

STABLE LOW ORDER NONCONFORMING
QUADRILATERAL FINITE ELEMENTS
FOR THE STOKES PROBLEM

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ABSTRACT. Stability result is obtained for the approximation of the stationary Stokes problem with nonconforming elements proposed by Douglas et al [1] for the velocity and discontinuous piecewise constants for the pressure on quadrilateral elements. Optimal order H^1 and L^2 error estimates are derived.

1. Introduction

In this paper we obtain the stability result for the finite elements approximation of the stationary Stokes problem with nonconforming elements proposed by Douglas et al [1] for the velocity and discontinuous piecewise constants for the pressure on quadrilateral elements. The stability result was obtained by the macroelement technique. Also, optimal order error estimates on quadrilateral elements will be given.

The finite element space $Q_1 - P_0$ combination for the velocity and pressure for the Stokes problem does not satisfy “inf-sup” condition: it strongly depends on the mesh. For a regular mesh the kernel of $(\operatorname{div} \mathbf{v}, p)$ is two dimensional. More precisely, the well known “checkerboard” pressure is in the kernel. This instability can be cured by using nonconforming velocity as will be shown in Theorem 2. The $Q_2 - P_1$ element is probably the most popular quadrilateral two dimensional element for the Stokes problem at the present time. It appeared as a cure for the instability of the $Q_2 - Q_1$ element which appears quite naturally in the use of reduced integration penalty methods.

Conforming $Q_1 - Q_1$ combination for the velocity and pressure with *three* internal degrees of freedom added to the velocity space for Stokes problem was given by [7].

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Nonconforming finite elements [3,8] are attractive for discretizing the Stokes problem since they possess favorable stability properties. For example, it removes the “checkerboard” pressures. Rannacher and Turek [4], in the setting of the Stokes problem, analyzed two forms of nonconforming elements based on simply rotating the usual bilinear element to employ $\text{Span}\{1, x, y, x^2 - y^2\}$ as the local basis. On rectangles, they construct a very clever argument that uses a cancellation property on each rectangle, plus a series of an inverse property, to show optimal order approximation of the solution of the Stokes problem; however, if the usual definition of the global nonconforming space by requiring continuity at interfacial midpoints is adopted, there is a loss of optimality for truly quadrilateral partitions of the domain.

Douglas et al [1], modified the rotated bilinear local basis $\text{Span}\{1, x, y, x^2 - y^2\}$ as $\text{Span}\{1, x, y, (3x^2 - 5x^4) - (3y^2 - 5y^4)\}$ which has the following orthogonalities $\langle 1, w_j - w_k \rangle_{\Gamma_{jk}} = 0$ and $\langle 1, w_j \rangle_{\Gamma_j} = 0$ on quadrilateral elements where $\Gamma_{jk} = \partial\Omega_j \cap \Omega_k$ and $\Gamma_j = \partial\Omega \cap \Omega_k$. By imposing these orthogonalities to local basis they can avoid that loss of optimality for quadrilateral elements.

A stability result for the stationary Stokes problem with the following velocity-pressure finite element spaces for quadrilateral elements was given in [2]:

$$\begin{aligned} \mathbf{V}_b &= \text{Span}\{1, x, y, (3x^2 - 5x^4) - (3y^2 - 5y^4), (1 - x^2)(1 - y^2)\}^2, \\ P_1 &= \text{Span}\{1, x, y\}. \end{aligned}$$

The conforming bubble function $(1 - x^2)(1 - y^2)$ was necessary for the stability, because they use linear function for pressures. The bubble function can be removed by using constant function for pressures. This is the objective of this paper. Now, let

$$\begin{aligned} \mathbf{V} &= \text{Span}\{1, x, y, (3x^2 - 5x^4) - (3y^2 - 5y^4)\}^2, \\ P &= \text{Span}\{1\}. \end{aligned}$$

In this paper, we use \mathbf{V} as an approximation space for velocity and P for pressure of the Stokes problem. Then, applying a standard macroelement technique [2, 4, 5, 6], we obtain the “inf-sup” or “Babuška and Brezzi” stability condition which is essential to finite elements approximation of Stokes problem. Optimal order error estimates derived for H^1 -norm and L^2 -norm, respectively.

An outline of this paper is as follows. In Section 2, we briefly review Stokes problem. In Section 3, finite elements approximation space for the Stokes problem on quadrilateral elements was given. Stability results

will be given in Section 4 and optimal order error estimates will be derived in Section 5.

2. Stokes problem

In this section we will consider nonconforming finite element method for the Stokes equations: Find the velocity \mathbf{u} and the pressure p such that

$$(1) \quad \begin{aligned} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, \mathbf{f} is the given body force, and $\nu > 0$ is the viscosity. From now on we will assume $\nu = 1$ for simplicity.

The usual variational formulation of (1) is the following. Find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$(2) \quad \begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^2, \\ b(q, \mathbf{u}) &= 0 \quad \forall q \in L_0^2(\Omega), \end{aligned}$$

where $a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v})$, $b(p, \mathbf{v}) = -(p, \operatorname{div} \mathbf{v})$ and (\cdot, \cdot) denotes the inner product in $L^2(\Omega)^2$ or $L^2(\Omega)^{2 \times 2}$, and $L_0^2(\Omega)$ is the space

$$L_0^2(\Omega) = \left\{ p \in L^2(\Omega) \mid \int_{\Omega} p \, dx = 0 \right\}.$$

The finite element methods we consider may be described abstractly as follows. We let \mathcal{T}_h denote a partition of Ω by quasi-regular convex quadrilaterals K of diameter $\leq h$. We then denote by \mathbf{V}_h a finite-dimensional approximation of $H_0^1(\Omega)^2$. Since we are considering nonconforming methods, $\mathbf{V}_h \not\subset H_0^1(\Omega)^2$, but $\mathbf{v}_h \in H^1(K)^2$ for all $\mathbf{v}_h \in \mathbf{V}_h$ and $K \in \mathcal{T}_h$. We assume that

$$\|\mathbf{v}_h\|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |\mathbf{v}_h|_{1,K} \right)^{1/2}$$

is a norm on \mathbf{V}_h . Let P_h denote a finite-dimensional subspace of $L_0^2(\Omega)$. The approximation scheme is then: Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in P_h$ satisfying

$$(3) \quad \begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(q, \mathbf{u}_h) &= 0 \quad \forall q \in P_h, \end{aligned}$$

where

$$a_h(\mathbf{u}, \mathbf{v}) = \sum_{K \in \mathcal{T}_h} (\nabla \mathbf{u}, \nabla \mathbf{v})_K,$$

$$b_h(p, \mathbf{v}) = - \sum_{K \in \mathcal{T}_h} (p, \operatorname{div} \mathbf{v})_K = -(p, \operatorname{div}_h \mathbf{v}).$$

And $\operatorname{div}_h \mathbf{v}$ is the $L^2(\Omega)$ function whose restriction to each quadrilateral $K \in \mathcal{T}_h$ is given by $(\operatorname{div} \mathbf{v})|_K$.

Combining (2) and (3), we obtain the following consistency error identity

$$(4) \quad \begin{aligned} & a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b_h(p - p_h, \mathbf{v}_h) + b_h(q_h, \mathbf{u} - \mathbf{u}_h) \\ &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left(\frac{\partial \mathbf{u}}{\partial \nu_K} - p \nu_K \right) \cdot \mathbf{v}_h, \quad \mathbf{v}_h \in \mathbf{V}_h, q_h \in P_h, \end{aligned}$$

where ν_K denotes the unit outward normal to ∂K .

The analysis of this type of method is well understood. If the space \mathbf{V}_h were conforming, the general theory of saddle-point problems developed by ‘‘Babuška and Brezzi’’ could be applied directly. In that case the only difficulty in the analysis is the verification of the ‘‘inf-sup’’ condition

$$(5) \quad \inf_{0 \neq q_h \in P_h} \sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b(q_h, \mathbf{v}_h)}{\|q_h\|_0 \|\mathbf{v}_h\|_1} \geq \gamma > 0.$$

When (5) holds, one obtain the quasi-optimal error estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leq C \inf (\|\mathbf{u} - \mathbf{v}_h\|_1 + \|p - q_h\|_0),$$

where the ‘‘inf’’ is taken over all $\mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in P_h$. For nonconforming methods, a straightforward modification of this result leads to the conclusion that if

$$(6) \quad \inf_{0 \neq q_h \in P_h} \sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b_h(q_h, \mathbf{v}_h)}{\|q_h\|_0 \|\mathbf{v}_h\|_{1,h}} \geq \gamma > 0,$$

then one obtain the error estimate

$$(7) \quad \begin{aligned} & \|\mathbf{u} - \mathbf{u}_h\|_{1,h} + \|p - p_h\|_0 \\ & \leq C \left(\inf (\|\mathbf{u} - \mathbf{v}_h\|_1 + \|p - q_h\|_0) + \sup \frac{\sum_K \int_{\partial K} \left(\frac{\partial \mathbf{u}}{\partial \nu_K} - p \nu_K \right) \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{1,h}} \right) \end{aligned}$$

where the ‘‘inf’’ is taken over all $\mathbf{v}_h \in \mathbf{V}_h$ and the ‘‘sup’’ is taken over all $\mathbf{w}_h \in \mathbf{V}_h$.

3. Finite elements spaces on quadrilateral elements

Let us partition the domain Ω to quasi-regular convex quadrilateral elements denoted by Ω_j . Let

$$\Omega = \bigcup_j \Omega_j, \quad \Gamma = \partial\Omega, \quad \Gamma_j = \Gamma \cap \partial\Omega_j, \quad \Gamma_{jk} = \Gamma_{kj} = \partial\Omega_j \cap \partial\Omega_k,$$

and denote the centers of Γ_j and Γ_{jk} by c_j and c_{jk} , respectively. Let $\mathcal{P}_l(E)$ denote the class of polynomials of degree l on the set E . For an arbitrary quadrilateral K with vertices a_i , $1 \leq i \leq 4$, let \hat{R} be the reference square $[-1, 1] \times [-1, 1]$ in the (ξ, η) reference space, with vertices \hat{a}_i , $1 \leq i \leq 4$, as in Fig. 1.

Let F_K be the bilinear mapping from \hat{R} onto K , that is, $(x, y) = F_K(\xi, \eta) = (F_K^1, F_K^2) = (\sum_{i=1}^4 x_i \hat{\phi}_i, \sum_{i=1}^4 y_i \hat{\phi}_i)$, where $\hat{\phi}_1 = (1 - \xi)(1 - \eta)/4$, $\hat{\phi}_2 = (1 + \xi)(1 - \eta)/4$, $\hat{\phi}_3 = (1 + \xi)(1 + \eta)/4$, $\hat{\phi}_4 = (1 - \xi)(1 + \eta)/4$. We will use nonconforming finite elements proposed by [1] as an approximation space for velocity of Stokes problem. The nodal basis on reference element is the following:

$$\begin{aligned} \hat{\phi}_1(\xi, \eta) &= \frac{1}{4} - \frac{1}{2}\eta + \frac{1}{8}\{(3\xi^2 - 5\xi^4) - (3\eta^2 - 5\eta^4)\}, \\ (8) \quad \hat{\phi}_2(\xi, \eta) &= \frac{1}{4} + \frac{1}{2}\xi - \frac{1}{8}\{(3\xi^2 - 5\xi^4) - (3\eta^2 - 5\eta^4)\}, \\ \hat{\phi}_3(\xi, \eta) &= \frac{1}{4} + \frac{1}{2}\eta + \frac{1}{8}\{(3\xi^2 - 5\xi^4) - (3\eta^2 - 5\eta^4)\}, \\ \hat{\phi}_4(\xi, \eta) &= \frac{1}{4} - \frac{1}{2}\xi - \frac{1}{8}\{(3\xi^2 - 5\xi^4) - (3\eta^2 - 5\eta^4)\}. \end{aligned}$$

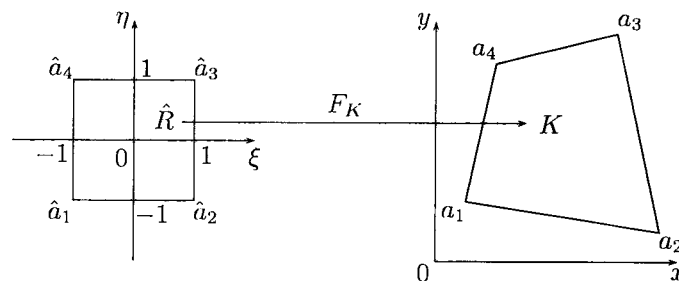


Fig. 1. Illustration of quadrilateral K and reference square \hat{R} .

With the above notations, we construct the following velocity-pressure finite element spaces:

$$(9) \quad \begin{aligned} \tilde{\mathbf{V}}_h &= \{\mathbf{v}(\mathbf{x}) = \hat{\mathbf{v}}(F_K^{-1}(\mathbf{x})), \hat{\mathbf{v}} \in \mathcal{NCQ}_1(\hat{R}), K \in \mathcal{T}_h\}, \\ \mathbf{V}_h &= \{\mathbf{v} \in \tilde{\mathbf{V}}_h \mid \mathbf{v}(c_j) = \mathbf{0}, \text{ for all midpoints } c_j \text{ on } \Gamma\}, \\ P_h &= \{p \in L_0^2(\Omega) \mid p \in \mathcal{P}_0, K \in \mathcal{T}_h\}, \end{aligned}$$

where

$$\mathcal{NCQ}_1(\hat{R}) = \text{Span}\{\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3, \hat{\varphi}_4\}^2.$$

From the definition of the space \mathbf{V}_h , any $\mathbf{v} \in \mathbf{V}_h$ has midpoints continuity between adjacent quadrilateral elements. Using the orthogonal properties for scalar function [1], that is, $\langle \mathbf{1}, w_j - w_k \rangle_{\Gamma_{jk}} = 0$, $\langle \mathbf{1}, w_j \rangle_{\Gamma_j} = 0$, we can derive the following orthogonal properties for vector functions,

$$(10) \quad \langle \mathbf{1}, \mathbf{w}_j - \mathbf{w}_k \rangle_{\Gamma_{jk}} = \mathbf{0}, \quad \mathbf{w} \in \mathbf{V}_h,$$

$$(11) \quad \langle \mathbf{1}, \mathbf{w}_j \rangle_{\Gamma_j} = \mathbf{0}, \quad \mathbf{w} \in \mathbf{V}_h,$$

which guarantees optimal order error estimates. For convenience in the analysis below, let

$$\begin{aligned} \Lambda_h &= \{\lambda \mid \lambda_{jk} = \text{tr}_{\Gamma_{jk}}(\lambda|_{\Omega_j}) \in \mathcal{P}_0(\Gamma_{jk}); \\ &\quad \lambda_{jk} + \lambda_{kj} = 0; \lambda_j = \text{tr}_{\Gamma_j}(\lambda|_{\Omega_j}) \in \mathcal{P}_0(\Gamma_j)\}. \end{aligned}$$

Define projections Π, P_0 and I by

$$(12) \quad \begin{aligned} \Pi : H^2(\Omega)^2 &\rightarrow \mathbf{V}_h : (\mathbf{v} - \Pi\mathbf{v})(c) = \mathbf{0}, \quad c = c_{jk} \text{ or } c_j; \\ P_0 : H^2(\Omega)^2 &\rightarrow \Lambda_h^2 : \langle \nabla \mathbf{v}_j \cdot \nu_j - P_0 \mathbf{v}_j, \mathbf{z} \rangle_{\Gamma} = \mathbf{0}, \quad \mathbf{z} \in \mathcal{P}_0(\Gamma)^2, \\ &\quad \Gamma = \Gamma_{jk} \text{ or } \Gamma_j; \\ I : L^2(\Omega) &\rightarrow P_h : (p - Ip, q_h)_{\Omega} = 0 \quad \forall q_h \in P_h, \\ &\quad \text{i.e., } Ip \text{ be the mean value of } p \text{ in each } K \in \mathcal{T}_h. \end{aligned}$$

Since Ip reproduces constant functions on elements we obtain the following:

$$(13) \quad \|p - Ip\|_{0,\Omega} \leq Ch \|p\|_{1,\Omega}, \quad p \in H^1(\Omega).$$

Moreover, since Π reproduces linear functions on elements and P_0 reproduces constants on faces, it follows from standard polynomial approximation results that

$$\begin{aligned}
 (14) \quad & \left(\sum_j \|\mathbf{v} - \Pi\mathbf{v}\|_{0,j}^2 \right)^{\frac{1}{2}} + h \left(\sum_j \|\mathbf{v} - \Pi\mathbf{v}\|_{1,j}^2 \right)^{\frac{1}{2}} \\
 & + h^2 \left(\sum_j \|\mathbf{v} - \Pi\mathbf{v}\|_{2,j}^2 \right)^{\frac{1}{2}} + h^{\frac{1}{2}} \left(\sum_j |\mathbf{v} - \Pi\mathbf{v}|_{0,j}^2 \right)^{\frac{1}{2}} \\
 & + h^{\frac{3}{2}} \left(\sum_j \left| \frac{\partial}{\partial \nu_j} (\mathbf{v} - \Pi\mathbf{v}) \right|_{0,j}^2 \right)^{\frac{1}{2}} + h^{\frac{3}{2}} \left(\sum_j \left| \frac{\partial \mathbf{v}}{\partial \nu_j} - P_0 \mathbf{v} \right|_{0,j}^2 \right)^{\frac{1}{2}} \\
 & \leq C h^2 \|\mathbf{v}\|_2, \quad \mathbf{v} \in H^2(\Omega)^2,
 \end{aligned}$$

where $\|\mathbf{z}\|_{m,j}^2 = \|\mathbf{z}\|_{H^m(\Omega_j)}^2$, $|\mathbf{z}|_{m,j}^2 = \sum_k \|\mathbf{z}\|_{H^m(\Gamma_{jk})}^2$ with Γ_j replacing Γ_{jk} for boundary faces. (See [1, Section 6])

4. Stability results using macroelement technique

In this section we slightly modify the macroelement technique [2, 4, 5, 6] and obtain the “inf-sup” stability condition.

By a macroelement M we define a connected set of elements of which the intersection of any two is either empty, a vertex, or an edge. Two macroelements M and \tilde{M} are said to be equivalent if they can be mapped continuously onto each other [6]. For a macroelement M we define the spaces

$$\begin{aligned}
 \mathbf{V}_{0,M} &= \{ \mathbf{v} \in \mathbf{V}_h \mid \mathbf{v}(c_j) = \mathbf{0} \text{ for all midpoints } c_j \text{ on } \partial M, \\
 & \quad \mathbf{v} = \mathbf{0} \text{ in } \Omega \setminus M \}, \\
 P_M &= \{ p \in P_h \mid p = 0 \text{ in } \Omega \setminus M \}, \\
 N_M &= \{ p \in P_M \mid (\text{div}_h \mathbf{v}, p)_M = 0 \ \forall \mathbf{v} \in \mathbf{V}_{0,M} \}.
 \end{aligned}$$

THEOREM 1. Suppose that there is a fixed set of equivalent classes ε_i , $i = 1, 2, \dots, l$, of macroelements, a positive integer L , and a macroelement partitioning \mathcal{M}_h such that

- (M1) For each $M \in \varepsilon_i, i = 1, 2, \dots, l$, the space N_M is one-dimensional, consisting of functions that are constant on M ;
- (M2) Each $M \in \mathcal{M}_h$ belongs to one of the classes $\varepsilon_i, i = 1, 2, \dots, l$;
- (M3) Each $K \in \mathcal{T}_h$ is contained in at least one and not more than L macroelements of \mathcal{M}_h ;
- (M4) Each $e \in \Gamma_h$ is contained in the interior of at least one and not more than L macroelements of \mathcal{M}_h , where Γ_h denotes the collection of edges, of the elements of \mathcal{T}_h , in the interior of Ω .

Then the inf-sup condition (6) is valid.

We now show that the finite elements approximation scheme (3) with \mathbf{V}_h and P_h defined by (9) satisfy the inf-sup condition by using the macroelement technique.

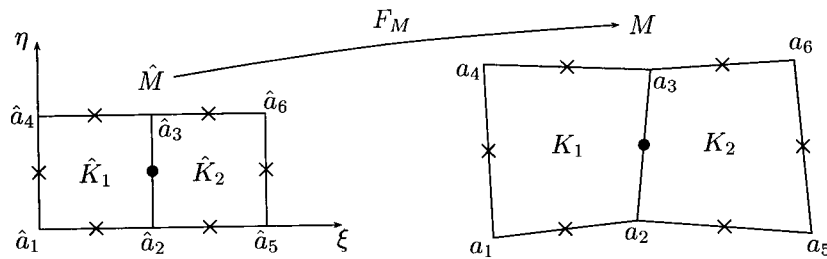


Fig. 2. Reference and arbitrary two elements macroelement.

THEOREM 2. For the Stokes problem (1), the velocity-pressure finite element formulation (3) satisfies the inf-sup stability condition (6).

Proof. Consider a macroelement consisting of two elements: $M = K_1 \cup K_2$ as in Fig. 2. By virtue of the macroelement technique, it is sufficient to show that

$$\begin{aligned}
 (15) \quad N_M &= \{w \in W_M | (\operatorname{div}_h \mathbf{v}, w)_M = 0, \mathbf{v} \in \mathbf{V}_{0,M}\} \\
 &= \{\text{constants on } M\}.
 \end{aligned}$$

By elementary calculation, we get

$$\begin{aligned}
& \int_M \operatorname{div}_h \mathbf{v} w \, dx \, dy = \int_{K_1 \cup K_2} \operatorname{div}_h \mathbf{v} w \, dx \, dy \\
&= \int_{\hat{R}} \left(\frac{\partial y}{\partial \eta} \frac{\partial u}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial u}{\partial \eta} \right) \alpha_1 \, d\xi \, d\eta + \int_{\hat{R}} \left(-\frac{\partial x}{\partial \eta} \frac{\partial v}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial v}{\partial \eta} \right) \alpha_2 \, d\xi \, d\eta \\
&= \int_{\hat{R}} \left\{ ((y_3 + y_4) - (y_1 + y_2) + \xi((y_1 + y_3) - (y_2 + y_4)) \times \frac{\partial u}{\partial \xi} \right. \\
&\quad \left. - ((y_2 + y_3) - (y_1 + y_4) + \eta((y_1 + y_3) - (y_2 + y_4)) \times \frac{\partial u}{\partial \eta} \right\} \alpha_1 \\
&+ \int_{\hat{R}} \left\{ -((x_3 + x_4) - (x_1 + x_2) + \xi((x_1 + x_3) - (x_2 + x_4)) \times \frac{\partial v}{\partial \xi} \right. \\
&\quad \left. + ((x_2 + x_3) - (x_1 + x_4) + \eta((x_1 + x_3) - (x_2 + x_4)) \times \frac{\partial v}{\partial \eta} \right\} \alpha_1 \\
&+ \int_{\hat{R}} \left\{ ((y_6 + y_3) - (y_2 + y_5) + \xi((y_2 + y_6) - (y_5 + y_3)) \times \frac{\partial u}{\partial \xi} \right. \\
&\quad \left. - ((y_5 + y_6) - (y_2 + y_3) + \eta((y_2 + y_6) - (y_5 + y_3)) \times \frac{\partial u}{\partial \eta} \right\} \alpha_2 \\
&+ \int_{\hat{R}} \left\{ -((x_6 + x_3) - (x_2 + x_5) + \xi((x_2 + x_6) - (x_5 + x_3)) \times \frac{\partial v}{\partial \xi} \right. \\
&\quad \left. + ((x_5 + x_6) - (x_2 + x_3) + \eta((x_2 + x_6) - (x_5 + x_3)) \times \frac{\partial v}{\partial \eta} \right\} \alpha_2.
\end{aligned}$$

Let φ be the function defined on M which has nodal value equal to one at the point denoted by \bullet and zeros at \times . (See Fig. 2.) By taking $\mathbf{v} = (u, v) = (\varphi, 0)$ we have

$$(16) \quad \alpha_1(x_2 - x_3) + \alpha_2(x_3 - x_2) = 0$$

and by taking $\mathbf{v} = (u, v) = (0, \varphi)$ we have

$$(17) \quad \alpha_1(y_2 - y_3) + \alpha_2(y_3 - y_2) = 0.$$

From these two equations we get

$$\alpha_1 \cdot \begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} x_3 - x_2 \\ y_2 - y_3 \end{pmatrix} = (\alpha_1 - \alpha_2) \begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \end{pmatrix} = \vec{0}.$$

This leads

$$(18) \quad \alpha_1 = \alpha_2.$$

From (18) we obtain (15). The theorem follows from Theorem 4.1

5. Error estimates

In this section we will estimate the energy and L^2 errors. At first by estimating the consistency error, we obtain optimal energy norm error. Next using the duality argument, we obtain optimal L^2 norm error. From now on we will use C denotes generic positive constants independent of the mesh parameter h and is different where it occurs. Now, let us estimate the consistency error given by the following:

$$\begin{aligned} \sum_j \left\langle \frac{\partial \mathbf{u}}{\partial \nu_j} - p\nu_j, \mathbf{w}_j \right\rangle_{\partial\Omega_j} &= \sum_j \left\langle \frac{\partial \mathbf{u}}{\partial \nu_j} - p\nu_j, \mathbf{w}_j \right\rangle_{\Gamma_j} \\ &\quad + \sum_j \left\langle \frac{\partial \mathbf{u}}{\partial \nu_j} - p\nu_j, \mathbf{w}_j \right\rangle_{\partial\Omega_j \setminus \Gamma_j}. \end{aligned}$$

From the definition of projection P_0 and the orthogonal properties (10), (11), the following orthogonalities hold:

$$\begin{aligned} \langle P_0 \mathbf{u}_j, \mathbf{w}_j \rangle_{\Gamma_{jk}} + \langle P_0 \mathbf{u}_k, \mathbf{w}_k \rangle_{\Gamma_{kj}} &= \langle P_0 \mathbf{u}_j, \mathbf{w}_j - \mathbf{w}_k \rangle_{\Gamma_{jk}} = \mathbf{0}, \quad \mathbf{w} \in \mathbf{V}_h, \\ \left\langle \frac{\partial \mathbf{u}_j}{\partial \nu_j} - P_0 \mathbf{u}_j, \mathbf{1} \right\rangle_{\Gamma} &= \mathbf{0}, \quad \Gamma = \Gamma_j \text{ or } \Gamma_{jk}, \\ \langle \mathbf{1}, \mathbf{w}_j \rangle_{\Gamma_j} &= \mathbf{0}, \quad \mathbf{w} \in \mathbf{V}_h. \end{aligned}$$

We will first estimate the term:

$$\begin{aligned} \sum_j \left\langle \frac{\partial \mathbf{u}}{\partial \nu_j} - p\nu_j, \mathbf{w}_j \right\rangle_{\partial\Omega_j \setminus \Gamma_j} &= \sum_j \left\langle \frac{\partial \mathbf{u}}{\partial \nu_j} - P_0 \mathbf{u}_j, \mathbf{w}_j - \mathbf{m}_j \right\rangle_{\partial\Omega_j \setminus \Gamma_j} \\ &\quad - \sum_j \langle (p - \bar{p})\nu_j, \mathbf{w}_j - \mathbf{w}_k \rangle_{\Gamma_{jk}}. \end{aligned}$$

By (14), a standard trace theorem, and certain approximation of \mathbf{w}_j , p by properly chosen constants \mathbf{m}_j , \bar{p} (its average over Ω_j),

$$\begin{aligned} &\sum_j \left\langle \frac{\partial \mathbf{u}}{\partial \nu_j} - P_0 \mathbf{u}_j, \mathbf{w}_j - \mathbf{m}_j \right\rangle_{\partial\Omega_j \setminus \Gamma_j} \\ (19) \quad &\leq C \|\mathbf{u}\|_2 h^{\frac{1}{2}} \cdot \left(\sum_j \|\mathbf{w}_j - \mathbf{m}_j\|_j \|\nabla(\mathbf{w}_j - \mathbf{m}_j)\|_j \right)^{\frac{1}{2}} \\ &\leq C \|\mathbf{u}\|_2 h^1 \left(\sum_j \|\nabla \mathbf{w}_j\|_j^2 \right)^{\frac{1}{2}} \leq C h \|\mathbf{u}\|_2 \|\mathbf{w}\|_{1,h}. \end{aligned}$$

Since $\mathbf{w}_j - \mathbf{w}_k$ vanishes at the midpoints of Γ_{jk} , we get

$$\sum_j \langle \mathbf{w}_j - \mathbf{w}_k, \mathbf{w}_j - \mathbf{w}_k \rangle_{\Gamma_{jk}} \leq C h \|\mathbf{w}\|_{1,h}^2,$$

and

$$(20) \quad \sum_j \langle (p - \bar{p})\nu_j, \mathbf{w}_j - \mathbf{w}_k \rangle_{\Gamma_{jk}} \leq C h \|p\|_1 \|\mathbf{w}\|_{1,h}.$$

Summing (19) and (20) we get

$$(21) \quad \sum_j \langle \frac{\partial \mathbf{u}}{\partial \nu_j} - p\nu_j, \mathbf{w}_j \rangle_{\partial\Omega_j \setminus \Gamma_j} \leq C h (\|\mathbf{u}\|_2 + \|p\|_1) \|\mathbf{w}\|_{1,h}.$$

Since $\mathbf{w}(c_j) = \mathbf{0}$ at the boundary midpoints, we get the following:

$$(22) \quad \langle \mathbf{w}, \mathbf{w} \rangle_{\Gamma_j} \leq C h (\nabla \mathbf{w}, \nabla \mathbf{w})_{\Omega_j}.$$

Then,

$$(23) \quad \begin{aligned} & \sum_j \langle \frac{\partial \mathbf{u}}{\partial \nu_j} - p\nu_j, \mathbf{w}_j \rangle_{\Gamma_j} \\ &= \sum_j \langle \frac{\partial \mathbf{u}}{\partial \nu_j} - P_0 \mathbf{u}_j, \mathbf{w}_j \rangle_{\Gamma_j} - \sum_j \langle (p - \bar{p})\nu_j, \mathbf{w}_j \rangle_{\Gamma_j} \\ &\leq C h (\|\mathbf{u}\|_2 + \|p\|_1) \|\mathbf{w}\|_{1,h}. \end{aligned}$$

Therefore, from (21) and (23) we get the following consistency error estimates,

$$(24) \quad \sum_j \langle \frac{\partial \mathbf{u}}{\partial \nu_j} - p\nu_j, \mathbf{w}_j \rangle_{\partial\Omega_j} \leq C h (\|\mathbf{u}\|_2 + \|p\|_1) \|\mathbf{w}\|_{1,h}.$$

THEOREM 3. Let (\mathbf{u}, p) be the solution of (2) and (\mathbf{u}_h, p_h) be the solution of (3). Then the following optimal error estimates holds:

$$(25) \quad \|\mathbf{u} - \mathbf{u}_h\|_{1,h} + \|p - p_h\|_0 \leq C h (\|\mathbf{u}\|_2 + \|p\|_1).$$

Proof. From the error estimates (7) and apply $\mathbf{v}_h = \Pi \mathbf{u}$ and $q_h = Ip$ and using approximation properties of (14), (13) and the consistency

estimates (24) we obtain:

$$\begin{aligned} & \|\mathbf{u} - \mathbf{u}_h\|_{1,h} + \|p - p_h\|_0 \\ & \leq C \left(\|\mathbf{u} - \Pi\mathbf{u}\|_1 + \|p - Ip\|_0 + \sup \frac{\sum_j \int_{\partial\Omega_j} (\frac{\partial\mathbf{u}}{\partial\nu_j} - p\nu_j) \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{1,h}} \right) \\ & \leq Ch (\|\mathbf{u}\|_2 + \|p\|_1). \end{aligned}$$

From now on we will estimate the L^2 error estimate using the duality arguments. Let $\eta = \mathbf{u} - \mathbf{u}_h$. Let $\varphi \in H_0^1(\Omega)^2$ and $\chi \in L_0^2(\Omega)$ be the unique solution of the auxiliary Stokes problem

$$(26) \quad \begin{aligned} a(\varphi, \mathbf{v}) + b(\chi, \mathbf{v}) &= (\eta, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^2, \\ b(q, \varphi) &= 0 \quad \forall q \in L_0^2(\Omega), \end{aligned}$$

which satisfies the following *a priori* estimate

$$(27) \quad \|\varphi\|_2 + \|\chi\|_1 \leq C \|\mathbf{u} - \mathbf{u}_h\|.$$

Using the above notation, there holds

$$\|\eta\|^2 = (\eta, -\Delta\varphi + \nabla\chi) = a_h(\eta, \varphi) + b_h(\chi, \eta) - \sum_j \langle \frac{\partial\varphi}{\partial\nu_j} - \chi\nu_j, \eta_j \rangle_{\partial\Omega_j}.$$

Using the error identity (4) and by taking $\mathbf{v}_h = \Pi\varphi$, $q_h = \chi_h$, we get

$$(28) \quad \begin{aligned} & a_h(\eta, \Pi\varphi) + b_h(p - p_h, \Pi\varphi) + b_h(\chi_h, \eta) \\ &= \sum_j \int_{\partial\Omega_j} (\frac{\partial\mathbf{u}}{\partial\nu_j} - p\nu_j) \cdot \Pi\varphi. \end{aligned}$$

Since η is in $L^2(\Omega)$ we have $\varphi \in H^2(\Omega)$, $\chi \in H^1(\Omega)$. From this and by (26), we get

$$(29) \quad b_h(p - p_h, \varphi) = 0, \quad \sum_j \langle \frac{\partial\mathbf{u}}{\partial\nu_j} - p\nu_j, \varphi_j \rangle_{\partial\Omega_j} = 0.$$

By (28) and (29),

$$\begin{aligned} \|\eta\|^2 &= a_h(\eta, \varphi - \Pi\varphi) + b_h(p - p_h, \varphi - \Pi\varphi) + b_h(\chi - \chi_h, \eta) \\ &\quad - \sum_j \langle \frac{\partial\varphi}{\partial\nu_j} - \chi\nu_j, \eta_j \rangle_{\partial\Omega_j} - \sum_j \langle \frac{\partial\mathbf{u}}{\partial\nu_j} - p\nu_j, (\varphi - \Pi\varphi)_j \rangle_{\partial\Omega_j}. \end{aligned}$$

Expand the fourth and fifth terms as following and using (22), (27) and (14), we get

$$\begin{aligned}
& \sum_j \langle \frac{\partial \varphi}{\partial \nu_j} - \chi \nu_j, \eta_j \rangle_{\partial \Omega_j} + \sum_j \langle \frac{\partial \mathbf{u}}{\partial \nu_j} - p \nu_j, (\varphi - \Pi \varphi)_j \rangle_{\partial \Omega_j} \\
= & \sum_j \langle (\frac{\partial \varphi}{\partial \nu_j} - P_0 \varphi) - (\chi - \bar{\chi}) \nu_j, \eta_j - \bar{\eta}_j \rangle_{\partial \Omega_j \setminus \Gamma_j} \\
& + \sum_j \langle (\frac{\partial \mathbf{u}}{\partial \nu_j} - P_0 \mathbf{u}) - (p - \bar{p}) \nu_j, (\varphi - \Pi \varphi)_j \rangle_{\partial \Omega_j \setminus \Gamma_j} \\
& + \sum_j \langle (\frac{\partial \varphi}{\partial \nu_j} - P_0 \varphi) - (\chi - \bar{\chi}) \nu_j, \eta_j \rangle_{\Gamma_j} \\
& + \sum_j \langle (\frac{\partial \mathbf{u}}{\partial \nu_j} - P_0 \mathbf{u}) - (p - \bar{p}) \nu_j, (\varphi - \Pi \varphi)_j \rangle_{\Gamma_j} \\
\leq & C h^2 (\|\mathbf{u}\|_2 + \|p\|_1) \|\eta\|.
\end{aligned}$$

Therefore, using the above result and (25), (27) and (14), we obtain

$$(30) \quad \|\mathbf{u} - \mathbf{u}_h\| \leq C h^2 (\|\mathbf{u}\|_2 + \|p\|_1).$$

THEOREM 4. Let (\mathbf{u}, p) be the solution of (2) and (\mathbf{u}_h, p_h) be the solution of (3). Then the following optimal L^2 error estimates holds:

$$(31) \quad \|\mathbf{u} - \mathbf{u}_h\| \leq C h^2 (\|\mathbf{u}\|_2 + \|p\|_1).$$

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