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INEQUALITIES FOR VECTOR-VALUED MAXIMAL FUNCTIONS OVER LOCALLY COMPACT VILENKIN GROUPS

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ABSTRACT. In this paper, some inequalities for vector-valued maximal functions over locally compact Vienkin groups are obtained.

1. Introduction

On Euclidean space $\mathbb{R}^n$, Fefferman-Stein [2] obtained the following well-known inequalities for vector-valued maximal functions:

THEOREM ([2]). *Let $1 < r < \infty, f = \{f_k\}_{k=1}^{\infty}$ be a sequence of locally integrable functions on $\mathbb{R}^n$, $M(f) = \{M(f_k)\}_{k=1}^{\infty}$ and $|f(x)|_r = (\sum_{k=1}^{\infty} |f_k(x)|^r)^{1/r}$. Then*

(i) $\int_{\mathbb{R}^n} |M(f)(x)|_r^q dx \le C_{r,q} \int_{\mathbb{R}^n} |f(x)|_r^q dx$, *where $1 < q < \infty$ and $C_{r,q}$ is a constant which only depends on r and q;*

(ii) $|\{x \in \mathbb{R}^n : |M(f)(x)|r > \alpha\}| \le C_r \alpha^{-1} \int_{\mathbb{R}^n} |f(x)|_r dx$, *for any $\alpha > 0$, where C_r is a constant which only depends on r.*

In this paper, we will establish some similar inequalities over locally compact Vilenkin groups. Furthermore, the relative inequalities on Herz spaces are also considered. First, let us introduce some definitions and notations.

Throughout this paper, we will denote by G a locally compact Abelian group containing a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n=-\infty}^{\infty}$ such that $\cup_{n=-\infty}^{\infty} G_n = G$, $\cap_{n=-\infty}^{\infty} G_n = \{0\}$ and $\sup\{order(G_n/G_{n+1}) : n \in \mathbb{Z}\} := B < \infty$. Let Γ denote the dual

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group of G and for each $n \in \mathbb{Z}$, let $\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1$ for all $x \in G_n\}$. Then $\{\Gamma_n\}_{n=-\infty}^{\infty}$ is a strictly increasing sequence of open compact subgroups of such that $\cup_{n=-\infty}^{\infty}\Gamma_n = \Gamma$, $\cap_{n=-\infty}^{\infty}\Gamma_n = \{1\}$, and order $(\Gamma_{n+1}/\Gamma_n) = order(G_n/G_{n+1})$. We choose Haar measure dx (or $d\mu$) on G and $d\gamma$ on Γ so that $|G_0| = |\Gamma_0| = 1$, where $|A|$ denotes the Haar measure of a measurable subset A of G, or Γ. Then $|G_n|^{-1} = |\Gamma_n| := m_n$ for each $n \in \mathbb{Z}$. Since $2m_n \leq m_{n+1} \leq Bm_n$ for each $n \in \mathbb{Z}$, it follows that $\sum_{n=k}^{\infty}(m_n)^{-\alpha} \leq C(m_k)^{-\alpha}$ and $\sum_{n=-\infty}^{k}(m_n)^{\alpha} \leq C(m_k)^{\alpha}$ for any $\alpha > 0, k \in \mathbb{Z}$. If we define the function $d : G \times G \to \mathbb{R}$ by $d(x, y) = 0$ when $x - y = 0$ and $d(x, y) = (m_n)^{-1}$ when $x - y \in G_n \setminus G_{n+1}$, then d defines a metric on $G \times G$ and the topology on G induced by this metric is the same as the original topology on G. For $x \in G$, we set $|x| = d(x, 0)$. For more details about G, see [1], [3-6].

DEFINITION 1. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{n\in\mathbb{Z}} m_n \int_{x+G_n} |f(y)|dy.$$

M is of strong type $(L^p, L^p)(1 < p < \infty)$ and of weak type (L^1, L^1) (see [3]).

DEFINITION 2 ([1]). Let $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$.

(a) The homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(G)$ are defined by

$$\dot{K}_q^{\alpha,p}(G) = \{f : f \text{ is a measurable function on } G \text{ and } ||f||_{\dot{K}_q^{\alpha,p}(G)} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}(G)} = \Big\{ \sum_{l=-\infty}^{\infty} m_l^{-\alpha p} ||f\chi_{G_l \setminus G_{l+1}}||_{L^q(G)}^p \Big\}^{1/p}.$$

(b) The non-homogeneous Herz spaces $K_q^{\alpha,p}(G)$ are defined by

$$K_q^{\alpha,p}(G) = \{f : f \text{ is a measurable function on } G \text{ and } ||f||_{K_q^{\alpha,p}(G)} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}(G)} = \Big\{ ||f\chi_{G_0}||_{L^q(G)}^p + \sum_{l=-\infty}^{-1} m_l^{-\alpha p} ||f\chi_{G_l \setminus G_{l+1}}||_{L^q(G)}^p \Big\}^{1/p}.$$

DEFINITION 3 ([7]). Let $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$.

(a) A measurable function $f(x)$ on G is said to belong to the homogeneous weak Herz space $W\dot{K}_q^{\alpha,p}(G)$ if

$$||f||_{W\dot{K}_q^{\alpha,p}(G)} = \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} m_k^{-\alpha p}[D_k(\lambda, f)]^{p/q} \Big\}^{1/p} < \infty,$$

where $D_k(\lambda, f) = |\{x \in G_k \setminus G_{k+1} : |f(x)| > \lambda\}|$, and the usual modification is made when $p = \infty$.

(b) A measurable function $f(x)$ on G is said to belong to the non-homogeneous weak Herz space $WK_q^{\alpha,p}(G)$ if

$$||f||_{WK_q^{\alpha,p}(G)} = \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} m_k^{-\alpha p}[\tilde{D}_k(\lambda, f)]^{p/q} \Big\}^{1/p} < \infty,$$

where for $k \in \{-1, -2, \cdots\}$, $\tilde{D}_k(\lambda, f) = D_k(\lambda, f)$, $\tilde{D}_0(\lambda, f) = |\{x \in G_0 : |f(x)| > \lambda\}|$, and the usual modification is made when $p = \infty$.

2. Inequalities for vector-valued maximal functions on Lebesgue spaces

THEOREM 1. *Let $1 < r < \infty, f = \{f_k\}_{k=1}^{\infty}$ be a sequence of locally integrable functions on G, $M(f) = \{M(f_k)\}_{k=1}^{\infty}$ and $|f(x)|_r = (\sum_{k=1}^{\infty} |f_k(x)|^r)^{1/r}$. Then*

$$\int_G |M(f)(x)|_r^q dx \leq C_{r,q} \int_G |f(x)|_r^q dx, \tag{1}$$

where $1 < q < \infty$, and $C_{r,q}$ is a constant only depending on r and q; and

$$|\{x \in G : |M(f)(x)|_r > \alpha\}| \leq C_r \alpha^{-1} \int_G |f(x)|_r dx, \tag{2}$$

where C_r is a constant only depending on r.

Proof. We consider three cases: $q = r$, $q < r$ and $q > r$.

Case I: $q = r$. The L^r-boundedness of M immediately implies inequality (1). For $\alpha > 0$ and the function $|f(x)|_r = (\sum_{k=1}^{\infty} |f_k(x)|^r)^{1/r}$, applying the Calderón-Zygmund decomposition theorem [6, Lemma 2], we obtain a collection $\{I_j : I_j = y_i + G_{n(j)}, y_j \in G, n(j) \in \mathbb{Z}\}$ of pairwise disjoint sets in G satisfying the following properties: (a) $\sum_j |I_j| \leq$

$\frac{C}{\alpha}\| \, |f|_r\|_1$; (b) $|f(x)|_r \le \alpha$ if $x \notin \Omega = \bigcup_j I_j$; (c) $\alpha \le \frac{1}{|I_j|}\int_{I_j} |f(y)|_r dy \le C\alpha$ for each I_j. Then we can decompose f_k as $f_k = f_k' + f_k''$, where $f_k' = f_k \cdot \chi_{G\setminus\Omega}$ and $f_k'' = f_k \cdot \chi_\Omega$. Since

$$\Big(\sum_{k=1}^{\infty} |Mf_k(x)|^r\Big)^{1/r} \le \Big(\sum_{k=1}^{\infty} |Mf_k'(x)|^r\Big)^{1/r} + \Big(\sum_{k=1}^{\infty} |Mf_k''(x)|^r\Big)^{1/r},$$

inequality (2) will be proved if we can show that

$$\Big|\Big\{x \in G : \Big(\sum_{k=1}^{\infty} |Mf_k'(x)|^r\Big)^{1/r} > \alpha\Big\}\Big| \le \frac{C}{\alpha}\| \, |f|_r\|_1 \tag{3}$$

and

$$\Big|\Big\{x \in G : \Big(\sum_{k=1}^{\infty} |Mf_k''(x)|^r\Big)^{1/r} > \alpha\Big\}\Big| \le \frac{C}{\alpha}\| \, |f|_r\|_1. \tag{4}$$

First, let us prove (3). From the inequality (b) and the obvious fact that

$$\Big\|\Big(\sum_{k=1}^{\infty} |f_k'(\cdot)|^r\Big)^{1/r}\Big\|_1 \le \| \, |f|_r\|_1,$$

we see

$$\Big\|\Big(\sum_{k=1}^{\infty} |f_k'(\cdot)|^r\Big)^{1/r}\Big\|_r^r \le \| \, |f|_r\|_1.$$

Therefore, by the easy case $p = r$ of inequality (1),

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^{\infty} |Mf_k'(\cdot)|^r\Big)^{1/r}\Big\|_r^r &\le C_r\Big\|\Big(\sum_{k=1}^{\infty} |f_k'(\cdot)|^r\Big)^{1/r}\Big\|_r^r \\ &\le C_r\alpha^{r-1}\| \, |f|_r\|_1.\end{aligned}$$

This immediately yields inequality (3) by the Chebyshev inequality.

To prove (4), we define the function $\tilde{f}_k$ by setting

$$\tilde{f}_k(x) = \begin{cases} \frac{1}{|I_j|}\int_{I_j} |f_k(y)|dy, & x \in I_j \\ 0, & x \notin \Omega. \end{cases}$$

For $x \in I_j$, by the vector-valued form of Minkowski's inequality and the inequality (c), we have

$$\Big(\sum_{k=1}^{\infty}|\tilde{f}_k(x)|^r\Big)^{1/r} = \Big(\sum_{k=1}^{\infty}\Big[\frac{1}{|I_j|}\int_{I_j}|f_k(y)|dy\Big]^r\Big)^{1/r}$$
$$\leq \frac{1}{|I_j|}\int_{I_j}\Big(\sum_{k=1}^{\infty}|f_k(y)|^r\Big)^{1/r}dy \leq C\alpha.$$

For $x \notin \Omega$, all $\tilde{f}_k$ are zero, so $\Big(\sum_{k=1}^{\infty}|\tilde{f}_k(x)|^r\Big)^{1/r} = 0$. Thus the function $\Big(\sum_{k=1}^{\infty}|\tilde{f}_k(\cdot)|^r\Big)^{1/r}$ supports in Ω and is bounded by A_α, which implies that

$$\Big\|\Big(\sum_{k=1}^{\infty}|\tilde{f}_k(\cdot)|^r\Big)^{1/r}\Big\|_r^r \leq C\alpha^r|\Omega| \leq C\alpha^{r-1}\|\,|f|_r\|_1.$$

As in the proof of (3), we now have that, by the case $p = r$ of (1) and the Chebyshev inequality,

$$\Big|\Big\{x \in G : \Big(\sum_{k=1}^{\infty}|M\tilde{f}_k(x)|^r\Big)^{1/r} > \alpha\Big\}\Big| \leq \frac{C_r}{\alpha}\|\,|f|_r\|_1. \tag{5}$$

For any set $I = y+G_n$, let $\tilde{I} = y+G_{n-2}$, and $\tilde{\Omega} = \bigcup_j \tilde{I}_j$. Obviously, $|\tilde{\Omega}| \leq C|\Omega| \leq \frac{C}{\alpha}\|\,|f|_r\|_1$. Therefore, to prove the weak type inequality (2), we only need to prove that $Mf_k''(x) \leq CM\tilde{f}_k(x)$ for any $x \notin \tilde{\Omega}$. In fact, $Mf_k''(x) \sim \sup_{x\in I}\Big\{\frac{1}{|I|}\int_I^k |f_k''(y)|dy\Big\}$, and for any fixed coset I containing x,

$$\frac{1}{|I|}\int_I |f_k''(y)|dy = \frac{1}{|I|}\sum_{j\in J}\int_{I_j\cap I}|f_k''(y)|dy,$$

where $J = \{j|I_j \cap I \neq \emptyset\}$. On the other hand, $I_j \cap I \neq \emptyset$ and $x \in I - \tilde{\Omega} \subseteq I - \tilde{I}_j$ imply that $I_j \subseteq \tilde{I}$. In fact, let $I_j = y_1 + G_{n_1}$, and $I = y_2 + G_{n_2}$. $I_j \cap I \neq \emptyset$ implies that $|y_1 - y_2| \leq m_{n_1}^{-1} + m_{n_2}^{-1}$ and $x \in I - \tilde{I}_j$ implies that $|x - y_1| > m_{n_1-2}^{-1} \geq 4m_{n_1}^{-1}$. Since $|x - y_1| \leq |x - y_2| + |y_1 - y_2| \leq 2m_{n_2}^{-1}$, we have $2m_{n_1}^{-1} \leq m_{n_2}^{-1}$. For any $y \in I_j$, $|y - y_2| \leq |y - y_1| + |y_1 - x| + |x - y_2| \leq 3m_{n_1}^{-1} + m_{n_2}^{-1} < 4m_{n_2}^{-1} \leq m_{n_2-2}^{-1}$ and thus $y \in y_2 + G_{n_2-2} = \tilde{I}$. It follows that $I_j \subseteq \tilde{I}$. Therefore,

$$\begin{aligned}\frac{1}{|I|}\sum_{j\in J}\int_{I_j\cap I}|f_k''(y)|dy &\le \frac{1}{|I|}\sum_{j\in J}\int_{I_j}|f_k''(y)|dy\\ &= \frac{1}{|I|}\sum_{j\in J}\int_{I_j}|\tilde{f}_k(y)|dy\\ &\le \frac{1}{|I|}\int_{\tilde{I}}|\tilde{f}_k(y)|dy\\ &\le \frac{C}{|\tilde{I}|}\int_{\tilde{I}}|\tilde{f}_k(y)|dy\\ &\le CM\tilde{f}_k(x).\end{aligned}$$

We conclude that for $x \in G - \tilde{\Omega}$, $\frac{1}{|I|}\int_I |f_k''(y)|dy \le CM\tilde{f}_k(x)$ holds for any I containing x, which implies that $Mf_k''(x) \le CM\tilde{f}_k(x)$. Thus, inequality (2) holds.

Case II: $q < r$. Inequality (1) is a simple consequence of the case $p = r$, the inequality (2) and the Marcinkiewicz interpolation theorem. Thus, to prove Theorem 1, we have only to demonstrate case $q > r$ in (1). To do this, first let us prove the following lemma.

LEMMA 1. *Let f and ϕ be positive real-valued functions on G. Then for $r > 1$,*

$$\int_G (Mf(x))^r\phi(x)dx \le C_r\int_G |f(x)|^r M\phi(x)dx$$

with C_r depending only on r.

Proof. Fix ϕ and consider the mapping $M : f \to Mf$. Clearly, M is bounded from $L^\infty(G, M\phi(x)dx)$ to $L^\infty(G, \phi(x)dx)$. If we can show that M is of weak type (1.1), then the lemma will follow immediately from the Marcinkiewicz interpolation theorem. Given $f(x)$ and $\alpha > 0$, the Calderón-Zygmund decomposition theorem tells us that $\{x \in G|Mf(x) > \alpha\} = \bigcup_j I_j$, where $\{I_j\}$ is a sequence of pairwise disjoint cosets satisfying the condition $\alpha \le \frac{1}{|I_j|}\int_{I_j}|f(x)|dx \le C\alpha$. Restricting attention to I_j for a moment, we see that

$$\begin{aligned}&\int_{I_j} f(x)M\phi(x)dx\\ &\geq \int_{I_j} f(x)\Big[\frac{1}{|I_j|}\int_{I_j}\phi(y)dy\Big]dx\\ &= \Big[\int_{I_j}\phi(y)dy\Big]\cdot\frac{1}{|I_j|}\int_{I_j} f(x)dx\\ &\leq C\alpha\cdot\int_{I_j}\phi(y)dy.\end{aligned}$$

Summing over j, we obtain

$$\begin{aligned}\alpha\int_{\{x\in G|Mf(x)>\alpha\}}\phi(y)dy &\leq C\int_{\{x\in G|Mf(x)>\alpha\}} f(x)M\phi(x)dx\\ &\leq C\int_G f(x)M\phi(x)dx.\end{aligned}$$

This finishes the proof of Lemma 1.

Let us now continue the proof of Theorem 1.

Case III: $q > r$. By Lemma 1, we have

$$\begin{aligned}&\int_G\Big(\sum_{k=1}^{\infty}(Mf(x))^r\Big)\phi(x)\\ &= \sum_{k=1}^{\infty}\int_G (Mf(x))^r\phi(x)dx\\ &\leq C_r\sum_{k=1}^{\infty}\int_G |f_k(x)|^r M\phi(x)dx\\ &= C_r\int_G\Big(\sum_{k=1}^{\infty}|f_k(x)|^r\Big)M\phi(x)dx.\end{aligned}\tag{6}$$

If in (6), letting ϕ take over the unit ball of $L^p(G)(1 < p < \infty)$, by duality theorem we obtain

$$\Big|\Big|\sum_{k=1}^{\infty}|Mf(\cdot)|^r\Big|\Big|_{p'} \leq C_{r,p}\Big|\Big|\sum_{k=1}^{\infty}|f_k(\cdot)|^r\Big|\Big|_{p'},$$

where $1/p + 1/p' = 1$. Let $p' = q/r > 1$, we then obtain

$$||\ |M(f)|_r||_q \leq C_{r,q}||\ |f|_r||_q.$$

This finishes the proof of Theorem 1. □

3. Inequalities for vector-valued maximal functions on Herz spaces

Recently, L. Tang and D. Yang studied the boundedness of vector-valued operators on Herz spaces of $\mathbb{R}^n$. Now, we consider a similar problem over locally compact Vilenkin groups.

THEOREM 2. *Let $1 < r < \infty$, $f = \{f_k\}_{k=1}^{\infty}$ be a sequence of locally integrable functions on G, $M(f) = \{M(f_k)\}_{k=1}^{\infty}$ and $|f(x)|_r = \left(\sum_{k=1}^{\infty} |f_k(x)|^r\right)^{1/r}$. Then*

(i) $\| \, |M(f)|_r \|_{\dot{K}_q^{\alpha,p}(G)} \le C \| \, |f|_r \|_{\dot{K}_q^{\alpha,p}(G)}$ *if* $-1/q < \alpha < 1 - 1/q$, $1 < q < \infty$ *and* $0 < p \le 1$;

(ii) $\| \, |M(f)|_r \|_{W\dot{K}_q^{1-1/q,p}(G)} \le C \| \, |f|_r \|_{\dot{K}_q^{1-1/q,p}(G)}$ *if* $1 \le q < \infty$ *and* $0 < p \le 1$.

Proof. Let us first prove (i). Suppose $|f|_r \in \dot{K}_q^{\alpha,p}(G)$. Write $\chi_i(x) = \chi_{G_i \setminus G_{i+1}}$, $C_i = G_i \setminus G_{i+1}$, $f_j(x) = \sum_{i=-\infty}^{\infty} f_j(x)\chi_i(x) = \sum_{i=-\infty}^{\infty} f_j^i(x)$, and $|f^i(x)|_r = \left(\sum_{j=1}^{\infty} |f_j^i(x)|^r\right)^{1/r}$. Then by Minkowski's inequality, we have

$$
\begin{aligned}
&\| \, |M(f)|_r \|_{\dot{K}_q^{\alpha,p}(G)} \\
&= \Big\{ \sum_{k=-\infty}^{\infty} m_k^{-\alpha p} \Big\| \chi_k \Big[\sum_{j=1}^{\infty} (M(f_j))^r \Big]^{1/r} \Big\|_{L^q(G)}^p \Big\}^{1/p} \\
&\le \Big\{ \sum_{k=-\infty}^{\infty} m_k^{-\alpha p} \Big\| \chi_k \Big[\sum_{j=1}^{\infty} \Big(\sum_{i=-\infty}^{\infty} M(f_j^i) \Big)^r \Big]^{1/r} \Big\|_{L^q(G)}^p \Big\}^{1/p} \\
&\le C \Big\{ \sum_{k=-\infty}^{\infty} m_k^{-\alpha p} \Big\| \chi_k \Big(\sum_{i=-\infty}^{\infty} \Big[\sum_{j=1}^{\infty} M^r(f_j^i) \Big]^{1/r} \Big) \Big\|_{L^q(G)}^p \Big\}^{1/p} \\
&\le C \Big\{ \sum_{k=-\infty}^{\infty} m_k^{-\alpha p} \Big\| \chi_k \Big(\sum_{i=k+1}^{\infty} \Big[\sum_{j=1}^{\infty} M^r(f_j^i) \Big]^{1/r} \Big) \Big\|_{L^q(G)}^p \Big\}^{1/p}
\end{aligned}
$$

$$+C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\Big\|\chi_k\Big[\sum_{j=1}^{\infty} M^r(f_j^k)\Big]^{1/r}\Big\|_{L^q(G)}^p\Big\}^{1/p}$$

$$+C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\Big\|\chi_k\Big(\sum_{i=-\infty}^{k-1}\Big[\sum_{j=1}^{\infty} M^r(f_j^i)\Big]^{1/r}\Big)\Big\|_{L^q(G)}^p\Big\}^{1/p}$$

$$= E_1 + E_2 + E_3,$$

where we denote $[M(f)]^r$ by $M^r(f)$ for simplicity.

For E_2, using (1) of Theorem 1, we have

$$\begin{aligned} E_2 &\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\Big\|\Big[\sum_{j=1}^{\infty} M_r(f_j^k)\Big]^{1/r}\Big\|_{L^q(G)}^p\Big\}^{1/p} \\ &\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\Big\|\Big[\sum_{j=1}^{\infty} (f_j^k)^r\Big]^{1/r}\Big\|_{L^q(G)}^p\Big\}^{1/p} \\ &= C\|\,|f|_r\|_{\dot K_q^{\alpha,p}(G)}. \end{aligned}$$

For E_1, noting that $i \ge k+1$, $x \in C_k$, $1 < r < \infty$, by the Minkowski inequality, we have

$$\begin{aligned} \Big[\sum_{j=1}^{\infty} M^r(f_j^i)\Big]^{1/r} &\le C\Big[\sum_{j=1}^{\infty}\Big(M^r\int_G |f_j^i(y)|dy\Big)^r\Big]^{1/r} \\ &\le Cm_k\int_G\Big(\sum_{j=1}^{\infty}|f_j^i(y)|^r\Big)^{1/r}dy. \end{aligned}$$

Therefore, on the condition of $\alpha < 1 - 1/q$ and $0 < p \le 1$, we have

$$\begin{aligned} &E_1 \\ &\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\Big[\sum_{i=k+1}^{\infty} m_k^{1-1/q}\int_G\Big(\sum_{j=1}^{\infty}|f_j^i(y)|^r\Big)^{1/r}dy\Big]^p\Big\}^{1/p} \\ &\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\Big[\sum_{i=k+1}^{\infty} m_k^{1-1/q}\|\,|f^i|_r\|_{L^q(G)}\Big(\int_{G_i\setminus G_{i+1}} dx\Big)^{1/q^1}\Big]^p\Big\}^{1/p} \\ &\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\Big[\sum_{i=k+1}^{\infty} m_k^{1-1/q} m_i^{1/q-1}\|\,|f^i|_r\|_{L^q}\Big]^p\Big\}^{1/p} \end{aligned}$$

$$
\begin{aligned}
&\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\sum_{i=k+1}^{\infty} m_k^{(1-1/q)p} m_i^{(1/q-1)p}\|\,|f^i|_r\|_{L^q}^p\Big\}^{1/p}\\
&\le C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{i-1} m_k^{(1-1/q)p} m_i^{(1/q-1)p}\|\,|f^i|_r\|_{L^q}^p\Big)\Big\}^{1/p}\\
&\le C\Big\{\sum_{k=-\infty}^{\infty} m_i^{-\alpha p}\|\,|f^i|_r\|_{L^q}^p\Big\}^{1/p}\\
&= C\|\,|f|_r\|_{\dot K_q^{\alpha,p}(G)},
\end{aligned}
$$

where $1/q + 1/q' = 1$ and when $0 < p \le 1$ we used the well-known inequality: $\Big(\sum\limits_{i=1}^{\infty}|\alpha_i|\Big)^p \le \sum\limits_{i=1}^{\infty}|\alpha_i|^p$.

For E_3, similar to E_1, when $i \le k-1$, $x \in C_k$, and $1 < r < \infty$, by the Minkowski inequality, we have

$$
\begin{aligned}
\Big[\sum_{j=1}^{\infty} M^r(f_j^i)(x)\Big]^{1/r} &\le C\Big[\sum_{j=1}^{\infty}\Big(M_i\int_G |f_j^i(y)|dy\Big)^r\Big]^{1/r}\\
&\le Cm_i\int_G\Big(\sum_{j=1}^{\infty}|f_j^i(y)|^r\Big)^{1/r}dy.
\end{aligned}
$$

Therefore, noting that $\alpha + 1/q > 0$ and $0 < p \le 1$, we have

$$
\begin{aligned}
&E_1\\
&\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\Big[\sum_{i=-\infty}^{k-1} m_i m_k^{1-1/q}\int_G\Big(\sum_{j=1}^{\infty}|f_j^i(y)|^r\Big)^{1/r}dy\Big]^p\Big\}^{1/p}\\
&\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\Big[\sum_{i=-\infty}^{k-1} m_i^{1-1/q'} m_k^{-1/q}\|\,|f^i|_r\|_{L^q(G)}\Big]^p\Big\}^{1/p}\\
&\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{-\alpha p}\sum_{i=-\infty}^{k-1} m_i^{p/q} m_k^{-p/q}\|\,|f^i|_r\|_{L^q(G)}^p\Big\}^{1/p}\\
&\le C\Big\{\sum_{i=-\infty}^{\infty} m_i^{p/q}\|\,|f^i|_r\|_{L^q(G)}^p\sum_{k=i+1}^{\infty} m_k^{-(\alpha+1/q)p}\Big\}^{1/p}\\
&\le C\Big\{\sum_{i=-\infty}^{\infty} m_i^{-\alpha p}\|\,|f^i|_r\|_{L^q}^p\Big\}^{1/p}\\
&= C\|\,|f|_r\|_{\dot K_q^{\alpha,p}(G)}.
\end{aligned}
$$

(i) is proved.

Now we turn to prove (ii). For $f \in \dot{K}_q^{1-1/q,p}(G)$, we write

$$
\begin{aligned}
&\Big\|\Big[\sum_{j=1}^{\infty} M^r(f_j)\Big]^{1/r}\Big\|_{W\dot{K}_q^{1-1/q,p}(G)} \\
&= \sup_{\lambda} \lambda\Big\{\sum_{k=-\infty}^{\infty} m_k^{(1/q-1)p}\Big|\Big\{x\in C_k : \Big[\sum_{j=1}^{\infty} M^r(f_j)(x)\Big]^{1/r} > \lambda\Big\}\Big|^{p/q}\Big\}^{1/p} \\
&\le C\sup_{\lambda>0}\lambda\Big\{\sum_{k=-\infty}^{\infty} m_k^{(1/q-1)p}\Big| \\
&\qquad \Big\{x\in C_k : \Big[\sum_{j=1}^{\infty} M^r\Big(\sum_{i=k+1}^{\infty} f_j^i\Big)(x)\Big]^{1/r} > \frac{\lambda}{3}\Big\}\Big|^{p/q}\Big\}^{1/p} \\
&\quad + C\sup_{\lambda>0}\lambda\Big\{\sum_{k=-\infty}^{\infty} m_k^{(1/q-1)p}\Big| \\
&\qquad \Big\{x\in C_k : \Big[\sum_{j=1}^{\infty} M^r(f_j^k)(x)\Big]^{1/r} > \frac{\lambda}{3}\Big\}\Big|^{p/q}\Big\}^{1/p} \\
&\quad + C\sup_{\lambda>0}\lambda\Big\{\sum_{k=-\infty}^{\infty} m_k^{(1/q-1)p}\Big| \\
&\qquad \Big\{x\in C_k : \Big[\sum_{j=1}^{\infty} M^r\Big(\sum_{i=-\infty}^{k-1} f_j^i\Big)(x)\Big]^{1/r} > \frac{\lambda}{3}\Big\}\Big|^{p/q}\Big\}^{1/p} \\
&:= F_1 + F_2 + F_3.
\end{aligned}
$$

For F_2, by Theorem 1, we have

$$
\begin{aligned}
F_2 &\le C\Big\{\sum_{k=-\infty}^{\infty} m_k^{(1/q-1)p}\Big\|\Big(\sum_{j=1}^{\infty}|f_j^k|^r\Big)^{1/r}\Big\|_{L^q}^p\Big\}^{1/p} \\
&\le C\|\,|f|_r\|_{\dot{K}_q^{1-1/q,p}(G)}.
\end{aligned}
$$

For F_1, noting that $i \geq k+1$, $x \in C_k$ and $0 < p \leq 1$, by Minkowski's inequality, we have

$$
\begin{aligned}
&\Big[\sum_{j=1}^{\infty} M^r\Big(\sum_{i=k+1}^{\infty} f_j^i\Big)(x)\Big]^{1/r} \\
\leq\ & \Big[\sum_{j=1}^{\infty}\Big(\sum_{i=k+1}^{\infty} M(f_j^i)(x)\Big)^r\Big]^{1/r} \\
\leq\ & \sum_{i=k+1}^{\infty}\Big[\sum_{j=1}^{\infty} M^r(f_j^i)(x)\Big]^{1/r} \\
\leq\ & C\sum_{i=k+1}^{\infty}\Big[\sum_{j=1}^{\infty}(m_k\int_G |f_j^i(y)|dy)^r\Big]^{1/r} \\
\leq\ & C\sum_{i=k+1}^{\infty} m_k\int_G |f^i(y)|_r dy \\
\leq\ & C\sum_{i=k+1}^{\infty} m_k\Big(\int_{G_i}|f^i(y)|_r^q dy\Big)^{1/q}\Big(\int_{G_i} dy\Big)^{1/q'} \\
\leq\ & C\sum_{i=k+1}^{\infty} m_k m_i^{-1/q'}\|\,|f^i|_r\|_{L^q(G)} \\
\leq\ & Cm_k\Big\{\sum_{i=k+1}^{\infty} m_i^{(1/q-1)p}\|\,|f^i|_r\|_{L^q(G)}^p\Big\}^{1/p} \\
\leq\ & Cm_k\|\,|f|_r\|_{\dot{K}_q^{1-1/q,p}(G)}.
\end{aligned}
$$

For F_3, similar to F_1, we have

$$
\begin{aligned}
&\Big[\sum_{j=1}^{\infty} M^r\Big(\sum_{i=-\infty}^{k-1} f_j^i\Big)(x)\Big]^{1/r} \\
\leq\ & C\sum_{i=-\infty}^{k-1} m_i m_i^{(1/q-1)}\|\,|f^i|_r\|_{L^q(G)} \\
\leq\ & Cm_k\sum_{i=-\infty}^{k-1} m_i^{(1/q-1)}\|\,|f^i|_r\|_{L^q(G)} \\
\leq\ & Cm_k\|\,|f|_r\|_{\dot{K}_q^{1-1/q,p}(G)}.
\end{aligned}
$$

Thus, for any fixed $\lambda > 0$, if

$$\left|\left\{x \in C_k : \left[M^r\Big(\sum_{i=k+1}^{\infty} f_j^i\Big)(x)\right]^{1/r} > \frac{\lambda}{3}\right\}\right| \neq 0$$

or

$$\left|\left\{x \in C_k : \left[M^r\Big(\sum_{i=-\infty}^{k-1} f_j^i\Big)(x)\right]^{1/r} > \frac{\lambda}{3}\right\}\right| \neq 0,$$

we have $\lambda/3 \le Cm_k\| \, |f|_r\|_{\dot{K}_q^{1-1/q,p}(G)}$, and we can find a minimal integer k_λ such that

$$m_{k_\lambda}^{-1} \le C\lambda^{-1}\| \, |f|_r\|_{\dot{K}_q^{1-1/q,p}(G)}.$$

Therefore when $\gamma = 1, 3$, we obtain

$$\begin{aligned} F_\gamma &\le C\sup_{\lambda>0}\lambda\Big\{\sum_{k=k_\lambda}^{\infty} m_k^{(1/q-1)p} m_k^{-p/q}\Big\}^{1/p} \\ &\le C\sup_{\lambda>0}\lambda m_{k\lambda}^{-1} \le C\| \, |f|_r\|_{\dot{K}_q^{1-1/q,p}(G)}. \end{aligned}$$

This finishes the proof of Theorem 2. □

REMARK. Theorem 2 also holds if we replace homogeneous Herz spaces by non-homogeneous Herz spaces. Since the proof is similar, we omit the details.

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