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A SUFFICIENT CONDITION FOR THE UNIQUENESS OF POSITIVE STEADY STATE TO A REACTION DIFFUSION SYSTEM

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ABSTRACT. In this paper, we concentrate on the uniqueness of the positive solution for the general elliptic system

$$\begin{cases} \Delta u + u(g_1(u) - g_2(v)) = 0 \\ \Delta v + v(h_1(u) - h_2(v)) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \quad \text{in } R^+ \times \Omega,$$

This system is the general model for the steady state of a competitive interacting system. The techniques used in this paper are upper-lower solutions, maximum principles and spectrum estimates. The arguments also rely on some detailed properties for the solution of logistic equations.

1. Introduction

A lot of research has been focused on reaction-diffusion equations modeling of various systems in mathematical biology, especially the elliptic steady states of competitive and predator-prey interacting processes with various boundary conditions.

In the earlier literature, investigations into mathematical biology models were concerned with studying those with homogeneous Neumann boundary conditions. From here on, the more important Dirichlet problems, which allow flux across the boundary, became the subject of study.(see [1], [2], [3], [7], [6], [9], [10]) While necessary and sufficient conditions for the existence of positive solutions to the steady states have been established for rather general types of systems(see [9], [10]), our knowledge about the uniqueness of positive solutions is limited to somewhat rather special systems, whose relative growth rates are linear;

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the results established are only for the following competition models (see [2], [3], [6], [7])

$$\begin{cases} \Delta u + u(a - bu - cv) = 0 \\ \Delta v + v(d - fv - eu) = 0 \end{cases} \text{ in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0.$$

The question in this paper concerns the uniqueness of positive coexistence when the relative growth rates are nonlinear, more precisely, the uniqueness of positive steady state of

$$\begin{cases} \Delta u + u(g_1(u) - g_2(v)) = 0 & \text{in } \Omega, \\ \Delta v + v(h_1(v) - h_2(u)) = 0 & \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, & \end{cases}$$

where $g_i's, h_i's$ are C^1 functions, Ω is a bounded domain in R^n and u, v are densities of two competitive species.

2. Preliminaries

In this section we will state some preliminary results which will be useful for our later arguments.

DEFINITION 2.1. (Super and sub solutions)

$$\begin{cases} \Delta u + f(x, u) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $f \in C^\alpha(\bar\Omega \times R)$ and Ω is a bounded domain in R^n.
(A) A function $\bar u \in C^{2,\alpha}(\bar\Omega)$ satisfying

$$\begin{cases} \Delta \bar u + f(x, \bar u) \le 0 \text{ in } \Omega, \\ \bar u|_{\partial\Omega} \ge 0 \end{cases}$$

is called an super solution to (1).
(B) A function $\underline{u} \in C^{2,\alpha}(\bar\Omega)$ satisfying

$$\begin{cases} \Delta \underline{u} + f(x, \underline{u}) \ge 0 \text{ in } \Omega, \\ \underline{u}|_{\partial\Omega} \le 0 \end{cases}$$

is called a sub solution to (1).

LEMMA 2.1. *Let $f(x, \xi) \in C^\alpha(\bar\Omega \times R)$ and let $\bar u, \underline{u} \in C^{2,\alpha}(\bar\Omega)$ be, respectively, super and sub solutions to (1) which satisfy $\underline{u}(x) \le \bar u(x), x \in \bar\Omega$. Then (1) has a solution $u \in C^{2,\alpha}(\bar\Omega)$ with $\underline{u}(x) \le u(x) \le \bar u(x), x \in \bar\Omega$.*

LEMMA 2.2. (The first eigenvalue)

$$\left\{ \begin{array}{l} -\Delta u + q(x)u = \lambda u \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, \end{array} \right. \tag{2}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain in R^n.

(A) *The first eigenvalue $\lambda_1(q)$, denoted by simply λ_1 when $q \equiv 0$, is simple with a positive eigenfunction.*

(B) *If $q_1(x) < q_2(x)$ for all $x \in \Omega$, then $\lambda_1(q_1) < \lambda_1(q_2)$.*

(C) *(Variational Characterization of the first eigenvalue)*

$$\lambda_1(q) = \min_{\phi \in W_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega (|\nabla \phi|^2 + q\phi^2)dx}{\int_\Omega \phi^2 dx}.$$

LEMMA 2.3. (Maximum Principles)

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u + \sum_{i=1}^{n} a_i(x) D_i u + a(x)u = f(x) \ \ in \ \ \Omega,$$

where Ω is a bounded domain in R^n.

(M1) $\partial\Omega \in C^{2,\alpha}(0 < \alpha < 1)$.

(M2) $|a_{ij}(x)|_\alpha, |a_i(x)|_\alpha, |a(x)|_\alpha \leq M(i,j = 1, ..., n)$.

(M3) *L is uniformly elliptic in Ω, with ellipticity constant γ, i.e., for every $x \in \bar{\Omega}$ and every real vector $\xi = (\xi_1, ..., \xi_n)$*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_{i=1}^{n} |\xi_i|^2.$$

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of $Lu \geq 0 (Lu \leq 0)$ in Ω.

(A) *If $a(x) \equiv 0$, then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u (\min_{\bar{\Omega}} u = \min_{\partial\Omega} u)$.*

(B) *If $a(x) \leq 0$, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+ (\min_{\bar{\Omega}} u \geq -\max_{\partial\Omega} u^-)$, where $u^+ = \max(u, 0), u^- = -\min(u, 0)$.*

(C) *If $a(x) \equiv 0$ and u attains its maximum (minimum) at an interior point of Ω, then u is identically a constant in Ω.*

(D) *If $a(x) \leq 0$ and u attains a nonnegative maximum (nonpositive minimum) at an interior point of Ω, then u is identically a constant in Ω.*

We also need some information on the solutions of the following logistic equations.

LEMMA 2.4. ([10])

$$\begin{cases} \Delta u + uf(u) = 0 \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$ and Ω is a bounded domain in R^n. If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary condition. We denote this unique positive solution as θ_f.

The main property about this positive solution is that θ_f is increasing as f is increasing.

Especially, for $a > \lambda_1$, we denote θ_a as the unique positive solution of

$$\begin{cases} \Delta u + u(a - u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0. \end{cases}$$

Hence, θ_a is increasing as $a > 0$ is increasing.

3. Uniqueness of steady state

We consider the elliptic system

$$\begin{cases} \left.\begin{array}{l} \Delta u + u(g_1(u) - g_2(v)) = 0 \\ \Delta v + v(h_1(v) - h_2(u)) = 0 \end{array}\right\} \text{ in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \tag{3}$$

Here Ω is a bounded, smooth domain in R^n and the functions $g_i's, h_i's$ satisfy

(U1) g_1, g_2, h_1 and h_2 are C^1 functions,
(U2) $g_1, h_1, -g_2, -h_2$ are strictly decreasing,
(U3) $g_2(0) = h_2(0) = 0$,
(U4) there are $c_0 > 0, c_1 > 0$ such that $g_1(u) < 0$ for $u > c_0$ and $h_1(v) < 0$ for $v > c_1$.

In 1991, Li and Logan ([10]) found some sufficient conditions to guarantee the existence of positive solution to (3). Here, we try to solve the question of uniqueness. The following is the main result:

THEOREM 3.1. *If*

(A) $g_1(0) - g_2(c_1) > \lambda_1, h_1(0) - h_2(c_0) > \lambda_1$ *and*

$$\text{(B)}\ 4\inf(-g_1')\inf(-h_1') \ge \sup(\frac{\theta_{g_1}}{\theta_{h_1-h_2(c_0)}})(\sup(g_2'))^2 + \sup(\frac{\theta_{h_1}}{\theta_{g_1-g_2(c_1)}})(\sup(h_2'))^2 + 2\sup(g_2')\sup(h_2'),$$

then (3) *has a unique coexistence state.*

Biologically, we can interpret the conditions in Theorem 3.1 as follows. The functions g_1, g_2, h_1, h_2 describe how species 1 (u) and 2 (v) interact among themselves and with each other. Hence, the both conditions (1) and (2) implies that species 1 interacts strongly among themselves and weakly with species 2. Similarly for species 2, they interact more strongly among themselves than they do with species 1. Especially, if we consider the linear case

$$\begin{cases} \Delta u + u(a - bu - cv) = 0 & \text{in } \Omega, \\ \Delta v + v(d - fv - eu) = 0 & \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases}$$

where a, b, c, d, e, f are positive constants, then $g_1(u) = a - bu, g_2(v) = cv, h_1(v) = d - fv$ and $h_2(u) = eu$ and so the condition (A) and (B) are reduced to

(A′) $\quad a - \frac{cd}{f} > \lambda_1, d - \frac{ae}{b} > \lambda_1,$

(B′) $\quad 4bf \ge \frac{c^2 f}{b}\sup(\frac{\theta_a}{\theta_{d-\frac{ac}{b}}}) + \frac{c^2 b}{f}\sup(\frac{\theta_d}{\theta_{a-\frac{cd}{f}}}) + 2ce.$

Proof. By the Maximum Principle, $\theta_{h_1} < c_1$. But, since g_2 is increasing, $g_2(\theta_{h_1}) < g_2(c_1)$, and so, $\lambda_1(\Delta + (g_1(0) - g_2(\theta_{h_1}))I) > \lambda_1(\Delta + (g_1(0) - g_2(c_1))I) > 0$, since $g_1(0) - g_2(c_1) > \lambda_1 > 0$. Similarly, we have $\lambda_1(\Delta + (h_1(0) - h_2(\theta_{g_1}))I) > 0$. Hence, by Theorem 1.1 in [10], (3) has a positive solution. We concentrate the uniqueness part. Suppose (u, v) is a positive solution to (3). Then

$$\begin{cases} \Delta u + ug_1(u) = ug_2(v) > 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Hence, u is a subsolution to

$$\begin{cases} \Delta z + zg_1(z) = 0 \quad \text{in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Any constant which is bigger than c_0 is a supersolution to

$$\begin{cases} \Delta z + zg_1(z) = 0 \quad \text{in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Therefore, by the super-sub solution method, we have

$$u \leq \theta_{g_1}. \tag{4}$$

The same argument shows

$$v \leq \theta_{h_1}. \tag{5}$$

Since $\theta_{h_1} < c_1$ by the Maximum Principle and g_2 is increasing, $g_2(v) \leq g_2(\theta_{h_1}) \leq g_2(c_1)$, we obtain

$$\begin{cases} \Delta u + u(g_1(u) - g_2(c_1)) \leq \Delta u + u(g_1(u) - g_2(v)) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Thus, u is a supersolution to

$$\begin{cases} \Delta z + z(g_1(z) - g_2(c_1)) = 0 \text{ in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Let ϕ_1 be the first eigenfunction of

$$\begin{cases} \Delta u + \lambda u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

If $\epsilon > 0$ is so small that $g_1(\epsilon\phi_1) - g_2(c_1) - \lambda_1 > 0$ on $\bar{\Omega}$ (Such ϵ exists, because $g_1(0) - g_2(c_1) > \lambda_1$ and g_1 is continuous.), then

$$\begin{aligned} \Delta\epsilon\phi_1 + \epsilon\phi_1(g_1(\epsilon\phi_1) - g_2(c_1)) &= \epsilon[\Delta\phi_1 + \phi_1(g_1(\epsilon\phi_1) - g_2(c_1))] \\ &> \epsilon(\Delta\phi_1 + \lambda_1\phi_1) \\ &= 0 \text{ in } \Omega, \end{aligned}$$

which implies that $\epsilon\phi_1$ is a subsolution to

$$\begin{cases} \Delta z + z(g_1(z) - g_2(c_1)) = 0 \text{ in } \Omega, \\ z|_{\partial\Omega} = 0. \end{cases}$$

Consequently,

$$\theta_{g_1 - g_2(c_1)} \leq u. \tag{6}$$

The same argument shows

$$\theta_{h_1 - h_2(c_0)} \leq v. \tag{7}$$

From (4)-(7), we get

$$\theta_{g_1 - g_2(c_1)} \leq u \leq \theta_{g_1}, \ \theta_{h_1 - h_2(c_0)} \leq v \leq \theta_{h_1}. \tag{8}$$

Consequently, for any positive solution (u, v) of (3), the inequalities (8) hold.

Now we are ready to prove the uniqueness.

Suppose (u_1, v_1) and (u_2, v_2) are positive solutions to (3). Let $p = u_1 - u_2$ and $q = v_1 - v_2$. Then

$$\begin{aligned}\Delta p + (g_1(u_1) - g_2(v_1))p &= \Delta u_1 - \Delta u_2 + (g_1(u_1) - g_2(v_1))(u_1 - u_2)\\ &= -\Delta u_2 - (g_1(u_1) - g_2(v_1))u_2\\ &= -\Delta u_2 - (g_1(u_2) - g_2(v_2) - g_1(u_2) + g_2(v_2)\\ &\quad +g_1(u_1) - g_2(v_1))u_2\\ &= -u_2(g_1(u_1) - g_1(u_2) + g_2(v_2) - g_2(v_1))\\ &\quad \text{in } \Omega.\end{aligned}$$

But, by the Mean Value Theorem, there is $\tilde{x}$ depending on u_1, u_2 such that $g_1(u_1) - g_1(u_2) = g_1'(\tilde{x})(-p)$. Hence,

$$\begin{aligned}\Delta p + (g_1(u_1) - g_2(v_1))p &= -u_2(g_1(u_1) - g_1(u_2) + g_2(v_2) - g_2(v_1))\\ &= -u_2[g_1'(\tilde{x})(-p) + g_2(v_2) - g_2(v_1)],\end{aligned}$$

i.e.,

(9) $\Delta p + (g_1(u_1) - g_2(v_1))p + u_2 p g_1'(\tilde{x}) - u_2(g_2(v_1) - g_2(v_2)) = 0$ in Ω.

The same argument shows that

(10) $\Delta q + (h_1(v_2) - h_2(u_2))q + v_1 q h_1'(\bar{x}) - v_1(h_2(u_1) - h_2(u_2)) = 0$ in Ω,

where $\bar{x}$ depends on v_1, v_2 by the Mean Value Theorem. Since $\lambda_1(\Delta + (g_1(u_1) - g_2(v_1))I) = 0$, by the Variational Characterization of the first eigenvalue,

$$\int_\Omega z(-\Delta z - (g_1(u_1) - g_2(v_1))z)dx \geq 0 \tag{11}$$

for any $z \in C^2(\bar{\Omega})$ and $z|_{\partial\Omega} = 0$. The same argument shows that

$$\int_\Omega w(-\Delta w - (h_1(v_2) - h_2(u_2))w)dx \geq 0 \tag{12}$$

for any $w \in C^2(\bar{\Omega})$ and $w|_{\partial\Omega} = 0$. From (9) and (10), we get

$$\begin{cases} -p\Delta p - (g_1(u_1) - g_2(v_1))p^2 - g_1'(\tilde{x})u_2 p^2 + u_2 p(g_2(v_1) - g_2(v_2)) = 0\\ -q\Delta q - (h_1(v_2) - h_2(u_2))q^2 - h_1'(\bar{x})v_1 q^2 + v_1 q(h_2(u_1) - h_2(u_2)) = 0\end{cases}$$

in Ω. Hence, from (11) and (12), we obtain

$$\begin{aligned}\int_\Omega & - g_1'(\tilde{x})u_2 p^2 + u_2 p(g_2(v_1) - g_2(v_2)) + v_1 q(h_2(u_1)\\ & - h_2(u_2)) - h_1'(\bar{x})v_1 q^2 dx \leq 0.\end{aligned}$$

By the Mean Value Theorem, for each $x \in \Omega$, there exist $\tilde{y}, \bar{y}$ such that

$$\begin{aligned} g_2(v_1) - g_2(v_2) &= g_2'(\tilde{y})(v_1 - v_2) = g_2'(\tilde{y})q, \\ h_2(u_1) - h_2(u_2) &= h_2'(\bar{y})(u_1 - u_2) = h_2'(\bar{y})p, \end{aligned}$$

which implies that

$$\int_\Omega -g_1'(\tilde{x})u_2p^2 + (u_2g_2'(\tilde{y}) + v_1h_2'(\bar{y}))pq - h_1'(\bar{x})v_1q^2dx \le 0.$$

Therefore, we find

$$\begin{aligned} p \equiv q \equiv 0 \quad \text{if} \quad & -g_1'(\tilde{x})u_2\zeta^2 + (u_2g_2'(\tilde{y}) + v_1h_2'(\bar{y}))\zeta\eta \\ & -h_1'(\bar{x})v_1\eta^2 \ \text{ is positive definite} \\ & \text{for each } x \in \Omega. \end{aligned}$$

This is the case if

$$u_2^2g_2'(\tilde{y})^2 + v_1^2h_2'(\bar{y})^2 + 2u_2v_1g_2'(\tilde{y})h_2'(\bar{y}) - 4g_1'(\tilde{x})h_1'(\bar{x})u_2v_1 \le 0$$

for each $x \in \Omega$, i.e.,

$$4g_1'(\tilde{x})h_1'(\bar{x}) \ge \frac{u_2}{v_1}g_2'(\tilde{y})^2 + 2g_2'(\tilde{y})h_2'(\bar{y}) + \frac{v_1}{u_2}h_2'(\bar{y})^2 \ \text{ for each } x \in \Omega.$$

But, from the inequality (8) and the hypothesis in the theorem,

$$\begin{aligned} \tfrac{u_2}{v_1}g_2'(\tilde{y})^2 \ + \ & 2g_2'(\tilde{y})h_2'(\bar{y}) \ + \ \tfrac{v_1}{u_2}h_2'(\bar{y})^2 \\ \le \ & \tfrac{\theta_{g_1}}{\theta_{h_1-h_2(c_0)}}[\sup(g_2')]^2 + \tfrac{\theta_{h_1}}{\theta_{g_1-g_2(c_1)}}[\sup(h_2')]^2 \\ & +2\sup(g_2')\sup(h_2') \\ \le \ & 4\inf(-g_1')\inf(-h_1') \\ \le \ & 4g_1'(\tilde{x})h_1'(\tilde{x}), \end{aligned}$$

and so $p \equiv q \equiv 0$. The uniqueness is established.

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References

[1] R. S. Cantrell and C. Cosner, *On the steady-state problem for the Volterra - Lotka competition model with diffusion*, Houston J. Math. **13** (1987), 337–352.

[2] ———, *On the uniqueness and stability of positive solutions in the Lotka-Volterra competition model with diffusion*, Houston J. Math. **15** (1989), 341–361.

[3] C. Cosner and A. C. Lazer, *Stable coexistence states in the Volterra-Lotka competition model with diffusion*, SIAM J. Appl. Math. **44** (1984), 1112–1132.

[4] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol II: Partial differential equations, Interscience Publishers, New York-London, 1962.
[5] D. Dunninger, *Lecture note for applied analysis in Michigan State University.*
[6] J. L.-Gomez and R. Pardo, *Existence and uniqueness for some competition models with diffusion*, C.R. Acad. Sci. Paris, Ser. I Math. **313** (1991), 933–938.
[7] C. Gui and Y. Lou, *Uniqueness and nonuniqueness of coexistence states in the Lotka-Volterra competition model*, Comm. Pure Appl. Math. **47** (1994), no. 12, 1571–1594.
[8] P. Hess, *On uniqueness of positive solutions of nonlinear elliptic boundary value problems*, Math. Z. **155** (1977), 17–18.
[9] A. Leung, *Equilibria and stabilities for competing-species, reaction-diffusion equations with Dirichlet boundary data*, J. Math. Anal. Appl. **73** (1980), 204–218.
[10] L. Li and R. Logan, *Positive solutions to general elliptic competition models*, Differential Integral Equations **4** (1991), 817–834.
[11] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1967.
[12] I. Stakgold and L. E. Payne, *Nonlinear problems in nuclear reactor analysis, in nonlinear problems in the physical sciences and biology*, Lecture Notes in Mathematics 322, Springer, Berlin, 1973, pp. 298–307.

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