ON A CONNECTION ON A
HYPERCONTACT MANIFOLD

HYUNSUK KIM

ABSTRACT. We construct the canonical connection associated with a hypercontact structure. Moreover, we discuss the canonical connection associated with a sub-Riemannian 3-structure. As an application, we study the sub-symmetry property in terms of the canonical connection.

1. Introduction

Several connections on contact structures have been studied by many geometers ([4], [5], [7]). Recently, Falbel-Gorodski ([2]) defined a connection on a contact structure in the sense of sub-Riemannian geometry, which can be considered as a generalization of the generalized Tanaka connection ([5]) on a contact Riemannian structure.

On the other hand, Geiges-Thomas ([3]) introduced a notion of a hypercontact structure as a quaternionic analogue of contact Riemannian structure.

In this paper, we shall construct a new connection on a hypercontact manifold from the viewpoint of foliated structure. That is, if \((\phi_\alpha, \xi, \eta_\alpha)_{\alpha=1,2,3}\) is an almost contact 3-structure compatible with a hypercontact structure, the foliation is defined by vector fields \(\{\xi_1, \xi_2, \xi_3\}\) which generate a Lie algebra locally isomorphic to \(so(3)\).

In Section 2, we give a brief review of several known connections on a contact Riemannian structure. In Section 3, we define a new connection \(D\) on a hypercontact manifold by a similar way as in [5]. In Section 4, we discuss a canonical connection associated with a sub-Riemannian 3-structure. As an application, we study the sub-symmetry property in terms of the canonical connection.

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2. The case of a contact Riemannian structure

An $m$-dimensional manifold $M$ ($m = 2n + 1$) is a contact manifold if it admits a 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$. There is a unique vector field $\xi$ on $M$ such that

$$\eta(\xi) = 1, \quad L_\xi \eta = 0,$$

where $L_\xi$ denotes the Lie derivation by $\xi$. It is well known that there is a contact Riemannian structure $(\phi, \eta, \xi, g)$ such that

$$g(\xi, X) = \eta(X), \quad 2g(\phi X, \phi Y) = d\eta(X, Y), \quad \phi^2 X = -X + \eta(X)\xi,$$

where $X, Y \in \Gamma(TM)$ on $M$. Here and hereafter $\Gamma(.)$ is denoted by the space of all sections of $(.)$. The followings hold:

$$\phi \xi = 0, \quad \eta(\phi X) = 0,$$

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y),
\quad d\eta(X, \phi Y) = -d\eta(\phi X, Y).$$

Let $E$ be the foliation of $TM$ generated by the Reeb vector field $\xi$. Then $E$ gives the orthogonal decomposition

$$TM = E \oplus \mathcal{D}$$

with respect to $g$. By (2.1), $E$ is a geodesic and transversally symplectic flow with exact transversal symplectic form $d\eta$ on a Riemannian manifold $(M, g)$. If, moreover, $\xi$ satisfies $L_\xi g = 0$, or equivalently, $L_\xi \phi = 0$, then $E$ can be considered as a geodesic almost Kähler flow on $(M, g)$.

Lemma 2.1 ([5]). On a contact Riemannian structure $(\phi, \xi, \eta, g)$, the Riemannian connection $\nabla$ satisfies the following properties:

(i) $\nabla_\xi \eta = 0$, $\nabla_\xi \xi = 0$, $\xi^\ast \nabla_r \eta_r = 0$,

(ii) $\nabla_r \xi^r = 0$, $\nabla_r \phi^i_r = -2n\eta_j$,

(iii) $\nabla_r \eta_r \phi^j_r \phi^i_r = -\nabla_j \eta_i$,

(iv) $\nabla_r \eta_i \phi^i_j$ and $\nabla_r \eta_r \phi^i_r$ are symmetric in $i, j$,

(v) $\nabla_\xi \phi = 0$. 
Tanno ([5]) defined the generalized Tanaka connection $^*\nabla$ on a contact Riemannian manifold $(M, \phi, \xi, \eta, g)$ by

\begin{equation}
^*\nabla_X Y = \nabla_X Y + \eta(X) \phi Y - \eta(Y) \nabla_X \xi + \langle \nabla_X \eta \rangle(Y) \xi
\end{equation}

for $X, Y \in \Gamma(TM)$. The torsion tensor $^*T$ of $^*\nabla$ is given by

\begin{equation}
^*T(X, Y) = \eta(X) \phi Y - \phi X \eta(Y) - \eta(Y) \nabla_X \xi + \eta(X) \nabla_Y \xi + 2g(X, \phi Y) \xi.
\end{equation}

**Proposition 2.2 ([5]).** With above notations, $^*\nabla$ satisfies the followings:

(i) $^*\nabla \eta = 0$, $^*\nabla \xi = 0$,
(ii) $^*\nabla g = 0$,
(iii) $^*T(X, Y) = d\eta(X, Y) \xi$ for $X, Y \in \Gamma(D)$,
(iv) $^*T(\xi, \phi Y) = -\phi^*T(\xi, Y)$ for $Y \in \Gamma(D)$,
(v) $^*T(X, \phi Y) = (\nabla_X \phi) Y + \eta(Y) \phi \nabla_X \xi + \langle \nabla_X \eta \rangle(\phi Y) \xi$ for $X, Y \in \Gamma(TM)$,
(vi) $^*\nabla \phi = 0$ if and only if $\phi$ is integrable.

This connection is a natural generalization of the Tanaka connection defined on a nondegenerate, pseudo-hermitian manifold ([6]).

We suppose that $M$ is oriented and $D$ is oriented. Let $g_D$ be a positive definite symmetric bilinear form on $D$. A triple $(M, D, g_D)$ becomes a sub-Riemannian manifold. A contact manifold admits a sub-Riemannian metric $d\eta(\phi \cdot, \cdot)$. Fable-Gorodski showed the following result.

**Proposition 2.3 ([2]).** There is a unique connection $\nabla^F$ on a contact sub-Riemannian manifold $(M, D, \xi, \eta, g_D)$ with following properties:

(i) $\nabla^F_X : \Gamma(D) \to \Gamma(D)$ for $X \in \Gamma(TM)$,
(ii) $\nabla^F \xi = 0$,
(iii) $\nabla^F g = 0$,
(iv) $T^F(X, Y) = d\eta(X, Y) \xi$ for $X, Y \in \Gamma(D)$,
(v) the sub-torsion $\tau^F$ of $\nabla^F$ defined by $\tau^F(X, \xi) = T^F(X, \xi)$ satisfies $\tau^F(\Gamma(D)) \subset \Gamma(D)$ and is symmetric.

The connection $\nabla^F$ may be regarded as a natural extension of the generalized Tanaka connection defined on a contact Riemannian structure in the sense of sub-Riemannian geometry.
3. The case of a hypercontact structure

We recall the definitions of the following quaternionic analogue of an almost contact structure.

**Definition 3.1.** A tensor field \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}\) is called an almost contact 3-structure if the following conditions are satisfied:

(i) \(\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}\),

(ii) \(\phi_\alpha \xi_\beta = \sum_\gamma \epsilon_{\alpha\beta\gamma} \xi_\gamma\),

(iii) \(\eta_\alpha \circ \phi_\beta = \sum_\gamma \epsilon_{\alpha\beta\gamma} \eta_\gamma\),

(iv) \(\phi_\alpha \phi_\beta = -\delta_{\alpha\beta} + \xi_\alpha \otimes \eta_\beta + \sum_\gamma \epsilon_{\alpha\beta\gamma} \phi_\gamma\),

where \(\epsilon_{\alpha\beta\gamma}\) is the sign of a permutation of \((1,2,3)\).

The \(\eta_\alpha\) define the subbundle \(D\) of codimension 3 in \(TM\) on which the \(\phi_\alpha\) satisfy the quaternionic identities. The existence of an almost contact 3-structure on a manifold \(M\) is equivalent to a reduction of the structure group of \(M\) to \(Sp(n) \times Sp(1)\). In particular, \(M\) has to be of dimension \(4n + 3\).

An almost contact 3-structure is said to be compatible with a Riemannian metric \(g\) if

\[
(3.1) \quad g(\phi_\alpha X, \phi_\alpha Y) = g(X, Y) - \eta_\alpha(X) \eta_\alpha(Y), \quad X, Y \in \Gamma(TM).
\]

**Definition 3.2.** A triple of contact forms \((\omega_1, \omega_2, \omega_3)\) on a manifold \(M\) is called a hypercontact structure if there is a Riemannian metric \(g\) and a compatible almost contact 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}\) such that

\[
(3.2) \quad g(\phi_\alpha X, Y) := d\omega_\alpha(X, Y), \quad X, Y \in \Gamma(TM).
\]

The following result was proved in [3].

**Proposition 3.3 ([3]).** With above notations,

(i) \(d\omega_\alpha(\phi_\alpha X, \phi_\alpha Y) = d\omega_\alpha(X, Y)\),

(ii) \(d\omega_\alpha(\xi_\beta, \xi_\gamma) = g(\xi_\gamma, \xi_\gamma) = 1\) for any cyclic permutation \(\{\alpha, \beta, \gamma\}\) of \(\{1,2,3\}\),

(iii) the \(\xi_\alpha\) are multiples of the Reeb vector fields of the \(\omega_\alpha\),

(iv) the underlying almost contact 3-structure \((\phi_\alpha, \eta_\alpha, \xi_\alpha)_{\alpha=1,2,3}\) is completely determined if \(\omega_\alpha(\xi_\alpha) > 0\).
The definition of a hypercontact structure involves a triple of contact forms, a Riemannian metric and an almost contact 3-structure. Proposition 3.3 shows that the almost contact 3-structure is completely determined (up to sign) by the contact forms \((\omega_\alpha)_{\alpha=1,2,3}\) and metric \(g\).

In the following, we consider a hypercontact structure \((\omega_\alpha, g)_{\alpha=1,2,3}\) satisfying assumptions \((A)\) and \((B)\).

(A) \(D := \bigcap_{\alpha=1}^3 \ker \eta_\alpha = \bigcap_{\alpha=1}^3 \ker \omega_\alpha\).

(B) Let \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=1,2,3}\) be the underlying almost contact 3-structure of a given hypercontact structure \((\omega_\alpha, g)_{\alpha=1,2,3}\). The vector field \(\xi_\alpha\) is a positive multiple of Reeb vector field of \(\omega_\alpha\) for each \(\alpha\).

Then we may \(g(\xi_\alpha, \xi_\beta) = \omega_\alpha(\xi_\beta) = \delta_{\alpha, \beta}(\alpha, \beta = 1, 2, 3)\) in the sense of Proposition 3.3 (iv). It is obvious from (3.2) and Proposition 3.3 that

\[
[\xi_\alpha, \xi_\beta] = 2\epsilon_{\alpha, \beta, \gamma} \xi_\gamma.
\]

It is well known that on contact Riemannian structure ([B]),

\[
(3.4) \quad \nabla_X \xi_\beta = -\phi_\beta X - \frac{1}{2} \phi_\beta(\xi_\beta, \phi_\beta) X \quad \text{for} \quad X \in \ker \omega_\beta.
\]

**Lemma 3.4.** Let \((\omega_\alpha, g)_{\alpha=1,2,3}\) be a hypercontact structure on \(M\) with assumptions A and B. Then we have:

(i) \(\nabla_{\xi_\beta} \xi_\gamma = \epsilon_{\alpha, \beta, \gamma} \xi_\alpha\).

(ii) \(\nabla_X \xi_\beta \in \Gamma(D)\) for \(X \in \Gamma(D)\).

**Proof.** By (3.4), we have

\[
\nabla_{\xi_\alpha, \xi_\beta} = -\phi_\beta \eta_\alpha - \frac{1}{2} \phi_\beta(\xi_\beta, \phi_\beta) \xi_\alpha
\]

for \(\xi_\alpha \in \ker \omega_\beta\). A direct computation gives rise to

\[
\phi_\beta(\xi_\beta, \phi_\beta) \xi_\alpha = \phi_\beta \xi_\alpha, (\phi_\beta \xi_\alpha) - \phi_\beta^2 \xi_\alpha \xi_\alpha
\]

\[
= \phi_\beta [\xi_\beta, \phi_\beta] \xi_\alpha + [\xi_\beta, \xi_\alpha] - \eta_\beta ([\xi_\beta, \xi_\alpha]) \xi_\beta
\]

\[
= \phi_\beta [\xi_\beta, \epsilon_{\alpha, \gamma} \xi_\gamma] + [\xi_\beta, \xi_\alpha]
\]

\[
= 2\epsilon_{\alpha, \beta, \gamma} \xi_\gamma - 2\epsilon_{\alpha, \gamma, \beta} \xi_\gamma = 0,
\]

which proves (i).
Similarly, we have
\[
g(\nabla_X \xi_\beta, \xi_\gamma) = g(X, \xi_\alpha) + g(\frac{1}{2} (L_{\xi_\alpha} \phi_\beta) X, \xi_\alpha)
\]
\[
= \frac{1}{2} g((L_{\xi_\alpha} \phi_\beta) X, \xi_\alpha)
\]
for \( X \in \Gamma(D) \). On the other hand, we note that \([\xi_\alpha, X] \in \Gamma(D)\) by means of \( d\omega_\beta(\xi_\alpha, X) = 0 \) for \( X \in \Gamma(D) \). Then
\[
g((L_{\xi_\alpha} \phi_\beta) X, \xi_\alpha) = g([\xi_\beta, \phi_\beta X] - \phi_\beta[\xi_\beta, X], \xi_\alpha) = 0,
\]
which completes the proof of \((ii)\). \( \square \)

Lemma 3.4 \((i)\) means that \( E \) is totally geodesic with respect to \( g \). Moreover, this, together with the metrical property of \( \nabla \), implies
\[
\nabla_{\xi_\alpha} \omega_\beta = \epsilon_{\alpha\beta\gamma} \omega_\gamma.
\]
Since \( d\omega_\alpha(\xi_\beta, X) = 0 \) for \( X \in \Gamma(D) \), we have
\[
(3.5) \quad g((\nabla_{\xi_\alpha} \phi_\beta) Y, Z) = g(\nabla_{\xi_\alpha} (\phi_\beta Y), Z) + g(\nabla_{\xi_\alpha} Y, \phi_\beta Z)
\]
for \( Y, Z \in \Gamma(D) \). Since \( \nabla \) can be expressed as
\[
2g(\nabla_{\xi_\alpha} Y, Z) = \xi_\alpha d\omega_\gamma(Y, \phi_\gamma Z) + Y d\omega_\gamma(\xi_\alpha, \phi_\gamma Z) - Z d\omega_\gamma(\xi_\alpha, \phi_\gamma Y)
\]
\[
- d\omega_\gamma([Y, Z], \phi_\gamma \xi_\alpha) + d\omega_\gamma([Z, \xi_\alpha], \phi_\gamma Y)
\]
\[
+ d\omega_\gamma([\xi_\alpha, Y], \phi_\gamma Z),
\]
(3.7)

Lemma 3.4 together with \((3.6)\) and \((3.7)\) gives rise to
\[
2g((\nabla_{\xi_\alpha} \phi_\beta) Y, Z) = 2g(\nabla_{\xi_\alpha} (\phi_\beta Y), Z) + 2g(\nabla_{\xi_\alpha} Y, \phi_\beta Z)
\]
\[
= \xi_\alpha d\omega_\gamma(\phi_\beta Y, \phi_\gamma Z) - \xi_\alpha d\omega_\gamma(\phi_\gamma Y, \phi_\beta Z)
\]
\[
+ d\omega_\gamma([Z, \xi_\alpha], \phi_\gamma \phi_\beta Y) + d\omega_\gamma([\phi_\beta Z, \xi_\alpha], \phi_\gamma Y)
\]
\[
+ d\omega_\gamma([\xi_\alpha, \phi_\beta Y], \phi_\gamma Z) + d\omega_\gamma([\xi_\alpha, Y], \phi_\gamma \phi_\beta Z).
\]

By using the Jacobi identity, the right hand side of the above formula becomes
\[
\xi_\alpha d\omega_\gamma(\phi_\beta Y, \phi_\gamma Z) - \xi_\alpha d\omega_\gamma(\phi_\gamma Y, \phi_\beta Z)
\]
\[
+ d\omega_\gamma([Z, \xi_\alpha], \phi_\gamma \phi_\beta Y) + d\omega_\gamma([\phi_\beta Z, \xi_\alpha], \phi_\gamma Y)
\]
\[
+ d\omega_\gamma([\xi_\alpha, \phi_\beta Y], \phi_\gamma Z) + d\omega_\gamma([\xi_\alpha, Y], \phi_\gamma \phi_\beta Z) = 0.
\]
Thus, we have
\begin{equation}
(\nabla_{\xi_i} \phi_j) Y \in \Gamma(E) \text{ for } Y \in \Gamma(D).
\end{equation}
It follows from Lemma 3.4 and (3.8) that
\[ (\nabla_{\xi_i} \phi_j) Y = 0 \text{ for } Y \in \Gamma(D). \]
On the other hand,
\begin{align*}
(\nabla_{\xi_i} \phi_j) \xi_\gamma &= \nabla_{\xi_i} (\phi_\beta \xi_\gamma) - \phi_\beta (\nabla_{\xi_i} \xi_\gamma) \\
&= \nabla_{\xi_i} \xi_{\beta \gamma} \xi_\alpha - \phi_\beta \xi_{\alpha \gamma} \xi_\beta = 0
\end{align*}
for distinct \( \{\alpha, \beta, \gamma\} \). It is easy to see that
\begin{align*}
(\nabla_{\xi_i} \phi_j) \xi_\alpha &= \nabla_{\xi_i} (\phi_\beta \xi_\alpha) - \phi_\beta (\nabla_{\xi_i} \xi_\alpha) \\
&= \nabla_{\xi_i} \xi_{\alpha \beta} \xi_\gamma = \xi_{\beta i}.
\end{align*}
By a similar way as in [4], we can show that \( (\nabla_X \phi_\alpha) Y = 0 \) if and only if \( \phi_\alpha \) is integrable. Summing up, we have

**Lemma 3.5.** Under the same station as in Lemma 3.4, the Ric-\mbox{mann}ian connection \( \nabla \) satisfies
\begin{enumerate}
\item \( \nabla_{\xi_i} \omega_{\beta} = \epsilon_{\alpha \beta \gamma} \omega_{\gamma} \),
\item \( (\nabla_{\xi_i} \phi_\beta) X = 0 \),
\item \( (\nabla_{\xi_i} \phi_\beta) \xi_\gamma = 0 \), \( (\nabla_{\xi_i} \phi_\beta) \xi_\alpha = \xi_{\alpha \beta} \) for distinct \( \alpha, \beta, \gamma \),
\item \( (\nabla_X \phi_\alpha) Y = 0 \) if and only if \( \phi_\alpha \) is integrable,
\end{enumerate}
where \( X, Y \in \Gamma(D) \).

Now, we can construct a new connection on a hypercontact structure by a similar way as in [5].

Define a connection \( D \) on a hypercontact structure \( (\omega_\alpha, g)_{\alpha = 1, 2, 3} \) by
\begin{equation}
D_Y Z = \nabla_Y Z + \sum_{\alpha = 1}^3 \{ \omega_\alpha (Y) \phi_\alpha Z - \omega_\alpha (Z) \omega_\alpha (Z) \nabla_Y \xi_\alpha + (\nabla_Y \omega_\alpha)(Z) \xi_\alpha \},
\end{equation}
where \( Y, Z \in \Gamma(TM) \). Then torsion tensor \( T^D \) of \( D \) is given by
\begin{equation}
T^D(Y, Z) = \sum_{\alpha = 1}^3 \{ \omega_\alpha (Y) \phi_\alpha Z - \omega_\alpha (Z) \phi_\alpha Y - \omega_\alpha (Z) \nabla_Y \xi_\alpha \\
+ \omega_\alpha (Y) \nabla_Z \xi_\alpha - 2g(\phi_\alpha Y, Z) \xi_\alpha \}
\end{equation}
for \( Y, Z \in \Gamma(TM) \).
Theorem 3.6. Let \((\omega_{\alpha}, g)_{\alpha=1,2,3}\) be as in Lemma 3.4. The connection \(D\) defined by (3.9) is a unique linear connection satisfying the followings:

(i) \(D\omega_{\alpha} = 0,\) \(D\xi_{\alpha} = 0,\)

(ii) \(Dg = 0,\)

(iii) \(T^{D}(X,Y) = \sum_{\alpha=1}^{3} d\omega_{\alpha}(X,Y)\xi_{\alpha},\)

(iv) \(T^{D}(\xi_{\alpha}, \phi_{\alpha}Y) = -\phi_{\alpha}T^{D}(\xi_{\alpha}, Y),\)

(v) \(T^{D}(\xi_{\alpha}, \phi_{\beta}Y) - \phi_{\beta}T^{D}(\xi_{\alpha}, Y)\)
\[= 2\epsilon_{\alpha\beta\gamma}\phi_{\gamma}Y - (L_{\xi_{\alpha}}\phi_{\beta})Y,\]
\(T^{D}(\xi_{\alpha}, \phi_{\beta}Y) + \phi_{\beta}T^{D}(\xi_{\alpha}, Y)\)
\[= \phi_{\beta}[\xi_{\alpha}, Y] + [\xi_{\alpha}, \phi_{\beta}Y] \text{ for } \alpha \neq \beta,\]

(vi) \(T^{D}(\xi_{\alpha}, \xi_{\beta}) = -2\epsilon_{\alpha\beta\gamma}\xi_{\gamma},\)

(vii) \((D\phi_{\alpha})\xi_{\beta} = 0,\)

(viii) \((D\xi_{\alpha})\phi_{\beta}X = 2\epsilon_{\alpha\beta\gamma}\phi_{\gamma}X,\)

(ix) \((DX\phi_{\alpha})Y = 0 \text{ if and only if } \phi_{\alpha} \text{ is integrable,}\)

where \(X,Y \in \Gamma(D).\)

Proof. \(D\xi_{\alpha} = 0\) is proved by (3.9) and Lemma 3.4. By (3.9) and Lemma 3.5, for \(X,Y,Z \in \Gamma(TM),\) we have

\[
(Dg)(X,Y) = Zg(X,Y) - g(DZ,X,Y) - g(X,DZY)
\]
\[
= Zg(X,Y) - g(\nabla_Z X,Y) - g(X,\nabla_Z Y)
\-
\[g(\{\omega_{\alpha}(Z)\}\phi_{\alpha}X - \omega_{\alpha}(X)\nabla_Z \xi_{\alpha} + (\nabla_Z \omega_{\alpha})(X)\xi_{\alpha}, Y)
\-
\[g(X,\sum_{\alpha=1}^{3} \{\omega_{\alpha}(Z)\phi_{\alpha}Y - \omega_{\alpha}(Y)\nabla_Z \xi_{\alpha} + (\nabla_Z \omega_{\alpha})(Y)\xi_{\alpha}\})
\=
\](\nabla_Z g)(X,Y) = 0.
\]

Thus (ii) is proved.

(iii) can be easily verified by (3.10). From Proposition 2.2 and (3.10), we see (iv).

For the case that \(\alpha \neq \beta,\) we have from (3.10) that

\[T^{D}(\xi_{\alpha}, \phi_{\beta}Y) = \epsilon_{\alpha\beta\gamma}\phi_{\gamma}Y + \nabla_{\phi_{\alpha}Y}\xi_{\alpha},\]

and

\[\phi_{\beta}T^{D}(\xi_{\alpha}, Y) = \phi_{\beta}\phi_{\alpha}Y + \phi_{\beta}\nabla_Y \xi_{\alpha}\]
\[= -\epsilon_{\alpha\beta\gamma}\phi_{\gamma}Y + \phi_{\beta}\nabla_Y \xi_{\alpha}.\]
Thus, we have
\[ T^D(\xi_\alpha, \phi_3 Y) - \phi_3 T^D(\xi_\alpha, Y) = 2\epsilon_{\alpha,\beta,\gamma} \phi_\gamma Y + \phi_3 [\xi_\alpha, Y] - [\xi_\alpha, \phi_3 Y] \]
\[ = 2\epsilon_{\alpha,\beta,\gamma} \phi_\gamma Y - (L_{\xi_\alpha} \phi_3) Y \]
and
\[ T^D(\xi_\alpha, \phi_3 Y) + \phi_3 T^D(\xi_\alpha, Y) = \phi_3 [\xi_\alpha, Y] + [\xi_\alpha, \phi_3 Y]. \]
By (3.3) and (i), we have (vi) and (vii). (viii) and (ix) are verified from Lemma 3.5. The uniqueness of the connection is obvious. □

**Proposition 3.7.** On a hypercontact manifold \((M, \omega, \xi, g)\) of dimension \(4n + 3\), the following holds:
\[ \|T^D\|^2 = \|L_{\xi_\alpha} g\|^2 + 16(4n) + 12. \]
In particular, \(\|T^D\|^2\) attains its minimum \(16(4n) + 12\) if and only if \(\xi_\alpha\) is a Killing vector field for each \(\alpha\).

**Proof.**
\[ \|T^D\|^2 = 2\|\nabla \xi_\alpha\|^2 + 6(4n) + 6. \]
By the formula \((L_{\xi_\alpha} g)(X, Y) = (\nabla_X \omega_n)(Y) + (\nabla_Y \omega_n)(X)\), we obtain
\[ \|L_{\xi_\alpha} g\|^2 = 2\|\nabla \xi_\alpha\|^2 + 2\nabla_X \xi_\alpha (\nabla_Y \omega_n) X \]
for \(X, Y \in \Gamma(D)\).
Since \(\nabla_X \xi_\alpha (\nabla_Y \omega_n) X = \|\nabla \xi_\alpha\|^2 - 2(4n)\), we get
\[ \|L_{\xi_\alpha} g\|^2 = 4\|\nabla \xi_\alpha\|^2 - 4(4n), \]
which yields (3.11). □

**4. The case of a sub-Riemannian 3-structure**

Let \((\varphi_\alpha, \eta_\alpha, \xi_\alpha)\) be an almost contact 3-structure on an oriented manifold \(M\). We suppose that \(d\eta_\alpha(\xi_\alpha, X) = 0\) for \(X \in \Gamma(D)\). Then the \(\varphi_\alpha\) satisfy the quaternionic identities on \(D\). We consider a smoothly varying positive definite symmetric bilinear form \(g_D\) on \(D\). Then \((D, g_D)\) is called a sub-Riemannian 3-structure on \(M\). From (3.3) and Lemma 3.4, a hypercontact structure \((\omega, g)\) satisfying the assumptions A and B, is an example of a sub-Riemannian 3-structure whose sub-Riemannian metric is the restriction of \(g\) to \(D\).

Note that the sub-Riemannian metric \(g\) has a natural extension to a Riemannian metric \(\langle \cdot, \cdot \rangle\) on \(M\) by setting \(\xi_\alpha (\alpha = 1, 2, 3)\) to be orthonormal to \(D\).
Theorem 4.1. Let \((M, \mathcal{D}, g_\mathcal{D})\) be a sub-Riemannian 3-structure with the underlying almost contact 3-structure \((\varphi_\alpha, \eta_\alpha, \xi_\alpha)_{\alpha=1,2,3}\). Then there is a unique connection \(D\) with following properties:

(i) \(D_X : \Gamma(\mathcal{D}) \rightarrow \Gamma(\mathcal{D})\) for \(X \in \Gamma(TM)\),

(ii) \(D\xi_\alpha = 0\),

(iii) \(Dg_\mathcal{D} = 0\),

(iv) \(T^D(X,Y) = \sum_{\alpha=1}^3 d\eta_\alpha(X,Y)\xi_\alpha, \; X,Y \in \Gamma(\mathcal{D})\),

(v) the sub-torsion \(\tau^D_\alpha\) of \(D\) defined by \(\tau^D_\alpha(X) := T^D(\xi_\alpha, X)\) satisfies \(\tau^D_\alpha(\Gamma(\mathcal{D})) \subset \Gamma(\mathcal{D})\) and symmetric,

(vi) \(T^D(\xi_\alpha, \xi_\beta) = -2\epsilon_{\alpha\beta\gamma} \xi_\gamma\).

Proof. Let \(X, Y, Z \in \Gamma(\mathcal{D})\). As is Riemannian geometry, (i), (iii) and (iv) uniquely determine \(D_X Y\) by virtue of the formula

\[
X(Y, Z) + Y(Z, X) - Z(X, Y) = 2(D_X Y, Z) + \langle Y, [X, Z] + T(X, Z) \rangle \\
+ \langle X, [Y, Z] + T(Y, Z) \rangle + \langle Z, [Y, X] + T(Y, X) \rangle.
\]

Because of (ii), it remains only to define \(D_\xi_\alpha X\). Since \(D_\xi_\alpha X = [\xi_\alpha, X] + \tau_\alpha(X)\), the formula

\[
\xi_\alpha \langle X, Y \rangle = \langle D_\xi_\alpha X, Y \rangle + \langle X, D_\xi_\alpha Y \rangle \\
= \langle [\xi_\alpha, X], Y \rangle + \langle [\xi_\alpha, Y], X \rangle + 2\tau^D_\alpha(X, Y)
\]

determines \(\tau^D_\alpha(X)\) By (ii), \(T^D(\xi_\alpha, \xi_\beta) = -[\xi_\alpha, \xi_\beta]\), which determines \(T^D(\xi_\alpha, \xi_\beta)\).

\[\square\]

Corollary 4.2. The connection \(D\) has following properties:

(i) \(d\eta_\alpha(X, Y) = \eta_\alpha(T^D(X, Y))\),

(ii) \(2(T^D(\xi_\alpha, X), Y) = (L_{\xi_\alpha} g_\mathcal{D})(X, Y)\)

for \(X, Y \in \Gamma(\mathcal{D})\).

The curvature of this connection is given by

\[
R^D(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z.
\]

Now we study the sub-symmetry property in a situation of a sub-Riemannian 3-structure. Recalled notion of sub-Riemannian symmetric space.
A local isometry between sub-Riemannian manifolds \((M, \mathcal{D}, g_D)\) and \((M', \mathcal{D}', g_D')\) is a diffeomorphism \(\psi : U \subset M \to U' \subset M'\) between open sets such that \(\psi_*(\mathcal{D}) = \mathcal{D}'\) and \(\psi^*g_D = g_D'\). In the sub-Riemannian 3-structure case, it can be seen that \(\psi^*(\omega_\alpha) = \pm \omega_\alpha'\) and \(\psi^*(\xi_\alpha) = \pm \xi_\alpha'\) for each \(\alpha\). Indeed, if \(\psi^*(\omega_\alpha) = \pm \omega_\beta'\) with \(\alpha \neq \beta\) then such a \(\psi^*\) contradicts to the sub-symmetric property. If \(\psi\) is globally defined of \(M\) to \(M'\), we say that \(\psi\) is isometry.

Note that an isometry \(\psi : M \to M'\) is an affine map with respect to the adapted connection, that is, \(D'_{\psi_*X} \psi_*Y = \psi_*(D_XY)\) for \(X, Y \in \Gamma(TM)\).

A sub-Riemannian symmetric space (or sub-symmetric space) is an homogeneous sub-Riemannian manifold \((M, \mathcal{D}, g_D)\) such that for every point \(x_0 \in M\) there is an isometry \(\psi\) such that \(\psi(x_0) = x_0\) and \(\psi_*|_{\mathcal{D}_{x_0}} = -1\), which is called a sub-symmetry at \(x_0\). Then we have:

**Theorem 4.3.** A sub-Riemannian manifold with sub-Riemannian 3-structure is a locally sub-symmetric space if and only if the following conditions are verified:

1. \(D_X T^D = 0\),
2. \(D_X R^D = 0\)

for all \(X \in \Gamma(\mathcal{D})\).

**Proof.** Suppose that \(M\) is a sub-symmetry space. The sub-symmetry \(\psi\) is an affine map with respect to the canonical connection \(D\) given in Theorem 4.1. We compute for \(X, Y, Z \in \Gamma(\mathcal{D})\)

\[
\psi_*(\nabla_Z T^D)(X, Y) = (D_{\psi_*Z} T^D)(\psi_*X, \psi_*Y) = -(D_Z T^D)(X, Y).
\]

By Theorem 4.1 (vi), we have that \(\psi_*(D_Z T^D(X, Y)) = D_Z T^D(X, Y)\). Therefore we have

\[
(D_Z T^D)(X, Y) = 0.
\]

Now, it follows from Theorem 4.1 (v) that

\[
\psi_*(D_Z T^D(X, \xi_\alpha)) = -D_Z T^D(X, \xi_\alpha).
\]

On the other hand,

\[
\psi_*(D_Z T^D(X, \xi_\alpha)) = D_{\psi_*Z} T^D(\psi_*X, \psi_*\xi_\alpha) = D_{(-Z)} T^D(-X, \xi_\alpha) = D_Z T^D(X, \xi_\alpha),
\]
so that \((DZT^D)(X, \xi_\alpha) = 0\). Finally by Theorem 4.1 (ii), we have
\[
DZT^D(\xi_\alpha, \xi_\beta) = DZ(D^D(\xi_\alpha, \xi_\beta)) - T^D(DZ\xi_\alpha, \xi_\beta) - T^D(\xi_\alpha, DZ\xi_\beta) = 0.
\]
Hence, \((DZT^D)(\xi_\alpha, \xi_\beta) = 0\). We have (ii) by a similar way.

Conversely, suppose the conditions (i) and (ii). We will find differential equation which must be satisfied by the curvature and torsion tensors of the connection \(D\) along the geodesic rays. Suppose \(\{X_i\} = \{X_1, \cdots, X_n, X_{4n+1} = \xi_1, X_{4n+2} = \xi_2, X_{4n+3} = \xi_3\}\) is an adapted frame at the point \(p \in M\) where \(d\eta_\alpha(X_1, X_2) \neq 0\) for each \(\alpha = 1, 2, 3\) and denote by the same symbols \(\{X_i\}\) the frame obtained by parallel translation along geodesic rays. Our basic arguments follow [2].

Let \(Z = \sum_j a^jX_j\) be a direction at \(p\). Then \(Z = \sum_j a^jX_j\) is the tangent along the geodesic ray in this direction. Write also \(Z = Z' + a\xi_1 + b\xi_2 + c\xi_3\) where \(Z' \in \Gamma(D)\). Using condition (i), we get
\[
DZ(R(X_i, X_j)X_l) = DZ_{Z'+a\xi_1+b\xi_2+c\xi_3}(R(X_i, X_j)X_l)
\]
\[= aD_{\xi_1}(R(X_i, X_j)X_l) + bD_{\xi_2}(R(X_i, X_j)X_l) + cD_{\xi_3}(R(X_i, X_j)X_l)
\]
\[= ah_1^{-1}D_{[X_1, X_2]}(R(X_i, X_j)X_l) + bh_2^{-1}D_{[X_1, X_2]}(R(X_i, X_j)X_l)
\]
\[+ ch_3^{-1}D_{[X_1, X_2]}(R(X_i, X_j)X_l)
\]
\[= (ah_1^{-1} + bh_2^{-1})h_3D_{\xi_3}(R(X_i, X_j)X_l)
\]
\[= (ah_1^{-1} + bh_2^{-1})h_3D_{\xi_3}(R(X_i, X_j)X_l)
\]
\[= (ah_1^{-1} + bh_2^{-1})h_3D_{\xi_3}(R(X_i, X_j)X_l)
\]
and analogously for the torsion tensor, where \(h_\alpha := \eta_\alpha([X_1, X_2])\) for each \(\alpha\) is a function. Moreover it may be simplified as followings:
\[= \frac{ah_2h_3 + bh_3h_1 + ch_1h_2}{h_1h_2h_3}D_{[X_1, X_2]}(R(X_i, X_j)X_l)
\]
\[= DZ(R(X_i, X_j)X_l).
\]
Therefore

$$2D_Z(R(X_i, X_j)X_l) = a h_i^{-1}D_{[X_i, X_j]}(R(X_i, X_j)X_l)$$

$$+ bh_2^{-1}D_{[X_i, X_j]}(R(X_i, X_j)X_l)$$

$$+ ch_3^{-1}D_{[X_i, X_j]}(R(X_i, X_j)X_l)$$

$$- \left( \frac{ah_2h_3 + bh_3h_1 + ch_1h_2}{h_1h_2h_3} \right) D_{[X_i, X_j]}(R(X_i, X_j)X_l).$$

Next, to find the function $h_n$ along the geodesic ray determined by $Z = Z' + a\xi_1 + b\xi_2 + c\xi_3$, we compute

$$2h_n' = -2\eta_n(D_ZT^D)(X_1, X_2)$$

$$= -\eta_n(ah_1^{-1}(D_{[X_1, X_2]}T^D)(X_1, X_2)$$

$$+ bh_2^{-1}(D_{[X_1, X_2]}T^D)(X_1, X_2)$$

$$+ ch_3^{-1}(D_{[X_1, X_2]}T^D)(X_1, X_2))$$

$$+ \eta_n \left( \frac{ah_2h_3 + bh_3h_1 + ch_1h_2}{h_1h_2h_3} \right) (D_{[X_1, X_2]}T^D)(X_1, X_2)$$

for each $\alpha = 1, 2, 3$. Notice (4.1) and (4.2), the rest of the proof follows [2, Theorem 2.1].

**References**


Department of Mathematics
Kanazawa University
Kanazawa 920-1192, Japan
E-mail: hyuns@orgio.net