ON HÖLDER-MCCARTHY-TYPE INEQUALITIES WITH POWERS

CHIA-SHIANG LIN AND YEOL JE CHO

Abstract. We extend the Hölder-McCarthy inequality for a positive and an arbitrary operator, respectively. The powers of each inequality are given and the improved Reid's inequality by Halmos is generalized. We also give the bound of the Hölder-McCarthy inequality by recursion.

Let $A$ be a positive (bounded and linear) operator (written $A \geq 0$) on a Hilbert space $H$. Then, for any $x \in H$ and a given positive real number $\gamma$,

(a) $\langle A^\gamma x, x \rangle \leq (\langle Ax, x \rangle)^\gamma \|x\|^{2(1-\gamma)}$, $\gamma \in (0, 1]$,

and

(b) $\langle A^\gamma x, x \rangle \geq (\langle Ax, x \rangle)^\gamma \|x\|^{2(1-\gamma)}$, $\gamma \geq 1$.

McCarthy [7] proved the inequalities above by using the spectral resolution of $A$ and the Hölder inequality. which justifies the terminology: the Hölder-McCarthy inequality. His proof is simple, but not elementary by no means.

In this paper, we shall generalize the inequalities (a), (b) and consider the powers of the inequalities for a positive and an arbitrary operator, respectively. Also, the improved Reid's inequality by Halmos is extended and the bound of $\langle A^n x, x \rangle - (\langle Ax, x \rangle)^n$ for $n = 1, 2, \cdots$ and $\|x\| = 1$ is
given recursively together with the equality condition. For other recent
improvements on Reid’s inequality, see [3] and [5].

Before we proceed, we need to know that, if $A \geq 0$, then

1. $A^\alpha \geq 0$ for any real number $\alpha \geq 0$,
2. $|\langle Ax, y \rangle|^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle$ for every $x, y \in H$.

The inequality (2) is known as the Cauchy-Schwarz inequality for a positive operator $A$. For more information on Cauchy-Schwarz inequality for high-order and high-power, one may refer to [6]. These two properties would be frequently used throughout this paper without mentioning them. The identity operator on $H$ is denoted by $I$, which is positive, and $A \geq 0$ means that $A \geq 0$ and $A$ is invertible.

**Theorem 1.** For $A \geq 0$, a given positive real number $\gamma \geq 1$ and for every $x, y \in H$, we have

$$
|\langle Ax, y \rangle|^{\gamma} \leq (A^{\gamma}x, x)^{\frac{1}{2}} (A^{\gamma}y, y)^{\frac{1}{2}} \|x\|^{\gamma - 1} \|y\|^{\gamma - 1}.
$$

More generally, for $n = 1, 2, \cdots$, we have

(2) \quad $$
|\langle Ax, y \rangle|^{\gamma} \leq (A^{2^n - 1(\gamma - 1) + 1}x, x)^{\frac{1}{2n}} (A^{2^n - 1(\gamma - 1) + 1}y, y)^{\frac{1}{2n}} (A^{2^n - 1 - \frac{1}{2n}}x, x)^{\frac{2^n - 1}{2n}} (A^{2^n - 1 - \frac{1}{2n}}y, y)^{\frac{2^n - 1 - \frac{1}{2n}}{2n}}
$$

satisfying the relation

$$
(A^{2^n - 1(\gamma - 1) + 1}x, x)^{\frac{1}{2n}} (A^{2^n - 1(\gamma - 1) + 1}y, y)^{\frac{1}{2n}} (A^{2^n - 1 - \frac{1}{2n}}x, x)^{\frac{2^n - 1}{2n}} (A^{2^n - 1 - \frac{1}{2n}}y, y)^{\frac{2^n - 1 - \frac{1}{2n}}{2n}}
$$

$$
\leq (A^{2^n(\gamma - 1) + 1}x, x)^{\frac{1}{2n + 1}} (A^{2^n(\gamma - 1) + 1}y, y)^{\frac{1}{2n + 1}}.
$$

**Proof.** (1) By the inequality (b), we have

$$
|\langle Ax, y \rangle|^{\gamma} = |\langle Ax, y \rangle|^{2 - \frac{\gamma}{2}} \leq (A^{\gamma}x, x)^{\frac{1}{2}} (A^{\gamma}y, y)^{\frac{1}{2}}
$$

$$
\leq (A^{\gamma}x, x)^{\frac{1}{2}} (A^{\gamma}y, y)^{\frac{1}{2}} \|x\|^{\gamma - 1} \|y\|^{\gamma - 1}.
$$

(2) \quad $$
(A^{\gamma}x, x)^{\frac{1}{2}} = (AA^{\gamma - 1}x, x)^{\frac{1}{2}} \leq (A^{2\gamma - 1}x, x)^{\frac{1}{4}} (A^{\gamma}x, x)^{\frac{1}{4}}
$$

$$
= (AA^{2\gamma - 2}x, x)^{\frac{1}{4}} (A^{\gamma}x, x)^{\frac{1}{4}}
$$

$$
\leq (A^{4\gamma - 3}x, x)^{\frac{1}{8}} (A^{\gamma}x, x)^{\frac{1}{8}}.
$$
For \( n \geq 2 \), suppose that
\[
(A^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{\gamma-1}} \leq (A^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{\gamma-1}} \leq (A^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{\gamma-1}}.
\]

Then we have
\[
(A^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{\gamma-1}} \leq (A^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{\gamma-1}} \leq (A^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{\gamma-1}}.
\]
as \((Ax, x)^{\frac{1}{\gamma-1}} + \frac{2^{n-1}-1}{2^{n-1}} = (Ax, x)^{\frac{2^{n-1}-1}{2^{n-1}}}.\) Similarly, we can consider the term \((A^n y, y)^{\frac{1}{\gamma}}\), and conclude that the proof is completed by induction. □

Remark 1. The Hölder-McCarthy inequality (a) and two inequalities (1) and (2) in Theorem 1 are all equivalent to one another.

Theorem 2. Let \( T \) be an arbitrary operator. If \( \gamma \) is a positive real number with \( \gamma \geq 1 \), then, for every \( x, y \in H \),

\[
|(Tx, y)|^\gamma \leq ((T^* T)^{\gamma} x, x)^{\frac{1}{\gamma}} \|x\|^\gamma \|y\|^\gamma.
\]

More generally, for \( n = 1, 2, \cdots \) we have

\[
|(Tx, y)|^\gamma \leq ((T^* T)^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{\gamma}} (T^* T x, x)^{\frac{2^{n-1}-1}{2^{n-1}}} \|x\|^\gamma \|y\|^\gamma
\]
satisfying the relation
\[
((T^* T)^{2^{n-1}(\gamma-1)+1} x, x)^{\frac{1}{\gamma}} (T^* T x, x)^{\frac{2^{n-1}-1}{2^{n-1}}} \leq ((T^* T)^{2^{n}(\gamma-1)+1} x, x)^{\frac{1}{\gamma}} (T^* T x, x)^{\frac{2^{n}-1}{2^{n+1}}}.
\]

Proof. (1) Clearly \( T^* T \geq 0 \). By the inequality (b), we have
\[
|(Tx, y)|^\gamma = |(ITx, y)|^{\frac{\gamma}{2}} \leq [(ITx, Tx)(Ty, y)]^{\frac{\gamma}{2}} = (T^* T x, x)^{\frac{\gamma}{2}} \|y\|^\gamma \leq ((T^* T)^{\gamma} x, x)^{\frac{1}{\gamma}} |x| \|y\|^\gamma.
\]
(2) We have

\[
\left( (T^*T)^{\gamma} x, x \right)^{\frac{1}{2}} \\
= \left( (T^*T) (T^*T)^{\gamma^{-1}} x, x \right)^{\frac{1}{2}} \\
\leq \left( (T^*T)^{2\gamma^{-1}} x, x \right)^{\frac{1}{2}} (T^*T x, x)^{\frac{1}{2}} \\
= \left( (T^*T)(T^*T)^{2\gamma-2} x, x \right)^{\frac{1}{2}} (T^*T x, x)^{\frac{1}{2}} \\
\leq \left( (T^*T)^{4\gamma-3} x, x \right)^{\frac{1}{2}} (T^*T x, x)^{\frac{1}{2}}.
\]

For \( n \geq 2 \), suppose that

\[
\left( (T^*T)^{2n^{-2}} x, x \right)^{\frac{2n^{-1}}{2n^{-1}}} (T^*T x, x) \leq \left( (T^*T)^{2n^{-1}} x, x \right)^{\frac{1}{2}} (T^*T x, x)^{\frac{1}{2}}.
\]

Then we have

\[
\left( (T^*T)^{2n^{-1}} x, x \right)^{\frac{1}{2}} (T^*T x, x) \leq \left( (T^*T)(T^*T)^{2n^{-1}} x, x \right)^{\frac{2n^{-1}}{2n^{-1}}} (T^*T x, x) \leq \left( (T^*T)^{2n^{-1}} x, x \right)^{\frac{1}{2}} (T^*T x, x)^{\frac{1}{2}}
\]

as \( (T^*T x, x)^{\frac{2n^{-1}}{2n^{-1}}} = (T^*T x, x)^{\frac{1}{2}} \) and so the proof is completed by induction. \( \square \)

**Remark 2.** The inequalities (1) and (2) in Theorem 2 are equivalent to one another. Also, if \( S \) is a self-adjoint operator (not necessarily positive), then Theorem 2 may be changed to the following and we shall omit the proof.

(1) \((S x, y)^{\gamma} \leq (S^2 x, x)^{\frac{1}{2}} \|x\|^{\gamma-1} \|y\|^\gamma.\)

More generally, for \( n = 1, 2, \cdots \), we have

(2) \((S x, y)^{\gamma} \leq (S^{2n} (\gamma-1)+ x, x)^{\frac{1}{2}} (S^2 x, x)^{\frac{2n^{-1}}{2n^{-1}}} \|x\|^{\gamma-1} \|y\|^\gamma\)

satisfying the relation

\[
(S^{2n} x, x)^{\frac{1}{2}} (S^2 x, x)^{\frac{2n^{-1}}{2n^{-1}}} \leq (S^{2n+1} x, x)^{\frac{1}{2}} (S^2 x, x)^{\frac{2n^{-1}}{2n^{-1}}}.
\]
Recall that the spectral radius of an operator $T$ is denoted by $r(T)$, which is defined by

$$r(T) = \sup\{\lambda : \lambda \in \sigma(T)\},$$

where $\sigma(T)$ is the spectrum of $T$. Note that clearly $0 \leq r(T) \leq \|T\|$ and $r(T)$ is known to be equal to $\lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}$.

The relation $\|(AEx, x)\| \leq \|E\|(Ax, x)$ for all $x \in H$ is known as the Reid inequality for $A \geq 0$, and an operator $E$ such that $AE$ is a self-adjoint operator ([8]). In [2], Halmos sharpened the inequality in that he has $r(E)$ instead of $\|E\|$. Our inequality (2) in Theorem 3 below is a further generalization with a different proof.

**Theorem 3.** Let $A \geq 0$ and let $E$, $F$ be any operators such that $AE$ and $AF$ are self-adjoint. Then, for every $x, y \in H$, a positive real number $\gamma \geq 1$ and $n = 1, 2, \cdots$, we have

$$\|(AEx, Fy)\| \leq (AE^{2^n} x, x)^{\frac{2^n - 1}{2^n}} (AF^{2^n} y, y)^{\frac{2^n - 1}{2^n}}$$

satisfying the relation

$$(AE^{2^n} x, x)^{\frac{2^n - 1}{2^n}} (Ax, x)^{\frac{2^n - 1}{2^n}} \leq (AE^{2^n} x, x)^{\frac{2^n - 1}{2^n}} (Ax, x)^{\frac{2^n - 1}{2^n}}$$

and

$$(AF^{2^n} x, y)^{\frac{2^n - 1}{2^n}} ( Ay, y)^{\frac{2^n - 1}{2^n}} \leq (AE^{2^n} x, y)^{\frac{2^n - 1}{2^n}} (Ay, y)^{\frac{2^n - 1}{2^n}}.$$

In particular,

$$\|(AEx, Fy)\| \leq r(E)r(F)(Ax, x)^{\frac{1}{2}} (Ay, y)^{\frac{1}{2}}.$$

**Proof.** (1) Notice first that $(E^*)^i AE^i = AE^{2i}$ and $(E^*)^i AF^i = AF^{2i}$ for $i = 1, 2, \cdots$ due to the self-adjointness of $AE$ and $AF$. Next, we see that

$$\|(AEx, Fy)\| \leq (AE^{2^n} x, x)^{\frac{2^n - 1}{2^n}} (AF^{2^n} y, y)^{\frac{2^n - 1}{2^n}} \leq (AE^{2^n} x, x)^{\frac{1}{2}} (AF^{2^n} y, y)^{\frac{1}{2}}.$$
Now, consider the term \((AE^2 x, x)^{\frac{1}{2}}\) as follows:

\[
(AE^2 x, x)^{\frac{1}{2}} = (AE^2 x, E^2 x)^{\frac{1}{2}} \leq (AE^2 x, E^2 x)^{\frac{1}{2}} (Ax, x)^{\frac{1}{4}}
\]

\[
= (AE^4 x, x)^{\frac{1}{4}} (Ax, x)^{\frac{1}{4}} = (AE^4 x, x)^{\frac{1}{4}} (Ax, x)^{\frac{1}{4}}
\]

\[
\leq (AE^8 x, x)^{\frac{1}{8}} (Ax, x)^{\frac{1}{8}}.
\]

For \(n \geq 2\), suppose that

\[
(AE^{2n-1} x, x)^{\frac{1}{2n-1}} (Ax, x)^{(2n-2-1)\gamma} \leq (AE^{2n} x, x)^{\frac{1}{2n}} (Ax, x)^{(2n-1-1)\gamma}.
\]

Then we have

\[
(AE^{2n} x, x)^{\frac{1}{2n}} (Ax, x)^{(2n-1-1)\gamma}
\]

\[
= (AE^{2n} x, x)^{\frac{1}{2n}} (Ax, x)^{(2n-1-1)\gamma}
\]

\[
\leq (AE^{2n} x, E^{2n} x)^{\frac{1}{2n}} (Ax, x)^{(2n-1-1)\gamma + \frac{1}{2n-1}}
\]

\[
= (AE^{2n+1} x, x)^{\frac{1}{2n+1}} (Ax, x)^{(2n-1)\gamma}.
\]

This together with a similar consideration for the term \((AF^2 y, y)^{\frac{1}{2}}\) implies the inequality (1) by induction.

(2) We may replace \(\gamma\) in (1) above by \(2^n\) to get

\[
|(AEx, Fy)|^{2^n}
\]

\[
\leq (AE^{2n} x, x)(Ax, x)^{2^{n-1} - 1} (AF^{2n} y, y)(Ay, y)^{2^{n-1} - 1}
\]

\[
\leq \|A\|^2 \|E^{2n}\| \|x\|^2 (Ax, x)^{2^{n-1} - 1} \|F^{2n}\| \|y\|^2 (Ay, y)^{2^{n-1} - 1}.
\]

The desired inequality follows by taking the \(2^n\)-th root of both sides above and passing to the limit as \(n \to \infty\). This completes the proof. \(\Box\)

**Remark 3.** The positive operator \(A\) in Theorem 3 may be relaxed to a self-adjoint operator \(S\). In other words, if \(E\) and \(F\) are any operators such that \(S^2E\) is self-adjoint, then, for any \(x, y \in H\), a positive real number \(\gamma \geq 1\) and \(n = 1, 2, \ldots\), we have

(1) \(|(SEx, Fy)|^{\gamma}\)

\[
\leq (S^2E^{2n} x, x)^{\frac{1}{2n}} (S^2 x, x)^{(2n-1-1)\gamma} ((F^* F)^{2n-1} y, y)^{\frac{1}{2n}} \|y\|^{(2n-1-1)\gamma}.
\]
satisfying the relation
\[
(S^2 E^{2n} x, x)^{\frac{1}{2n}} \leq (S^2 E^{2n+1} x, x)^{\frac{1}{2n+1}} (S^2 x, x)^{\frac{1}{2n+1}}.
\]

Note that
\[
|\langle SE x, F y \rangle| \leq \tau(E)\tau(F^* F)^{\frac{1}{2}} (S^2 x, x)^{\frac{1}{2}} \|y\|.
\]

We shall omit the proof.

The next result depends on the Hölder-McCarthy inequality (b) wherever is appropriate.

**Theorem 4.** For every \(x, y \in H\), we have the following:

1. If \(A \geq 0\), then, for a positive real number \(\gamma \in (0, 1)\),
   \[
   |(A^\gamma x, y)| \leq (Ax, x)^{\frac{\gamma}{2}} (Ay, y)^{\frac{1}{2}} \|x\|^{1-\gamma} \|y\|^{1-\gamma}.
   \]

2. If \(A > 0\), then, for \(n = 1, 2, \ldots\) and any real number \(\mu\),
   \[
   |(A^n x, y)|^{2^n} \leq (A^{2^n - 1} x, x)
   \times (Ax, x)^{2^n - 1} (A^{2^n - 1} y, y)(Ay, y)^2^{n-1}.
   \]

3. If \(A > 0\), then, for \(n = 1, 2, \ldots\) and a positive real number \(\gamma \in (\frac{2^{n-1}}{2^n - 1}, 1)\),
   \[
   |(A^{\gamma} x, y)|^{2^n} \leq (Ax, x)^{2^{n-1}} (Ay, y)^{2^{n-1}} \|x\|^{2^n(1-\gamma)} \|y\|^{2^n(1-\gamma)}.
   \]

**Proof.** (1) By the inequality (a), we have
   \[
   |(A^\gamma x, y)|^2 \leq (Ax, x)(A^\gamma y, y)
   \leq (Ax, x)^{\gamma} (Ay, y)^{\gamma} \|x\|^{2(1-\gamma)} \|y\|^{2(1-\gamma)}.
   \]

(2) Note that
   \[
   |(A^n x, y)|^4 \leq (A^n x, x)^2 (A^n y, y)^2 = (AA^{n-1} x, x)^2 (AA^{n-1} y, y)^2
   \leq (A^{2n-1} x, x)(Ax, x)(A^{2n-1} y, y)(Ay, y).
   \]
and
\[
|\langle A^\mu x, y \rangle|^2 \leq (A^{2\mu-1}x, x)^2(A^\mu, x)^2(A^{2\mu-1}y, y)^2(Ay, y)^2
\]
\[
= (AA^{2\mu-2}x, x)(A^\mu, x)^2(A^{2\mu-2}y, y)(Ay, y)^2
\]
\[
\leq (A^{4\mu-3}x, x)(A^\mu, x)^3(A^{4\mu-3}y, y)(Ay, y)^3.
\]

For \( n \geq 2 \), suppose that
\[
|\langle A^\mu x, y \rangle|^{2n-1} \leq (A^{2n-2\mu-2n-2+1}x, x)(A^\mu, x)^{2n-2-1}
\]
\[
\times (A^{2n-2\mu-2n-2+1}y, y)(Ay, y)^{2n-2-1}.
\]

Then it follows that
\[
|\langle A^\mu x, y \rangle|^{2n} \leq (A^{2n-2\mu-2n-2+1}x, x)^2(A^\mu, x)^{2n-1-2}
\]
\[
\times (A^{2n-2\mu-2n-2+1}y, y)^2(Ay, y)^{2n-1-2}
\]
\[
\leq (A^{2n-1-\mu-2n-1+1}x, x)(A^\mu, x)^{2n-1-1}
\]
\[
\times (A^{2n-1-\mu-2n-1+1}y, y)(Ay, y)^{2n-1-1}
\]

since we have
\[
(A^{2n-2\mu-2n-2+1}x, x)^2 = (AA^{2n-2\mu-2n-2}x, x)^2
\]
\[
\leq (A^{2n-1-\mu-2n-1+1}x, x)(A^\mu, x).
\]

The claim is thus proved.

(3) If \( 2^{n-1}\gamma - 2^{n-1} + 1 \in (0, 1) \), i.e., \( \gamma \in (\frac{2^{n-1}-1}{2^{n-1}}, 1) \), then we have
\[
(A^{2n-1-\gamma-2^{n-1}+1}x, x) \leq (A^\mu, x)^{2n-1-\gamma-2^{n-1}+1} \|x\|^{2n}(1-\gamma)
\]

and
\[
(A^{2n-1-\gamma-2^{n-1}+1}y, y) \leq (Ay, y)^{2n-1-\gamma-2^{n-1}+1} \|y\|^{2n}(1-\gamma)
\]

as \( 2[1 - (2^{n-1}\gamma - 2^{n-1} + 1)] = 2^n(1 - \gamma) \) for the power of \( \|x\| \) and \( \|y\| \). The required inequality is clear now due to (2) above. This completes the proof.
Remark 4. The Hölder-McCarthy inequality (a) and two inequalities (1) and (3) in Theorem 4 are all equivalent to one another.

Finally, we are going to find the bound of the Hölder-McCarthy inequality (b) by recursion. Now we assume that \( \|x\| = 1 \) in order to simplify the expression. First, we require the next lemma, for which the tool of the proof is the Cauchy-Schwarz inequality.

**Lemma.** Let \( A \geq 0 \) and let \( x \) be a unit vector. Then, for \( n = 1, 2, \cdots \),

\[
[(Ax, A^n x) - (Ax, x)(A^n x, x)]^2 \leq [\|Ax\|^2 - (Ax, x)^2][\|A^n x\|^2 - (A^n x, x)^2].
\]

The equality holds if and only if \( A^n x = cx + dAx \) for some real numbers \( c \) and \( d \).

**Proof.** Let \( u = \|A^n x\|^2 - (A^n x, x)^2 \), which is nonnegative by the Cauchy-Schwarz inequality. The required inequality is trivial if \( u = 0 \) (equivalently, \( x \) and \( A^n x \) are proportional). So, let \( u > 0 \) and put \( v = (Ax, A^n x) - (Ax, x)(A^n x, x) \). Then we have

\[
0 \leq \|uAx - vA^n x\|^2 - (uAx - vA^n x, x)^2
= u^2\|Ax\|^2 - 2uv(Ax, A^n x) + v^2\|A^n x\|^2
- [u^2(Ax, x)^2 - 2uv(Ax, x)(A^n x, x) + v^2(A^n x, x)^2]
= u\{\|Ax\|^2 - (Ax, x)^2\} - v^2,
\]

which yields \( u[\|Ax\|^2 - (Ax, x)^2] \geq v^2 \) and so we have the desired inequality.

The equality holds if and only if \( \|uAx - vA^n x\| = |(uAx - vA^n x, x)| \). Equivalently, \( uAx - vA^n x \) and \( x \) are proportional and so the equality condition follows. This completes the proof. \( \square \)

Remark 5. The equality condition in Lemma can be checked as follows:

Necessity is trivial by the proof. Since \( A^n x = cx + dAx \), a straightforward computation shows that both sides of the inequality are equal to \( d^2[\|Ax\|^2 - (Ax, x)^2]^2 \) and so sufficiency is proved.
THEOREM 5. Let $A \geq 0$ and let $x$ be a unit vector. Then, for $n = 1, 2, \cdots$,

$$(A^n x, x) - (Ax, x)^n$$

$$\leq [||Ax||^2 - (Ax, x)^2]^\frac{1}{2} [||A^{n-1} x||^2 - (A^{n-1} x, x)^2]^\frac{1}{2}$$

$$+ (Ax, x)[(A^{n-1} x, x) - (Ax, x)^{n-1}].$$

The equality holds if and only if $A^{n-1} x = cx + dAx$ for some real numbers $c$ and $d$.

Proof. The proof is a straightforward application of Lemma as follows:

$$(A^n x, x) - (Ax, x)^n$$

$$= (Ax, A^{n-1} x) - (Ax, x)(A^{n-1} x, x) + (Ax, x)(A^{n-1} x, x) - (Ax, x)^n$$

$$\leq [||Ax||^2 - (Ax, x)^2]^\frac{1}{2} [||A^{n-1} x||^2 - (A^{n-1} x, x)^2]^\frac{1}{2}$$

$$+ (Ax, x)[(A^{n-1} x, x) - (Ax, x)^{n-1}].$$

The equality holds if and only if

$$[(Ax, A^{n-1} x) - (Ax, x)(A^{n-1} x, x)]^2$$

$$= [||Ax||^2 - (Ax, x)^2][||A^{n-1} x||^2 - (A^{n-1} x, x)^2],$$

which, in turn, implies that if and only if $A^{n-1} x = cx + dAx$ for some real numbers $c$ and $d$ by Lemma again. \qed

REMARK 6. If $A \geq 0$ and $0 < m \leq A \leq M$ in particular for some real numbers $m$ and $M$, then it can be shown from Theorem 5 that $$(A^n x, x) - (Ax, x)^n$$ is bounded by a function of $m$ and $M$. This result was precisely obtained in [1, Theorem 2] by the fact that the covariance of $A$ and $A^n$ is bounded by a function of $m$ and $M$. For further developments of the variance-covariance inequality, refer to [4].

References


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Chin-Shiang Lin
Department of Mathematics
Bishop's University
Lennoxville
Quebec J1M 1Z7, Canada
E-mail: plin@ubishops.ca

Yeol Je Cho
Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea
E-mail: yjcho@nongae.gsmu.ac.kr