$. □

It follows from the above lemma that sending L to $L_{(0)}$ induces a correspondence

$$\operatorname{sur}(\mathbf{F}\oplus\mathbf{S},\mathbf{S})^G/\operatorname{aut}(\mathbf{F}\oplus\mathbf{S})^G \to \operatorname{sur}(\mathbf{F}\oplus\mathbf{S},\mathbf{S})^{G\times\mathbb{C}^*}/\operatorname{aut}(\mathbf{F}\oplus\mathbf{S})^{G\times\mathbb{C}^*}.$$

Here the left hand side is identified with $\mathrm{VEC}_G(B, F; S)$ while the right hand side is identified with $\mathrm{VEC}_G(B^{\mathbb{C}^*}, F; S)$ because $\mathcal{O}(B)^{\mathbb{C}^*} = \mathcal{O}(B^{\mathbb{C}^*})$ by Lemma 5.1. Through these identifications, the above map is nothing but the restriction of G-vector bundles over B to $B^{\mathbb{C}^*}$.

One can apply the above argument to F^n for each n in place of F, so all the statements above hold for F^∞ in place of F.

6. Analysis of $(R/I)^*/\Gamma^\infty$

Since the map ρ^∞ is bijective by Theorem 3.1, we are led to study its target group $(R/I)^*/\Gamma^\infty$. Henceforth we assume that R/I is commutative. Suppose that our $\mathbb{C}^*$-action on B commutes with the G-action and induces a positive grading on $\mathcal{O}(B)$. Then R has a positive grading and

I becomes a graded ideal in R because it is invariant under the induced $\mathbb{C}^*$-action on R. Therefore R/I inherits the grading from R. Since the grading on R/I is positive, the degree zero term of a unit in R/I is again a unit. We denote by $(R/I)^*_{(0)}$ the subgroup of $(R/I)^*$ consisting of elements of degree zero. Then we have a decomposition

$$(R/I)^* = (R/I)^*_{(0)} \times (1 + (R/I)_1)^*,$$

where $(R/I)_1$ denotes the set of elements in R/I whose degree zero terms vanish. On the other hand, $\Gamma^\infty_{(0)}$, which is the projection image of $\mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G_{(0)}$, is a subgroup of Γ^∞ and we have a decomposition

$$\Gamma^\infty = \Gamma^\infty_{(0)} \times \Gamma^\infty_*,$$

where Γ^∞_* denotes a subgroup of Γ^∞ with 1 as the degree zero term. The above two decompositions give rise to the following decomposition

$$(R/I)^*/\Gamma^\infty = (R/I)^*_{(0)}/\Gamma^\infty_{(0)} \times (1 + (R/I)_1)^*/\Gamma^\infty_*.$$

We note that $(R/I)^*_{(0)}/\Gamma^\infty_{(0)}$ is the target of the invariant ρ^∞ for $\mathrm{VEC}_G(B/\!\!/\mathbb{C}^*, F^\infty; S)$ and that $B/\!\!/\mathbb{C}^*$ can be identified with $B^{\mathbb{C}^*}$ by Lemma 5.2. Therefore the restriction map

$$\iota^*\colon \mathrm{VEC}_G(B, F^\infty; S) \to \mathrm{VEC}_G(B^{\mathbb{C}^*}, F^\infty; S),$$

where $\iota\colon B^{\mathbb{C}^*} \to B$ is the inclusion map, corresponds to the projection

$$(R/I)^*/\Gamma^\infty = (R/I)^*_{(0)}/\Gamma^\infty_{(0)} \times (1 + (R/I)_1)^*/\Gamma^\infty_* \to (R/I)^*_{(0)}/\Gamma^\infty_{(0)},$$

and thus we have

LEMMA 6.1. *If* $\mathrm{VEC}_G(B^{\mathbb{C}^*}, F; S)$ *consists of one element, then* $\mathrm{VEC}_G(B,\ F^\infty;\ S)$ *is isomorphic to* $(1 + (R/I)_1)^*/\Gamma^\infty_*$.

An element $x \in (R/I)_1$ is nilpotent if and only if $1+x \in (1+(R/I)_1)^*$, (see [1], Exercise 2 in p.11). Therefore we have a logathimic map

$$\log\colon (1 + (R/I)_1)^* \to \mathrm{Nil}(R/I)_1$$

where $\mathrm{Nil}(R/I)_1$ denotes the set of nilpotent elements in $(R/I)_1$. $\mathrm{Nil}(R/I)_1$ is an $\mathcal{O}(B)^G$-submodule of $(R/I)_1$ and hence of R/I. The map log is an isomorphism, the inverse being an exponential map.

LEMMA 6.2. $\log \Gamma^\infty_*$ *is an* $\mathcal{O}(B)^G$*-submodule of* $\mathrm{Nil}(R/I)_1$.

Proof. The groups $(1 + (R/I)_1)^*$ and $\mathrm{Nil}(R/I)_1$ have the $\mathbb{C}^*$-actions and the map log are equivariant with respect to the actions. Therefore, $\log \Gamma^\infty_*$ is a $\mathbb{C}^*$-invariant additive subgroup of $\mathrm{Nil}(R/I)_1$. It follows that if x is an element of $\log \Gamma^\infty_*$, then all its homogeneous terms $x_{(d)}$ lie in

$\log \Gamma_*^\infty$. In fact, since $x = \sum_{d=1}^\infty x_{(d)}$, where $x_{(d)} = 0$ for sufficiently large d, is an element of the $\mathbb{C}^*$-invariant additive subgroup $\log \Gamma_*^\infty$, $\sum z^d x_{(d)}$ lies in $\log \Gamma_*^\infty$ for any $z \in \mathbb{C}^*$. Suppose that $x_{(d)} = 0$ for all $d > m$ where m is a certain positive integer. Then we take m nonzero different integers for z. For those m values of z, $\sum z^d x_{(d)}$ lie in $\log \Gamma_*^\infty$. Using the non-singularity of Vandermonde matrix and the fact that $\log \Gamma_*^\infty$ is an additive group, one sees that $x_{(d)}$'s lie in $\log \Gamma_*^\infty$ for all d.

In the sequel, it suffices to show that if $x \in \log \Gamma_*^\infty$ is homogeneous, then fx lies again in $\log \Gamma_*^\infty$ for any $f \in \mathcal{O}(B)^G$. This can be seen as follows. Since the exponetional map exp: $\mathrm{Nil}(R/I)_1 \to (1 + (R/I)_1)^*$ is the inverse of log, $\exp(x)$ is an element of Γ_*^∞. Remember that an element in Γ_*^∞ is the (S, S)-component of an element of $\mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G$ with 1 as the degree zero term. Suppose that $\exp(x)$ is the (S, S)-component of such an element $A = \sum_{d=0}^\infty A_{(d)}$ where $A_{(0)} = 1$. Then, $\sum_{d=0}^\infty f^d A_{(d)}$ again lies in $\mathrm{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G$ for $f \in \mathcal{O}(B)^G$. In fact, if $A' = \sum_{d=0}^\infty A'_{(d)}$ is the inverse of A, then one checks that $\sum_{d=0}^\infty f^d A'_{(d)}$ is the inverse of $\sum_{d=0}^\infty f^d A_{(d)}$. Taking degrees into account, one sees that the (S, S)-component of $\sum_{d=0}^\infty f^d A_{(d)}$ is equal to $\exp(fx)$. Therefore fx lies again in $\log \Gamma_*^\infty$, proving the lemma. □

LEMMA 6.3. *The group $(1 + (R/I)_1)^*/\Gamma_*^\infty$ is isomorphic to a finitely generated $\mathcal{O}(B)^G$-module.*

Proof. The group $(1+(R/I)_1)^*/\Gamma_*^\infty$ is isomorphic to $\mathrm{Nil}(R/I)_1/\log \Gamma_*^\infty$ through the map log. As is well known, $R = \mathrm{Mor}(B, \mathrm{Hom}(S, S))^G$ is finitely generated as $\mathcal{O}(B)^G$-module and hence so is the quotient R/I. Since the ring $\mathcal{O}(B)^G$ is Noetherian and $\mathrm{Nil}(R/I)_1$ is an $\mathcal{O}(B)^G$-submodule of R/I, $\mathrm{Nil}(R/I)_1$ is finitely generated as $\mathcal{O}(B)^G$-module, (see Propositions 6.2 and 6.5 in [1]) and hence so is the quotient $\mathrm{Nil}(R/I)_1/\log \Gamma_*^\infty$. This proves the lemma. □

Proof of Theorem 1.2. We take the $\mathbb{C}^*$-action on B defined by scalar multiplication. Then $B^{\mathbb{C}^*}$ is a point, that is the origin, so $\mathrm{VEC}_G(B^{\mathbb{C}^*}, F^\infty; S)$ consists of one element. Therefore the theorem follows from Lemmas 6.1 and 6.3. □

7. Product formula

We shall prove Theorem 1.3. We use the notation R, I and Γ^∞ for the base space B as before and $\bar{R}, \bar{I}$ and $\bar{\Gamma}^\infty$ for the base space $B \oplus \mathbb{C}^m$.

LEMMA 7.1. *$\bar{R} = R \otimes \mathcal{O}(\mathbb{C}^m)$ and $\bar{I} = I \otimes \mathcal{O}(\mathbb{C}^m)$.*

Proof. As is well known,

(7.1) $\quad \mathrm{Mor}(B, V)^G$ is canonically isomorphic to $(V \otimes \mathcal{O}(B))^G$

for any G-module. In fact, an element $f \in \mathrm{Mor}(B, V)^G$ induces an equivariant algebra homomorphism $f^*\colon \mathcal{O}(V) \to \mathcal{O}(B)$. Since V is a module, $\mathcal{O}(V)$ is a symmetric tensor algebra of $V^* = \mathrm{Hom}(V, \mathbb{C})$. Therefore, f^* is determined by its restriction to V^* and hence f^* can be identified with an element of $\mathrm{Hom}(V^*, \mathcal{O}(B))^G = (V \otimes \mathcal{O}(B))^G$. This is the correspondence giving the isomorphism (7.1). Applying (7.1) to $B \oplus \mathbb{C}^m$ in place of B, we get

$$\begin{aligned} \mathrm{Mor}(B \oplus \mathbb{C}^m, V)^G &= (V \otimes \mathcal{O}(B \oplus \mathbb{C}^m))^G \\ &= (V \otimes \mathcal{O}(B) \otimes \mathcal{O}(\mathbb{C}^m))^G \\ &= (V \otimes \mathcal{O}(B))^G \otimes \mathcal{O}(\mathbb{C}^m) \\ &= \mathrm{Mor}(B, V)^G \otimes \mathcal{O}(\mathbb{C}^m). \end{aligned} \tag{7.2}$$

Since $\bar{R} = \mathrm{Mor}(B \oplus \mathbb{C}^m, \mathrm{Hom}(S, S))^G$ and $R = \mathrm{Mor}(B, \mathrm{Hom}(S, S))^G$, the isomorphism (7.2) applied with $V = \mathrm{Hom}(S, S)$ proves the first identity in the lemma.

As for the latter identity, we remember that I is generated by composition of elements in $\mathrm{mor}(\mathbf{F}, \mathbf{S})^G$ and $\mathrm{mor}(\mathbf{S}, \mathbf{F})^G$. Since $\mathrm{mor}(\mathbf{F}, \mathbf{S})^G = \mathrm{Mor}(B, \mathrm{Hom}(F, S))^G$ and $\mathrm{mor}(\mathbf{S}, \mathbf{F})^G = \mathrm{Mor}(B, \mathrm{Hom}(S, F))^G$, the isomorphism (7.2) applied with $V = \mathrm{Hom}(F, S)$ or $\mathrm{Hom}(S, F)$ implies the latter identity in the lemma. □

Now we consider the $\mathbb{C}^*$-action on $B \oplus \mathbb{C}^m$ defined by scalar multiplication on the factor B. This action commutes with the G-action on $B \oplus \mathbb{C}^m$, where the G-action on $\mathbb{C}^m$ is trivial, and $\mathcal{O}(B \oplus \mathbb{C}^m) = \mathcal{O}(B) \otimes \mathcal{O}(\mathbb{C}^m)$ is positively graded by the $\mathbb{C}^*$-action, so that we can apply the results in Section 6. Then, since $(B \oplus \mathbb{C}^m)^{\mathbb{C}^*} = \{0\} \oplus \mathbb{C}^m$ and $\mathrm{VEC}_G(\mathbb{C}^m, F^\infty; S)$ consists of one element (because any G-vector bundle over $\mathbb{C}^m$ is trivial, which follows from the Quillen-Suslin Theorem, see Corollary in p.113 of [7]), we have

$$(\bar{R}/\bar{I})^*/\bar{\Gamma}^\infty = (1 + (\bar{R}/\bar{I})_1)^*/\bar{\Gamma}^\infty_*,$$

and the logarithmic map

$$\log\colon (1 + (\bar{R}/\bar{I})_1)^* \to \mathrm{Nil}(\bar{R}/\bar{I})_1$$

is an isomorphism.

LEMMA 7.2. (1) $\mathrm{Nil}(\bar{R}/\bar{I})_1 = \mathrm{Nil}(R/I)_1 \otimes \mathcal{O}(\mathbb{C}^m)$.

(2) $\log \bar{\Gamma}_*^\infty = \log \Gamma_*^\infty \otimes \mathcal{O}(\mathbb{C}^m)$.

Proof. (1) Since R/I is commutative and $\mathcal{O}(\mathbb{C}^m)$ is a polynomial ring in m variables, it follows from a theorem of E. Snapper (see p.70 in [9]) and Lemma 7.1 that

$$\mathrm{Nil}(\bar{R}/\bar{I}) = \mathrm{Nil}(R/I) \otimes \mathcal{O}(\mathbb{C}^m). \tag{7.3}$$

Here elements in $\mathcal{O}(\mathbb{C}^m)$ have degree zero with respect to our $\mathbb{C}^*$-action, so the identity in the lemma follows by taking elements whose degree zero terms vanish in (7.3).

(2) Through the projection from $B \oplus \mathbb{C}^m$ on B, one can think of Γ_*^∞ as a subgroup of $\bar{\Gamma}_*^\infty$, hence $\log \bar{\Gamma}_*^\infty \supset \log \Gamma_*^\infty$. By Lemma 6.2 (applied with $B \oplus \mathbb{C}^m$ in place of B), $\log \bar{\Gamma}_*^\infty$ is a module over $\mathcal{O}(B \oplus \mathbb{C}^m)^G = \mathcal{O}(B)^G \otimes \mathcal{O}(\mathbb{C}^m)$. It follows that $\log \bar{\Gamma}_*^\infty \supset \log \Gamma_*^\infty \otimes \mathcal{O}(\mathbb{C}^m)$.

We shall prove the converse inclusion relation. By definition, an element in $\bar{\Gamma}_*^\infty$ is represented by the (S,S)-component of a G-vector bundle automorphism $\bar{A}$ of the trivial bundle $(B \oplus \mathbb{C}^m) \times (F \oplus S)$ over $B \oplus \mathbb{C}^m$ such that $\bar{A}$ restricted to $\{0\} \oplus \mathbb{C}^m$ is the identity. Since $\log[\bar{A}(S,S)]$ is contained in $\mathrm{Nil}(\bar{R}/\bar{I})_1 = \mathrm{Nil}(R/I)_1 \otimes \mathcal{O}(\mathbb{C}^m)$, one can express

$$\log[\bar{A}(S,S)] = \sum_{i=1}^{q} r_i p_i$$

with $r_i \in \mathrm{Nil}(R/I)_1$ and $p_i \in \mathcal{O}(\mathbb{C}^m)$. We may assume that the polynomials p_i's are linearly independent over $\mathbb{C}$. Then there are points $x_1, \ldots, x_q$ in $\mathbb{C}^m$ such that q vectors $(p_1(x_j), \ldots, p_q(x_j))$ for $j = 1, \ldots, q$ are linearly independent. We consider the restriction of $\bar{A}$ to $B \times \{x_j\}$, denoted by A_j, and think of A_j as a G-vector bundle automorphism of $B \times (F \oplus S)$. We have that $\log[A_j(S,S)] = \sum_{i=1}^{q} p_i(x_j) r_i$ and $\log[A_j(S,S)]$ is an element of $\log \Gamma_*^\infty$ for each j. It follows that r_i is an element of $\log \Gamma_*^\infty$ for each i because the q vectors $(p_1(x_j), \ldots, p_q(x_j))$ for $j = 1, \ldots, q$ are linearly independent and $\log \Gamma_*^\infty$ is a vector space over $\mathbb{C}$. Therefore, $\log[\bar{A}(S,S)]$ is an element of $\log \Gamma_*^\infty \otimes \mathcal{O}(\mathbb{C}^m)$. Since $\bar{A}$ is arbitrary, this proves the desired converse inclusion relation. □

Proof of Theorem 1.3. The theorem follows from Theorem 3.1 and Lemma 7.2. □

References

[1] M. Atiyah and I. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[2] H. Bass, *Algebraic group actions on affine spaces*, Group actions on rings (Brunswick, Maine, 1984), Contemp. Math. **43**, Amer. Math. Soc., Providence, R.I., 1985, pp. 1–23.

[3] H. Bass and W. Haboush, *Linearizing certain reductive group actions*, Trans. Amer. Math. Soc. **292** (1985), 463–482.

[4] ______, *Some equivariant K-theory of affine algebraic group actions*, Comm. Algebra **15** (1987), 181–217.

[5] P. Heinzner and F. Kutzschbauch, *An equivariant version of Grauert's Oka principle*, Invent. Math. **119** (1995), 317–346.

[6] F. Knop, *Nichtlinearisierbare Operationen halbeinfacher Gruppen auf affinen Räumen*, Invent. Math. **105** (1991), 217–220.

[7] H. Kraft, *G-vector bundles and the linearization problem*, Proceedings of the Conference on Group Actions and Invariant Theory, Montreal 1988, CMS Conference Proceedings **10** (1988), 111–124.

[8] H. Kraft and G. Schwarz, *Reductive group actions with one dimensional quotient*, Inst. Hautes Études Sci. Publ. Math. **76** (1992), 1–97.

[9] T. Y. Lam, *A First Course in Noncommutative Rings*, GTM **131**, Springer-Verlag, New York, 1991.

[10] D. Luna, *Slices Etales*, Bull. Soc. Math. France, Paris, Memoire **33** (1973), 81–105.

[11] K. Masuda, *Moduli of equivariant algebraic vector bundles over a product of affine varieties*, Duke Math. J. **88** (1997), 181–199.

[12] ______, *Equivariant algebraic vector bundles over representations - a survey*, Current Trends in Transformation Groups, K-Monographs in Mathematics, (to appear).

[13] M. Masuda, L. Moser-Jauslin, and T. Petrie, *Equivariant algebraic vector bundles over representations of reductive groups: Applications*, Proc. Nat. Acad. Sci. U.S.A. **88** (1991), 9065–9066.

[14] ______ *The equivariant Serre problem for abelian groups*, Topology **35** (1996), 329–334.

[15] ______ *Invariants of equivariant algebraic vector bundles and inequalities for dominant weights*, Topology **37** (1998), 161–177.

[16] M. Masuda and T. Petrie, *Equivariant algebraic vector bundles over representations of reductive groups: Theory*, Proc. Nat. Acad. Sci. U.S.A. **88** (1991), 9061–9064.

[17] ______ *Stably trivial equivariant algebraic vector bundles*, J. Amer. Math. Soc. **8** (1995), 687–714.

[18] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.

[19] G. Schwarz, *Exotic algebraic group actions*, C.R. Acad. Sci. Paris **309** (1989), 89–94.

[20] ______ *The topology of algebraic quotients*, Topological methods in algebraic transformation groups, Progress in Math. **80**, Birkhäuser Verlag, 1989, pp. 135–151.

[21] A. Suslin, *Projective modules over a polynomial ring*, Dokl. Akad. Nauk SSSR **26** (1976), (=Soviet Math. Doklady, **17** (1976), 1160–1164).

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