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SEPARABILITY PROPERTIES OF CERTAIN POLYGONAL PRODUCTS OF GROUPS

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Abstract. Let $G = E *_A F$, where A is a finitely generated abelian subgroup. We prove a criterion for G to be $\{A\}$-double coset separable. Applying this result, we show that polygonal products of central subgroup separable groups, amalgamating trivial intersecting central subgroups, are double coset separable relative to certain central subgroups of their vertex groups. Finally we show that such polygonal products are conjugacy separable. It follows that polygonal products of polycyclic-by-finite groups, amalgamating trivial intersecting central subgroups, are conjugacy separable.

1. Introduction

Let S be a subset of a group G. Then G is said to be *S-separable* if, for each $g \in G \backslash S$, there exists a normal subgroup N of finite index in G such that $g \notin NS$. Equivalently, S is a closed subset in the profinite topology of G. In particular, if $S = \{1\}$ then G is *residually finite.* If G is $\{x\}^G$-separable for all $x \in G$, where $\{x\}^G = \{g^{-1}xg \mid g \in G\}$, then G is called *conjugacy separable.* If G is H-separable for all finitely generated subgroups H of G, then G is called *subgroup separable.* These kinds of separability properties are directly related to important problems in group theory. Malcev [14] and Mostowski [16] showed that: (1) finitely presented residually finite groups have solvable word problem,

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(2) finitely presented conjugacy separable groups have solvable conjugacy problem, and (3) finitely presented subgroup separable groups have solvable generalized word problem.

In general it is difficult to show whether a given subset of a group is separable. Free groups and surface groups are known to be subgroup separable and conjugacy separable (Hall [6], Scott [19] and Stebe [20]). Fine and Rosenberger [3] showed that Fuchsian groups are conjugacy separable. Niblo [17] showed that these groups are also double coset separable. It is also known that polycyclic-by-finite groups are subgroup separable and conjugacy separable (Malcev [15] and Formanek [4]).

In [11], Kim and Tang showed that tree products of polycyclic-by-finite groups, amalgamating central subgroups, are conjugacy separable. In the case of polygonal products the problem is much more difficult, since some polygonal products of finitely generated groups need not have proper subgroups of finite index (Higman [7]). Allenby and Tang [1] constructed an example of polygonal product of finitely generated nilpotent groups of class 2, with trivial intersecting cyclic amalgamated subgroups, which is not residually finite. In this paper we study certain separability properties of polygonal products of central subgroup separable groups, amalgamating trivial intersecting central subgroups.

In Section 3, we introduce the concept of $\{A_1, \ldots, A_n\}$-double coset separability. We prove that if $G = E *_A F$, where E and F satisfy certain conditions and A is a finitely generated abelian subgroup then G is $\{A, A_1, A_2\}$-double coset separable for $A_1 \subset E$ and $A_2 \subset F$ (Theorem 3.6). In Section 4, we prove that polygonal products of central subgroup separable groups, amalgamating trivial intersecting central subgroups, $H_1, H_2, \ldots, H_m$, are $\{A_i, A_j\}$-double coset separable where $A_i = H_{i-1} \times H_i$ (Theorem 4.17). In Section 5, we prove that polygonal products of central subgroup separable and conjugacy separable groups, amalgamating trivial intersecting central subgroups, are conjugacy separable (Theorem 5.5). From this, it immediately follows that polygonal products of polycyclic-by-finite groups, amalgamating trivial intersecting central subgroups, are conjugacy separable.

2. Preliminaries

Throughout this paper we use standard terms and notations.

The letter G always denotes a group.

For $x \in G$, $\{x\}^G$ denotes the set of all conjugates of x in G.

$x \sim_G y$ means x, y are conjugate in G.

$N \lhd_f G$ means N is a normal subgroup of finite index in G.

$Z(G)$ denotes the center of G and $Z_A(x) = \{a \in A \mid [a, x] = 1\}$.

If $x \in G = A *_H B$ then $\|x\|$ denotes the free product length of x in G.

We use $\mathcal{RF}$ to denote the class of *residually finite* groups. By abuse of notation, we also use $\mathcal{RF}$ to mean *residually finite*.

Let Γ be a tree. To each vertex v of Γ assign a group G_v called a vertex group. Similarly to each edge e of Γ assign a group G_e. Let u, v be vertices at the ends of e. Let α_e and β_e be monomorphisms of G_e to G_u and to G_v, respectively. Then $\alpha_e(G_e)$ is called an edge group of the vertex group G_u and $\beta_e(G_e)$ an edge group of G_v. The *tree product* of Γ is defined to be the group generated by all the generators and relations of the vertex groups of Γ together with the extra relations obtained by identifying $\alpha_e(g_e)$ and $\beta_e(g_e)$ for each $g_e \in G_e$ and each e in Γ.

Let P be a polygon. Assign a vertex group G_v to each vertex v and a group G_e to each edge e of P. Let α_e and β_e be monomorphisms which embed G_e as a subgroup of the two vertex groups at the ends of the edge e. Then the *polygonal product* G is defined to be the group presented by the generators and relations of the vertex groups together with the extra relations obtained by identifying $\alpha_e(g_e)$ and $\beta_e(g_e)$ for each $g_e \in G_e$. As mentioned in [1], in the study of residual properties of polygonal products, we usually restrict ourselves to the case where the embedded subgroups at each vertex group meet trivially and where the polygon has at least 4 vertices. By abuse of language, we say that G is the polygonal product of the (vertex) groups $G_1, G_2, \ldots, G_m$, amalgamating the (edge) subgroups $H_1, H_2, \ldots, H_m = H_0$, with *trivial intersections*, if $G_i \cap G_{i+1} = H_i$ and $H_{i-1} \cap H_i = 1$, where $1 \leq i \leq m$ and the subscripts i are taken modulo m. We shall only consider the case $m \geq 4$.

The following two well-known results will be used extensively in this paper:

THEOREM 2.1. [13, Theorem 4.6] *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced, and that $x \sim_G y$.*

(1) *If $\|x\| = 0$, then $\|y\| \leq 1$ and, if $y \in A$, then there is a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_B h_r = x$.*
(2) *If $\|x\| = 1$, then $\|y\| = 1$ and, either $x, y \in A$ and $x \sim_A y$, or $x, y \in B$ and $x \sim_B y$.*
(3) *If $\|x\| \geq 2$, then $\|x\| = \|y\|$ and $y \sim_H x^*$ where x^* is a cyclic permutation of x.*

THEOREM 2.2. [2, Theorem 4] *If A and B are conjugacy separable and H is finite, then $A *_H B$ is conjugacy separable.*

In general, a polygonal product of at least four vertex groups, with trivial intersections, can be considered as a generalized free product of two groups, say E and F, amalgamating a subgroup (see [9, 12]). Then the groups E and F are tree products of the vertex groups of the polygonal product. Hence, in the study of polygonal products, we need some results on tree products of groups. We shall use the following results.

DEFINITION 2.3. Let G be a group and H be a subgroup of G. We say H is *finitely compatible* in G or G is *H-finite* if, for every $D \lhd_f H$, there exists $N_D \lhd_f G$ such that $N_D \cap H = D$.

LEMMA 2.4. [11, Lemma 3.3] *Let $G = A *_C B$ where A, B are C-finite. Let H be a subgroup of A such that A is H-finite. Then G is H-finite.*

LEMMA 2.5. [9, Theorem 2.3] *Let $G = A *_C B$, where A, B are C-finite and C-separable. Let H be a subgroup of A such that A is H-separable. Then G is H-separable.*

The following is an easy application of finiteness and separability:

THEOREM 2.6. *Let $H \leq G$ and H be $\mathcal{RF}$. If G is H-finite and H-separable, then G is $\mathcal{RF}$.*

DEFINITION 2.7. A group G is *central subgroup separable* if G is H-separable for any finitely generated subgroup H in the center $Z(G)$ of G.

In particular, every subgroup separable group is central subgroup separable. Using Lemma 2.4 and 2.5 repeatedly, we have the following:

COROLLARY 2.8. [11, Corollary 3.10] *Let G be a tree product of central subgroup separable groups amalgamating central edge groups. Let H be a finitely generated central subgroup of a vertex group. Then G is H-finite and H-separable. In particular, G is $\mathcal{RF}$.*

LEMMA 2.9. [11, Corollary 3.12] *Let G be a tree product of central subgroup separable groups amalgamating central edge groups. Let H, K, J be finitely generated central subgroups of some vertex groups. Then, for each $U \lhd_f H$, there exists $N \lhd_f G$ such that $N \cap H = U$, $NH \cap NK = N(H \cap K)$ and $NH \cap NJ = N(H \cap J)$.*

LEMMA 2.10. [11, Lemma 3.15] *Let G be a tree product of any groups A_i $(1 \leq i \leq n)$ amalgamating central edge groups. Let $x \in Z(A_i)$ and $y \in Z(A_j)$. If $x \sim_G y$ then $x = y$.*

3. Double coset separability

In this section we shall study the double coset separability of generalized free products of double coset separable groups amalgamating finitely generated abelian subgroups. Using this, we derive that tree products of central subgroup separable groups, amalgamating central subgroups, are HxK-separable for central subgroups H, K in some vertex groups which is a main result in [11].

LEMMA 3.1. *Let $G = E *_H F$. Suppose E, F are H-finite and H-separable. Let $U, V \leq E$. If E is UxV-separable for $x \in E$, then G is UxV-separable for $x \in E$.*

Proof. Let $g \in G$ such that $g \notin UxV$, where $x \in E$. We shall find $L \lhd_f E$ and $M \lhd_f F$ with $L \cap H = M \cap H$ such that in $\overline{G} = \overline{E} *_{\overline{H}} \overline{F}$, where $\overline{E} = E/L$ and $\overline{F} = F/M$, $\|\overline{g}\| = \|g\|$ and $\overline{g} \notin \overline{U}\overline{x}\overline{V}$. Since $\overline{G}$ is $\mathcal{RF}$ and $\overline{U}\overline{x}\overline{V}$ is finite, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{g} \notin \overline{N}\overline{U}\overline{x}\overline{V}$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_f G$ and $g \notin NUxV$.

Case 1. $g \in E$. Since E is UxV-separable, there exists $L \lhd_f E$ such that $g \notin LUxV$. Since F is H-finite, there exists $M \lhd_f F$ such that $L \cap H = M \cap H$. Let $\overline{G} = \overline{E} *_{\overline{H}} \overline{F}$, where $\overline{E} = E/L$ and $\overline{F} = F/M$. Then $\overline{g} \notin \overline{U}\overline{x}\overline{V}$.

Case 2. $g \notin E$. Let $g = f_1 e_1 \cdots f_n e_n$, where $e_i \in E \backslash H$ and $f_i \in F \backslash H$. Since E, F are H-separable and H-finite, G is H-separable by Lemma 2.5. Hence there exists $N \lhd_f G$ such that $e_i, f_i \notin NH$. Let $L = N \cap E$ and $M = N \cap F$ and let $\overline{G}$ be as above. Then $\|\overline{g}\| = \|g\|$, hence $\overline{g} \notin \overline{E}$. Thus $\overline{g} \notin \overline{U}\overline{x}\overline{V}$. □

DEFINITION 3.2. Let $A_1, \ldots, A_n$ be subgroups of a group G. We say that G is $\{A_1, \ldots, A_n\}$-*double coset separable*, briefly G is $\{A_1, \ldots, A_n\}$-*d-separable*, if G is UxV-separable for any $x \in G$ and for any subgroups U, V of $A_1, \ldots,$ or A_n.

In particular, if G is $\{A_1, \ldots, A_n\}$-d-separable, then G is $\{A_i, A_j\}$-d-separable and $\{A_i\}$-d-separable for $i, j = 1, \ldots, n$. Clearly if G is $\{A_i\}$-d-separable then G is $A_i x A_i$-separable for all $x \in G$.

Comparing with Lemma 2.5, the following criterion for double coset separability of generalized free products is of interest.

THEOREM 3.3. *Let $G = E *_A F$, where A is a finitely generated abelian subgroup. Suppose E and F satisfy the following:*

D1. *E and F are $\{A\}$-d-separable.*

D2. *For any $x, y \in A$, if $x \sim_{E \cup F} y$ then $x = y$.*

D3. *For every $U \lhd_f A$, there exists $L \lhd_f E$ such that $L \cap A = U$ and, in $\overline{E} = E/L$, if $\overline{x} \sim_{\overline{E}} \overline{y}$ for any $x, y \in A$ then $\overline{x} = \overline{y}$. Similarly, for every $U \lhd_f A$, there exists $M \lhd_f F$ such that $M \cap A = U$ and, in $\overline{F} = F/L$, if $\overline{x} \sim_{\overline{F}} \overline{y}$ for any $x, y \in A$ then $\overline{x} = \overline{y}$.*

Then G is $\{A\}$-d-separable.

To prove this result we need the following lemma. Since the lemma is very similar to a general case (Lemma 3.7 below), we omit the proof.

LEMMA 3.4. *Let $G = E *_A F$ be as in Theorem 3.3. Let $S_1, S_2 \leq A$ and let $1 \neq z \in A$, $c_i \in E \backslash A$ and $d_i \in F \backslash A$. Then there exists $N \lhd_f G$ such that*

(1) *if $x = c_1 d_1 \cdots c_n d_n$ and $c_1 z d_1 \cdots c_n d_n \notin S_1 x S_2$, then $c_1 z d_1 \cdots c_n d_n \notin N S_1 x S_2$;*

(2) *if $x = c_1 d_1 \cdots d_{n-1} c_n$ and $c_1 z d_1 \cdots d_{n-1} c_n \notin S_1 x S_2$, then $c_1 z d_1 \cdots d_{n-1} c_n \notin N S_1 x S_2$;*

(3) *if $x = d_1 c_1 \cdots c_{n-1} d_n$ and $d_1 z c_1 \cdots c_{n-1} d_n \notin S_1 x S_2$, then $d_1 z c_1 \cdots c_{n-1} d_n \notin N S_1 x S_2$; and*

(4) *if $x = d_1 c_1 \cdots d_n c_n$ and $d_1 z c_1 \cdots d_n c_n \notin S_1 x S_2$, then $d_1 z c_1 \cdots d_n c_n \notin N S_1 x S_2$.*

Proof of Theorem 3.3: Let $S_1, S_2 \leq A$ and let $g, x \in G$ such that $g \notin S_1 x S_2$. We shall find $N \lhd_f G$ such that $g \notin N S_1 x S_2$ or we shall find $L \lhd_f E$ and $M \lhd_f F$ with $L \cap A = M \cap A$ such that, in $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$ where $\overline{E} = E/L$ and $\overline{F} = F/M$, $\|\overline{x}\| = \|x\|$, $\|\overline{g}\| = \|g\|$ and $\overline{g} \notin \overline{S_1}\,\overline{x}\,\overline{S_2}$. Since $\overline{G}$ is $\mathcal{RF}$ and $\overline{S_1}\,\overline{x}\,\overline{S_2}$ is finite, we can find $N \lhd_f G$ such that $g \notin N S_1 x S_2$.

Case 1. $x \in E$. By Lemma 3.1, we can find $N \lhd_f G$ such that $g \notin N S_1 x S_2$.

Case 2. $x \in F \backslash A$.

(a) $g \in E$. Since $g \notin S_1 x S_2$, clearly $x \notin S_1 g S_2$. By Case 1, we can find $N \lhd_f G$ such that $x \notin N S_1 g S_2$. Hence $g \notin N S_1 x S_2$.

(b) $g \in F \backslash A$. Since F is $\{A\}$-d-separable by D1, there exists $M \lhd_f F$ such that $g \notin M S_1 x S_2$. By D3, there exists $L \lhd_f E$ such that $L \cap A = M \cap A$. Let $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$ where $\overline{E} = E/L$ and $\overline{F} = F/M$. Then $\overline{g} \notin \overline{S_1}\,\overline{x}\,\overline{S_2}$.

(c) $\|g\| \geq 2$. WLOG, let $g = e_1 f_1 \cdots e_n f_n$, where $e_i \in E \backslash A$ and $f_i \in F \backslash A$. By D1 and D3, G is A-separable (Lemma 2.5). Hence there exists $N \triangleleft_f G$ such that $e_i \notin NA$ and $f_i \notin NA$. Let $\overline{E} = E/(N \cap E)$, $\overline{F} = F/(N \cap F)$ and $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$. Then $\|\overline{g}\| = \|g\| \geq 2$. Thus $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

Case 3. $\|x\| \geq 2$. By induction, we assume G is UyV-separable for all $y \in G$ with $\|y\| < \|x\|$ and for any subgroup $U, V \leq A$. WLOG, let $x = c_1 d_1 \cdots c_n d_n$ where $c_i \in E \backslash A$ and $d_i \in F \backslash A$.

(a) $\|g\| < \|x\|$. Clearly $x \notin S_1 g S_2$. Since $\|g\| < \|x\|$, by induction, there exists $N \triangleleft_f G$ such that $x \notin NS_1 g S_2$. Then $g \notin NS_1 x S_2$.

(b) $\|g\| = \|x\|$. Suppose $g = f_1 e_1 \cdots f_n e_n$, where $e_i \in E \backslash A$ and $f_i \in F \backslash A$. Since G is A-separable (Lemma 2.5), there exists $N \triangleleft_f G$ such that $c_i, e_i, d_i, f_i \notin NA$. Let $\overline{G}$ be as in (c) above. Then $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. This implies $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

Suppose $g = e_1 f_1 \cdots e_n f_n$. Let $u = f_1 \cdots e_n f_n$. If either $e_1 \notin S_1 c_1 A$ or $u \notin A d_1 c_2 \cdots c_n d_n S_2$ then, by induction, we can find $\overline{G}$ such that $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$ and $\overline{e}_1 \notin \overline{S}_1 \overline{c}_1 \overline{A}$ or $\overline{u} \notin \overline{A d_1 c_2 \cdots c_n d_n S_2}$. This implies $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$. So let $e_1 = s_1 c_1 a_1$, where $s_1 \in S_1$ and $a_1 \in A$, and $u = a_2 d_1 c_2 \cdots c_n d_n s_2$, where $a_2 \in A$ and $s_2 \in S_2$. This implies $c_1 a_1 a_2 d_1 c_2 \cdots c_n d_n \notin S_1 x S_2$. Thus, by Lemma 3.4, there exists $N \triangleleft_f G$ such that $c_1 a_1 a_2 d_1 c_2 \cdots c_n d_n \notin NS_1 x S_2$ and $c_i, d_i, e_i, f_i \notin NA$. Let $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$ be as in (c) above. Then we have $\overline{c_1 a_1 a_2 d_1 c_2 \cdots c_n d_n} \notin \overline{S}_1 \overline{x} \overline{S}_2$, $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Thus $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

(c) $\|g\| > \|x\|$. Let $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$ such that $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Since the length of each element in $\overline{S_1 x S_2}$ is equal to $\|\overline{x}\|$ and since $\|\overline{x}\| < \|\overline{g}\|$, $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$. □

THEOREM 3.5. *Let $G = E *_A F$, where A is a finitely generated abelian subgroup. Let $A_1, A_2 \leq E$. Suppose E and F satisfy D1, D2 and D3 in Theorem 3.3 and satisfy the following:*

D4′. For every $U \triangleleft_f A$ and any subgroup S of A_1 or A_2, there exists $L \triangleleft_f E$ such that $L \cap A = U$ and $LA \cap LS = L(A \cap S)$.

If E is $\{A, A_1, A_2\}$-d-separable, then G is $\{A, A_1, A_2\}$-d-separable.

To avoid repetition we omit the proof of Theorem 3.5. The proof is quite long and similar to the proofs of Theorems 3.3 and 3.6. Although we use Theorem 3.5 to prove Theorem 3.6, the proof of Theorem 3.5 is independent of Theorem 3.6.

THEOREM 3.6. *Let $G = E *_A F$, where A is a finitely generated abelian subgroup. Let $A_1 \leq E$ and $A_2 \leq F$. Suppose E and F satisfy D1, D2 and D3 in Theorem 3.3 and satisfy the following:*

D4. *For every* $U \lhd_f A$ *and any subgroup* $S_1 \leq A_1$, *there exists* $L \lhd_f E$ *such that* $L \cap A = U$ *and* $LA \cap LS_1 = L(A \cap S_1)$. *Similarly, for every* $U \lhd_f A$ *and any subgroup* $S_2 \leq A_2$, *there exists* $M \lhd_f F$ *such that* $M \cap A = U$ *and* $MA \cap MS_2 = M(A \cap S_2)$.

If E *is* $\{A, A_1\}$*-d-separable and* F *is* $\{A, A_2\}$*-d-separable, then* G *is* $\{A, A_1, A_2\}$*-d-separable.*

To prove this result we first show the following:

LEMMA 3.7. *Let* $G = E *_A F$ *be as in Theorem 3.6. Let* $S_1 \leq A_1 \leq E$ *and* $S_2 \leq A_2 \leq F$. *Let* $1 \neq z \in A$, $c_i \in E \backslash A$ *and* $d_i \in F \backslash A$. *Then there exists* $N \lhd_f G$ *such that*

(1) *if* $x = c_1 d_1 \cdots c_n d_n$ *and* $c_1 z d_1 \cdots c_n d_n \notin S_1 x S_2$, *then* $c_1 z d_1 \cdots c_n d_n \notin N S_1 x S_2$;

(2) *if* $x = c_1 d_1 \cdots d_{n-1} c_n$ *and* $c_1 z d_1 \cdots d_{n-1} c_n \notin S_1 x S_2$, *then* $c_1 z d_1 \cdots d_{n-1} c_n \notin N S_1 x S_2$;

(3) *if* $x = d_1 c_1 \cdots c_{n-1} d_n$ *and* $d_1 z c_1 \cdots c_{n-1} d_n \notin S_1 x S_2$, *then* $d_1 z c_1 \cdots c_{n-1} d_n \notin N S_1 x S_2$; *and*

(4) *if* $x = d_1 c_1 \cdots d_n c_n$ *and* $d_1 z c_1 \cdots d_n c_n \notin S_1 x S_2$, *then* $d_1 z c_1 \cdots d_n c_n \notin N S_1 x S_2$.

Proof. We shall only prove (4), since the others are similar and relatively simple. Let $d_1 z c_1 \cdots d_n c_n \notin S_1 x S_2$, where $x = d_1 c_1 \cdots d_n c_n$. Clearly $z \notin Z_{A \cap S_1}(d_1) Z_{A \cap S_2}(c_1 \cdots d_n c_n)$. Let $D_1 = Z_{A \cap S_1}(d_1)$, $C_\alpha = Z_{A \cap S_2}(c_\alpha)$ and $D_\beta = Z_{A \cap S_2}(d_\beta)$ where $1 \leq \alpha \leq n$ and $2 \leq \beta \leq n$. Let $T = C_1 \cap D_2 \cap C_2 \cap \cdots \cap D_n \cap C_n$. We shall show $Z_{A \cap S_2}(c_1 \cdots d_n c_n) = T$. Clearly $T \subset Z_{A \cap S_2}(c_1 \cdots d_n c_n)$. For the converse, suppose $s \in Z_{A \cap S_2}(c_1 \cdots d_n c_n)$. Then $s \in A \cap S_2$ and $c_1 \cdots d_n c_n = s^{-1} c_1 \cdots d_n c_n s$. Thus there exist $u_\alpha, v_\beta \in A$, where $1 \leq \alpha \leq n$ and $2 \leq \beta \leq n$, such that

$$c_1 = s^{-1} c_1 u_1, \tag{3.1}$$

$$d_2 = u_1^{-1} d_2 v_2, \tag{3.2}$$

$$\vdots$$

$$d_n = u_{n-1}^{-1} d_n v_n, \text{ and} \tag{3.3}$$

$$c_n = v_n^{-1} c_n s. \tag{3.4}$$

From (3.1), we have $s \sim_E u_1$. This implies from D2 that $s = u_1$. Also, from (3.2), we have $u_1 = v_2$. Continuing this process, we have $v_2 = u_2$, $\ldots$, $u_{n-1} = v_n$ and $v_n = s$. Hence $u_\alpha = v_\beta = s$ for all α, β. This implies that $s \in C_{A \cap S_2}(c_\alpha) = C_\alpha$ and $s \in C_{A \cap S_2}(d_\beta) = D_\beta$. Hence $s \in T$, proving that $Z_{A \cap S_2}(c_1 \cdots d_n c_n) \subset T$. Therefore $T = Z_{A \cap S_2}(c_1 \cdots d_n c_n)$.

Hence $z \notin D_1T$. Since A is a finitely generated abelian group, it is not difficult to find $U \triangleleft_f A$ such that $T \subset U$, $z \notin UD_1$ and $UC_1 \cap UD_2 \cap \cdots \cap UD_n \cap UC_n = U$ (Lemma 4.4 in [11]).

Let $x_0, x_1, \ldots, x_{m_1}$ be coset representatives of $UD_1 \cap S_1$ in $A \cap S_1$ and $y_0, y_1, \ldots, y_{m_2}$ be coset representatives of $U \cap S_2$ in $A \cap S_2$, where $x_0 = 1 = y_0$.

(I) We note that $x_i d_1 x_j^{-1} \notin (UD_1 \cap S_1)d_1(UD_1 \cap S_1)$ for all i, j except $i = j = 0$. For, if $x_i d_1 x_j^{-1} = sd_1s'$ for $s, s' \in UD_1 \cap S_1$, then $d_1^{-1}s^{-1}x_i d_1 = s'x_j$. Hence, by D2, $s^{-1}x_i = s'x_j$. Since x_i and x_j are coset representatives of $UD_1 \cap S_1$ in $A \cap S_1$, we have $x_i = x_j$ and $s^{-1} = s'$. Thus $x_i d_1 x_i^{-1} = sd_1s^{-1}$. This implies $s^{-1}x_i \in Z_{A\cap S_1}(d_1) = D_1 \subset UD_1$. Since $s \in UD_1 \cap S_1$, we have $x_i \in UD_1 \cap S_1$. Hence $x_i = 1 = x_j$, that is, $i = j = 0$. Therefore $x_i d_1 x_j^{-1} \notin (UD_1 \cap S_1)d_1(UD_1 \cap S_1)$ for all i, j except $i = j = 0$.

(II) Let j be fixed. For each α, β where $1 \leq \alpha \leq n$ and $2 \leq \beta \leq n$, if there exist i_β and j_α, depending on α, β, such that $y_{i_\beta} d_\beta y_j^{-1} \in (U \cap S_2)d_\beta(U \cap S_2)$ and $y_{j_\alpha} c_\alpha y_j^{-1} \in (U \cap S_2)c_\alpha(U \cap S_2)$, then $y_j = 1$. To prove this, suppose $y_{i_\beta} d_\beta y_j^{-1} = sd_\beta s'$, where $s, s' \in U \cap S_2$. Then, as in (I), we can show $y_{i_\beta} = y_j$ and $s^{-1} = s'$. Hence $s'y_j \in Z_{S_2\cap A}(d_\beta) = D_\beta \subset UD_\beta$. Since $s' \in U \subset UD_\beta$, we have $y_j \in UD_\beta$. Similarly, since $y_{j_\alpha} c_\alpha y_j^{-1} \in (U \cap S_2)c_\alpha(U \cap S_2)$, $y_j \in UC_\alpha$. Hence $y_j \in \cap_{\beta=2}^n UD_\beta \cap (\cap_{\alpha=1}^n UC_\alpha) = U$. Thus $y_j \in U \cap S_2$. Since y_j is a coset representative of $U \cap S_2$ in $A \cap S_2$, $y_j = 1$.

Since E is $\{A\}$-d-separable and $z \notin UD_1$, there exists $L_1 \triangleleft_f E$ such that $z \notin L_1UD_1$ and $y_i c_\alpha y_j^{-1} \notin L_1(U \cap S_2)c_\alpha(U \cap S_2)$ for all possible i, j such that $y_i c_\alpha y_j^{-1} \notin (U \cap S_2)c_\alpha(U \cap S_2)$. Since F is $\{A\}$-d-separable, there exists $M_1 \triangleleft_f F$ such that $x_i d_1 x_j^{-1} \notin M_1(UD_1 \cap S_1)d_1(UD_1 \cap S_1)$ for all i, j except $i = j = 0$ and $y_i d_\beta y_j^{-1} \notin M_1(U \cap S_2)d_\beta(U \cap S_2)$ for all possible i, j such that $y_i d_\beta y_j^{-1} \notin (U \cap S_2)d_\beta(U \cap S_2)$. By Lemma 2.5, there exists $N_1 \triangleleft_f G$ such that $c_i, d_i \notin N_1A$ for $1 \leq i \leq n$. By D3, there exists $L_2 \triangleleft_f E$ such that $L_2 \cap A = L_1 \cap M_1 \cap N_1$ and, for $x, y \in A$, if $L_2x \sim_{E/L_2} L_2y$ then $L_2x = L_2y$. Similarly, there exists $M_2 \triangleleft_f F$ such that $M_2 \cap A = L_1 \cap M_1 \cap N_1$ and, for $x, y \in A$, if $M_2x \sim_{F/M_2} M_2y$ then $M_2x = M_2y$. By D4, there exists $L_3 \triangleleft_f E$ such that $L_3 \cap A = L_2 \cap A$ and $L_3S_1 \cap L_3A = L_3(S_1 \cap A)$. Similarly, there exists $M_3 \triangleleft_f F$ such that $M_3 \cap A = M_2 \cap A$ and $M_3S_2 \cap M_3A = M_3(S_2 \cap A)$. Let $L = L_1 \cap L_2 \cap L_3 \cap N_1$ and $M = M_1 \cap M_2 \cap M_3 \cap N_1$. Then $L \cap A =$

$L_1 \cap M_1 \cap N_1 = M \cap A$. Let $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$, where $\overline{E} = E/L$ and $\overline{F} = F/M$. We shall show that $\overline{S}_1 \cap \overline{A} = \overline{S_1 \cap A}$. It suffices to show that $S_1 L \cap AL = (S_1 \cap A)L$. Clearly $(S_1 \cap A)L \subset S_1 L \cap AL$. To show the converse, let $sl_1 = al_2$ where $s \in S_1$, $a \in A$ and $l_1, l_2 \in L$. Since $l_1, l_2 \in L \subset L_3$, $sL_3 = aL_3 \in S_1 L_3 \cap AL_3 = (S_1 \cap A)L_3$. Let $sL_3 = aL_3 = dL_3$ where $d \in S_1 \cap A$. Then $d^{-1}a \in L_3 \cap A = L_2 \cap A = L_1 \cap M_1 \cap N_1$. Thus $d^{-1}a \in L_1 \cap L_2 \cap L_3 \cap N_1 = L$. Hence $aL = dL$ for $d \in S_1 \cap A$. This shows that $S_1 L \cap AL \subset (S_1 \cap A)L$. Hence $S_1 L \cap AL = (S_1 \cap A)L$. This proves $\overline{S}_1 \cap \overline{A} = \overline{S_1 \cap A}$ in E/L. Similarly, $\overline{S}_2 \cap \overline{A} = \overline{S_2 \cap A}$. By the choice of N_1, $\|\overline{x}\| = \|x\|$ and if $\overline{x} \sim_{\overline{E} \cup \overline{F}} \overline{y}$ for $x, y \in A$, then $\overline{x} = \overline{y}$. Using these properties, we shall show that $\overline{d_1 z c_1 \cdots d_n c_n} \notin \overline{S_1 x S_2}$.

Suppose $\overline{d_1 z c_1 \cdots d_n c_n} = \overline{s_1 d_1 c_1 \cdots d_n c_n s_2}$ for some $s_i \in S_i$. Then we have

$$\overline{d}_1 \overline{z} = \overline{s}_1 \overline{d}_1 \overline{a}_1, \tag{3.5}$$

$$\overline{c}_1 = \overline{a}_1^{-1} \overline{c}_1 \overline{\alpha}_1, \tag{3.6}$$

$$\overline{d}_2 = \overline{\alpha}_1^{-1} \overline{d}_2 \overline{a}_2, \tag{3.7}$$

$$\vdots$$

$$\overline{d}_n = \overline{\alpha}_{n-1}^{-1} \overline{d}_n \overline{a}_n, \text{ and} \tag{3.8}$$

$$\overline{c}_n = \overline{a}_n^{-1} \overline{c}_n \overline{s}_2, \tag{3.9}$$

for some $\overline{a}_i, \overline{\alpha}_i \in \overline{A}$. From (3.6), $\overline{a}_1 \sim_{\overline{E}} \overline{\alpha}_1$. Hence, by the choice of L_2, $\overline{a}_1 = \overline{\alpha}_1$. Similarly, from equations (3.7), $\ldots$, (3.9), we have $\overline{\alpha}_1 = \overline{a}_2 = \cdots = \overline{a}_n = \overline{s}_2$. Hence $\overline{s}_2 \in \overline{S}_2 \cap \overline{A} = \overline{S_2 \cap A}$. Let $s \in S_2 \cap A$ such that $\overline{s} = \overline{s}_2$. Suppose $s = k_1 y_j$ where $k_1 \in U \cap S_2$. From (3.6), $\overline{c}_1^{-1} \overline{k}_1 \overline{y}_j \overline{c}_1 = \overline{k}_1 \overline{y}_j$. Thus $\overline{y}_j \overline{c}_1 \overline{y}_j^{-1} = \overline{k}_1^{-1} \overline{c}_1 \overline{k}_1$. Hence $y_j c_1 y_j^{-1} \in L(U \cap S_2) c_1 (U \cap S_2)$. Similarly, $y_j c_\alpha y_j^{-1} \in L(U \cap S_2) c_\alpha (U \cap S_2)$ and $y_j d_\beta y_j^{-1} \in M(U \cap S_2) d_\beta (U \cap S_2)$ for $1 \leq \alpha \leq n$ and $2 \leq \beta \leq n$. Since $L \subset L_1$ and $M \subset M_1$, $y_j c_\alpha y_j^{-1} \in L_1 (U \cap S_2) c_\alpha (U \cap S_2)$ and $y_j d_\beta y_j^{-1} \in M_1 (U \cap S_2) d_\beta (U \cap S_2)$. Thus, by the choice of L_1 and M_1, we have $y_j c_\alpha y_j^{-1} \in (U \cap S_2) c_\alpha (U \cap S_2)$ and $y_j d_\beta y_j^{-1} \in (U \cap S_2) d_\beta (U \cap S_2)$. It follows from (II) that $y_j = 1$. Hence $s = k_1 \in U \cap S_2$.

From (3.5), $\overline{d}_1^{-1} \overline{s}_1 \overline{d}_1 = \overline{z} \overline{a}_1^{-1}$. Hence $\overline{z} \overline{a}_1^{-1} = \overline{s}_1 \in \overline{S}_1 \cap \overline{A} = \overline{S_1 \cap A}$. Let $\overline{s}_1 = \overline{s'}$ where $s' \in S_1 \cap A$. Let $s' = k_2 x_i$ where $k_2 \in UD_1 \cap S_1$. Since $\overline{d}_1^{-1} \overline{s}_1 \overline{d}_1 = \overline{s}_1$, $\overline{d}_1^{-1} \overline{s'} \overline{d}_1 = \overline{s'}$. Hence $\overline{d}_1^{-1} \overline{k}_2 \overline{x}_i \overline{d}_1 = \overline{k}_2 \overline{x}_i$, thus $\overline{x}_i \overline{d}_1 \overline{x}_i^{-1} = \overline{k}_2^{-1} \overline{d}_1 \overline{k}_2$. This implies $x_i d_1 x_i^{-1} \in M(UD_1 \cap S_1) d_1 (UD_1 \cap S_1)$. Since $M \subset$

M_1, by the choice of M_1, we have $x_i d_1 x_i^{-1} \in (UD_1 \cap S_1) d_1 (UD_1 \cap S_1)$. It follows from (I) that $x_i = 1$. Hence $s' = k_2 \in UD_1 \cap S_1$.

Consequently $\overline{z} = \overline{s}_1 \overline{a}_1 = \overline{s'} \overline{s} \in (\overline{UD_1 \cap S_1})(\overline{U \cap S_2}) \subset \overline{UD_1 U} = \overline{UD_1}$, contradicting the choice of L_1. This shows that $\overline{d_1 z c_1 \cdots d_n c_n} \notin \overline{S_1 x S_2}$. Since $\overline{G}$ is $\mathcal{RF}$, and since $\overline{S_1 x S_2}$ is finite, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{d_1 z c_1 \cdots d_n c_n} \notin \overline{N S_1 x S_2}$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_f G$ and $d_1 z c_1 \cdots d_n c_n \notin N S_1 x S_2$ as required. □

Proof of Theorem 3.6: Since E is $\{A, A_1\}$-d-separable, by Theorem 3.5, G is $\{A, A_1\}$-d-separable. Similarly G is $\{A, A_2\}$-d-separable. Hence we need to show that G is $\{A_1, A_2\}$-d-separable. Let $g, x \in G$ such that $g \notin S_1 x S_2$ where $S_1 \leq A_1 \leq E$ and $S_2 \leq A_2 \leq F$. We shall find $L \lhd_f E$ and $M \lhd_f F$ with $L \cap A = M \cap A$ such that, in $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$ where $\overline{E} = E/L$ and $\overline{F} = F/M$, $\|\overline{x}\| = \|x\|$, $\|\overline{g}\| = \|g\|$ and $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$. Since $\overline{G}$ is $\mathcal{RF}$ and $\overline{S}_1 \overline{x} \overline{S}_2$ is finite, we can find $N \lhd_f G$ such that $g \notin N S_1 x S_2$.

Case 1. $x \in E$ (similarly $x \in F$).

(a) $g \in E$. Clearly $g \notin S_1 x (S_2 \cap A)$. Since E is $\{A, A_1\}$-d-separable, there exists $L \lhd_f E$ such that $g \notin L S_1 x (S_2 \cap A)$. By D4, there exists $M \lhd_f F$ such that $M \cap A = L \cap A$ such that $MA \cap MS_2 = M(A \cap S_2)$. Let $\overline{G}$ be as above. Then, by the choice of L, $\overline{g} \notin \overline{S_1 x (S_2 \cap A)}$. If $\overline{g} = \overline{s_1 x s_2} \in \overline{S_1 x S_2}$, then $(\overline{s_1 x})^{-1} \overline{g} = \overline{s}_2 \in \overline{E} \cap \overline{F} = \overline{A}$. Hence $\overline{s}_2 \in \overline{A} \cap \overline{S}_2 = \overline{S_2 \cap A}$. Thus $\overline{g} \in \overline{S_1 x (S_2 \cap A)}$, a contradiction. Hence $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

(b) $g \in F \backslash A$. Suppose $g \in AS_2$, say $g = au$ for $a \in A$ and $u \in S_2$. Then $g \notin S_1 x S_2$ implies $a \notin S_1 x S_2$. Since $a \in A \subset E$, by (a) above, we can find $\overline{G}$ such that $\overline{a} \notin \overline{S}_1 \overline{x} \overline{S}_2$. Hence $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

Suppose $g \notin AS_2$. Since F is $\{A, A_2\}$-d-separable, there exists $M \lhd_f F$ such that $g \notin MAS_2$. Let $L \lhd_f E$ such that $L \cap A = M \cap A$. Let $\overline{G}$ be as above. If $\overline{g} = \overline{s_1 x s_2} \in \overline{S_1 x S_2}$, then $\overline{g s_2^{-1}} = \overline{s_1 x} \in \overline{E} \cap \overline{F} = \overline{A}$. Hence $\overline{g} \in \overline{AS_2}$, a contradiction. Thus $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

(c) $g = ef$, where $e \in E \backslash A$ and $f \in F \backslash A$. Suppose $e \notin S_1 x A$ (or similarly $f \notin AS_2$). Since E is $\{A, A_1\}$-d-separable, there exists $L_1 \lhd_f E$ such that $e \notin L_1 S_1 x A$. By D3, there exists $M_1 \lhd_f F$ such that $M_1 \cap A = L_1 \cap A$. Since G is A-separable by Lemma 2.5, there exists $N \lhd_f G$ such that $e, f \notin NA$. Let $L = L_1 \cap N$, $M = M_1 \cap N$ and $\overline{G}$ be as above. Then $\|\overline{g}\| = 2$ and $\overline{e} \notin \overline{S_1 x A}$. If $\overline{g} = \overline{e f} \in \overline{S}_1 \overline{x} \overline{S}_2$, then $\overline{e} \in \overline{S}_1 \overline{x} \overline{A}$ and $\overline{f} \in \overline{AS_2}$, contradicting the choice of L_1. Hence $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

Suppose $e = s_1 x a_1$ and $f = a_2 s_2$ where $a_1, a_2 \in A$, $s_1 \in S_1$ and $s_2 \in S_2$. Then $s_1^{-1} g s_2^{-1} = x a_1 a_2 \notin S_1 x S_2$. Since $x a_1 a_2 \in E$, by (a) above there exists $\overline{G}$ such that $\overline{x a_1 a_2} \notin \overline{S_1 x S_2}$. Then $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

(d) $g = fe$ where $e \in E\backslash A$ and $f \in F\backslash A$ or $\|g\| \geq 3$. Since G is A-separable by Lemma 2.5, we can find $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$ such that $\|\overline{g}\| = \|g\|$. Then $\overline{g} \notin \overline{S}_1\overline{x}\overline{S}_2$.

Case 2. $\|x\| \geq 2$. By induction, we assume G is UyV-separable for all $y \in G$ with $\|y\| < \|x\|$ and for any subgroups $U \leq A_1$ and $V \leq A_2$. If $\|g\| < \|x\|$ then, by induction, we can find $N \lhd_f G$ such that $x \notin NS_1gS_2$. Hence $g \notin NS_1xS_2$. Thus we need only consider the cases $\|g\| \geq \|x\|$.
Subcase 1. Suppose $x = c_1d_1 \cdots c_nd_n$ where $c_i \in E\backslash A$ and $d_i \in F\backslash A$.

(a) $\|g\| = \|x\|$.

(i) Suppose $g = f_1e_1 \cdots f_ne_n$, where $e_i \in E\backslash A$ and $f_i \in F\backslash A$. If $c_1 \notin S_1A$ then there exists $L \lhd_f E$ such that $c_1 \notin LS_1A$. Moreover, L can be chosen such that $c_i, e_i \notin LA$. Let $M \lhd_f F$ such that $d_i, f_i \notin MA$ and $L \cap A = M \cap A$. Thus, in $\overline{G} = E/L *_{\overline{A}} T/M$, we have $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$ and $\overline{c}_1 \notin \overline{S}_1\overline{A}$. This implies $\overline{g} \notin \overline{S}_1\overline{x}\overline{S}_2$. So let $c_1 = sa$ where $s \in S_1$ and $a \in A$. This implies $g \notin S_1ad_1 \cdots c_nd_nS_2$. Since $ad_1 \in F$, we have $\|ad_1 \cdots c_nd_n\| < \|x\|$. By induction, there exists $\overline{G}$ such that $\overline{g} \notin \overline{S}_1\overline{ad_1 \cdots c_nd_n}\overline{S}_2$, $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$, whence $\overline{g} \notin \overline{S}_1\overline{x}\overline{S}_2$.

(ii) Suppose $g = e_1f_1 \cdots e_nf_n$, where $e_i \in E\backslash A$ and $f_i \in F\backslash A$. Let $u = f_1 \cdots e_nf_n$. If $e_1 \notin S_1c_1A$ or $u \notin Ad_1c_2 \cdots c_nd_nS_2$ then, by Theorem 3.5, we can find $\overline{G}$ such that $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$ and $\overline{e}_1 \notin \overline{S}_1\overline{c}_1\overline{A}$ or $\overline{u} \notin \overline{Ad_1c_2 \cdots c_nd_nS_2}$. This implies $\overline{g} \notin \overline{S}_1\overline{x}\overline{S}_2$. So let $e_1 = s_1c_1a_1$, where $s_1 \in S_1$ and $a_1 \in A$, and $u = a_2d_1c_2 \cdots c_nd_ns_2$, where $a_2 \in A$ and $s_2 \in S_2$. This implies $c_1a_1a_2d_1c_2 \cdots c_nd_n \notin S_1xS_2$. Thus, by Lemma 3.7, there exists $N \lhd_f G$ such that $c_1a_1a_2d_1c_2 \cdots c_nd_n \notin NS_1xS_2$ and $c_i, d_i, e_i, f_i \notin NA$. Let $\overline{E} = E/(N \cap E)$, $\overline{F} = F/(N \cap F)$ and $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$. Then we have $\overline{c_1a_1a_2d_1c_2 \cdots c_nd_n} \notin \overline{S}_1\overline{x}\overline{S}_2$, $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Thus $\overline{g} \notin \overline{S}_1\overline{x}\overline{S}_2$.

(b) $\|g\| > \|x\|$. Let $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$ such that $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Since the length of each element in $\overline{S_1xS_2}$ is equal to $\|\overline{x}\|$ and since $\|\overline{x}\| < \|\overline{g}\|$, $\overline{g} \notin \overline{S}_1\overline{x}\overline{S}_2$.
Subcase 2. Suppose $x = d_1c_1 \cdots d_nc_n$ where $c_i \in E\backslash A$ and $d_i \in F\backslash A$.

(a) $\|g\| = \|x\|$. Suppose $g = e_1f_1 \cdots e_nf_n$, where $e_i \in E\backslash A$ and $f_i \in F\backslash A$. We can find $\overline{G}$ such that $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Since $S_1 \leq E$ and $S_2 \leq F$, $\overline{g} = \overline{e_1f_1 \cdots e_nf_n} \notin \overline{S}_1\overline{x}\overline{S}_2$.

Suppose $g = f_1e_1 \cdots f_ne_n$, where $e_i \in E\backslash A$ and $f_i \in F\backslash A$.

(i) Suppose $f_1 \notin S_1d_1A$, equivalently, $f_1 \notin (S_1 \cap A)d_1A$. Since F is $\{A\}$-d-separable, there exists $M_1 \lhd_f F$ such that $f_1 \notin M_1(S_1 \cap A)d_1A$. Let $L_1 \lhd_f E$ such that $L_1 \cap A = M_1 \cap A$. By Lemma 2.5, there exists $N_1 \lhd_f G$ such that $e_i, f_i, c_i, d_i \notin N_1A$. By D4, we can find $L_2 \lhd_f E$ such that

$L_2 \cap A = L_1 \cap N_1 \cap A$ and $L_2 A \cap L_2 S_1 = L_2(A \cap S_1)$. Let $L = L_1 \cap L_2 \cap N_1$ and $M = M_1 \cap N_1$. Then, in $\overline{G} = E/L *_{\overline{A}} F/M$, we have $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$ and $\overline{f}_1 \notin \overline{(S_1 \cap A) d_1 A}$. If $\overline{g} = \overline{f_1 e_1 \cdots f_n e_n} \in \overline{S}_1 \overline{x} \overline{S}_2$, then $\overline{f}_1 = \overline{s} \overline{d}_1 \overline{a}$ for some $s \in S_1$ and $a \in A$. Since $S_1 \subset E$, we have $\overline{s} \in \overline{S}_1 \cap \overline{A} = \overline{S_1 \cap A}$. This implies $\overline{f}_1 \in \overline{(S_1 \cap A) d_1 A}$, contradicting the choice of M_1. Hence $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

(ii) Suppose $e_1 \cdots f_n e_n \notin A c_1 \cdots d_n c_n S_2$. By Theorem 3.5, there exists $N_1 \lhd_f G$ such that $e_1 \cdots f_n e_n \notin N_1 A c_1 \cdots d_n c_n S_2$. Let $N_2 \lhd_f G$ such that $c_i, d_i, e_i, f_i \notin N_2 A$. Let $L = N_1 \cap N_2 \cap E$, $M = N_1 \cap N_2 \cap F$ and $\overline{G} = E/L *_{\overline{A}} F/M$. Then we have $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$ and $\overline{e_1 \cdots f_n e_n} \notin \overline{A c_1 \cdots d_n c_n S_2}$. This implies $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

(iii) Suppose $f_1 = s_1 d_1 a_1$ and $e_1 \cdots f_n e_n = a_2 c_1 \cdots d_n c_n s_2$ for some $a_1, a_2 \in A$, $s_1 \in S_1$ and $s_2 \in S_2$. Then $d_1 a_1 a_2 c_1 \cdots d_n c_n \notin S_1 d_1 c_1 \cdots d_n c_n S_2$. By Lemma 3.7, there exists $N_1 \lhd_f G$ such that $d_1 a_1 a_2 c_1 \cdots d_n c_n \notin N_1 S_1 d_1 c_1 \cdots d_n c_n S_2$ and $c_i, d_i, e_i, f_i \notin N_1 A$. Let $L = N_1 \cap E$, $M = N_1 \cap F$ and $\overline{G} = E/L *_{\overline{A}} F/M$. Then we have $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$ and $\overline{d_1 a_1 a_2 c_1 \cdots d_n c_n} \notin \overline{S}_1 \overline{d_1 c_1 \cdots d_n c_n} \overline{S}_2$. This implies $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

(b) $\|g\| = \|x\| + 1$. Suppose $g = e_1 f_1 \cdots e_n f_n e_{n+1}$, where $e_i \in E \backslash A$ and $f_i \in F \backslash A$. If $e_1 \notin S_1 A$, then we can find $\overline{G}$ such that $\overline{e}_1 \notin \overline{S_1 A}$, $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Then $\overline{g} = \overline{e_1 f_1 \cdots e_n f_n e_{n+1}} \notin \overline{S}_1 \overline{x} \overline{S}_2$. If $e_1 = sa$ for some $s \in S_1$ and $a \in A$, then $s^{-1} g = a f_1 \cdots e_n f_n e_{n+1} \notin S_1 x S_2$. By (a) above, we can find $\overline{G}$ such that $\overline{s^{-1} g} \notin \overline{S_1 x S_2}$. Hence $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

The case $g = f_1 e_1 \cdots f_n e_n f_{n+1}$ can be similarly treated according to either $f_{n+1} \in A S_2$ or $f_{n+1} \notin A S_2$.

(c) $\|g\| \geq \|x\| + 2$. Suppose $g = e_1 f_1 \cdots e_n f_n e_{n+1} f_{n+1}$. If $g \in S_1 x S_2$ then we have $e_1 \in S_1 A$ and $f_{n+1} \in A S_2$. Hence, as in (b) above, we can find $\overline{G}$ such that $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

Suppose $g = f_1 e_1 \cdots f_n e_n f_{n+1} e_{n+1}$ or $\|g\| > \|x\| + 2$. We can find $\overline{G} = \overline{E} *_{\overline{A}} \overline{F}$ such that $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Then $\overline{g} \notin \overline{S}_1 \overline{x} \overline{S}_2$.

Subcase 3. Suppose $x = d_1 c_1 \cdots d_n$ where $c_i \in E \backslash A$ and $d_i \in F \backslash A$.

Subcase 4. Suppose $x = c_1 \cdots d_n c_n$ where $c_i \in E \backslash A$ and $d_i \in F \backslash A$.

The above two cases can be proved by a similar method. □

LEMMA 3.8. *Let G be a tree product of m central subgroup separable groups amalgamating central edge groups. Let H be a finitely generated central subgroup of a vertex group. For every $U \lhd_f H$, there exists $N \lhd_f G$ such that $N \cap H = U$ and, in $\overline{G} = G/N$, if $\overline{x} \sim_{\overline{G}} \overline{y}$ for any $x, y \in H$ then $\overline{x} = \overline{y}$.*

Proof. We shall prove this by induction on m. Clearly the lemma is true for $m = 1$. Let $G = A *_C T$ where A is the vertex group of an extremal vertex of the tree and T is the tree product on the remaining $m-1$ vertices and C is the central edge subgroup of A joining to its adjacent vertex group of T. By induction, we can assume T has the property stated in the lemma.

Case 1. $H \subset T$. By induction, there exists $M \lhd_f T$ such that $M \cap H = U$ and $\tilde{h}_1 \not\sim_{\tilde{T}} \tilde{h}_2$ in $\tilde{T} = T/M$ for all $\tilde{h}_1 \neq \tilde{h}_2$ in $\tilde{H}$. Let $L \lhd_f A$ such that $L \cap C = M \cap C$. Let $\tilde{G} = \tilde{A} *_{\tilde{C}} \tilde{T}$, where $\tilde{A} = A/L$. We first show that $\tilde{h} \not\sim_{\tilde{G}} \tilde{k}$ for all $\tilde{h} \neq \tilde{k}$ in $\tilde{H}$. Suppose $\tilde{h} \sim_{\tilde{G}} \tilde{k}$.

(1) If $\tilde{h} \sim_{\tilde{G}} \tilde{c}$ for some $c \in C$ then, by Theorem 2.1, there exist $\tilde{c}_1, \ldots, \tilde{c}_r \in \tilde{C}$ such that $\tilde{h} \sim_{\tilde{T}} \tilde{c}_1 \sim_{\tilde{A}} \tilde{c}_2 \sim_{\tilde{T}} \cdots \sim_{\tilde{A}} \tilde{c}_r = \tilde{c}$. Since $C \subset Z(A)$, $\tilde{c}_i \sim_{\tilde{A}} \tilde{c}_{i+1}$ implies $\tilde{c}_i = \tilde{c}_{i+1}$. Hence $\tilde{h} \sim_{\tilde{T}} \tilde{c}$. Similarly, $\tilde{k} \sim_{\tilde{T}} \tilde{c}$. Thus $\tilde{h} \sim_{\tilde{T}} \tilde{k}$, implying $\tilde{h} = \tilde{k}$ by induction.

(2) If $\tilde{h} \not\sim_{\tilde{G}} \tilde{c}$ for any $c \in C$, then $\tilde{h}$ has the minimal length 1 in its conjugacy class in $\tilde{G}$. Thus, by Theorem 2.1, $\tilde{h} \sim_{\tilde{T}} \tilde{k}$, implying $\tilde{h} = \tilde{k}$ by induction.

This proves that $\tilde{h} \not\sim_{\tilde{G}} \tilde{k}$ for all $\tilde{h} \neq \tilde{k}$ in $\tilde{H}$. Since $\tilde{H}$ is finite and $\tilde{G}$ is conjugacy separable (Theorem 2.2), there exists $\tilde{N} \lhd_f \tilde{G}$ such that, in $\tilde{G}/\tilde{N}$, $\tilde{N} \cap \tilde{H} = 1$ and $\tilde{N}\tilde{h} \not\sim_{\tilde{G}/\tilde{N}} \tilde{N}\tilde{k}$ for all $\tilde{h} \neq \tilde{k}$ in $\tilde{H}$. Let N be the preimage of $\tilde{N}$ in G. Then $N \lhd_f G$ and $N \cap H = U$. Moreover, if $\overline{h} \sim_{\overline{G}} \overline{k}$ for $h, k \in H$, where $\overline{G} = G/N$, then $\tilde{N}\tilde{h} \sim_{\tilde{G}/\tilde{N}} \tilde{N}\tilde{k}$. Hence $\tilde{h} = \tilde{k}$ by the choice of $\tilde{N}$. This implies $hk^{-1} \in M \cap H = U \subset N$. hence $\overline{h} = \overline{k}$, as required.

Case 2. $H \subset A$. There exists $L \lhd_f A$ such that $L \cap H = U$. By induction, there exists $M \lhd_f T$ such that $M \cap C = L \cap C$ and $\tilde{c}_1 \not\sim_{\tilde{T}} \tilde{c}_2$ in $\tilde{T} = T/M$ for all $\tilde{c}_1 \neq \tilde{c}_2$ in $\tilde{C}$. Consider $\tilde{G} = \tilde{A} *_{\tilde{C}} \tilde{T}$, where $\tilde{A} = A/L$. As in Case 1, we can show that $\tilde{h} \not\sim_{\tilde{G}} \tilde{k}$ for all $\tilde{h} \neq \tilde{k}$ in $\tilde{H}$. Now, choose $N \lhd_f G$ as in Case 1. Then $N \cap H = U$ and, in $\overline{G} = G/N$, $\overline{h} \not\sim_{\overline{G}} \overline{k}$ for all $\overline{h} \neq \overline{k}$ in $\overline{H}$. □

The following is one of the main results of [11]

THEOREM 3.9. *Let G be a tree product of m central subgroup separable groups amalgamating central edge groups. Let H and K be finitely generated central subgroups of some vertex groups. Then G is HxK-separable for $x \in G$.*

Proof. We prove this by induction on m. Clearly the theorem is true for $m = 1$. Let $G = A *_C T$ as in the proof of Lemma 3.8. Then A and T satisfy D2, D3 and D4 by Lemma 2.10, Lemma 3.8 and Lemma 2.9. By induction, we assume that any tree product T_1 of at most $m - 1$ central subgroup separable groups, amalgamating central edge groups, is UtV-separable for any $t \in T_1$ and for any central subgroups U, V in the vertex groups of T_1.

Case 1. $H, K \subset T$ (similarly $H, K \subset A$). By induction, T is $\{C, H, K\}$-d-separable. Hence, by Theorem 3.5, G is $\{C, H, K\}$-d-separable.

Case 2. $H \subset T$ and $K \subset A$. By induction, T is $\{C, H\}$-d-separable and A is $\{C, K\}$-d-separable. Hence, by Theorem 3.6, G is $\{C, H, K\}$-d-separable.

In particular, G is HxK-separable for $x \in G$. □

4. Double coset separability of polygonal products

Let P be a polygonal product of groups $G_1, \ldots, G_m$ ($m \geq 4$), amalgamating finitely generated central subgroups $H_1, \ldots, H_m$, with trivial intersections, where $H_{i-1}, H_i \subset G_i$ and $H_m = H_0$. Then the *reduced polygonal product* P_0 is the subgroup generated by $H_1, \ldots, H_m$, which is the polygonal product of finitely generated abelian groups $A_1, \ldots, A_m$, amalgamating $H_1, \ldots, H_m$, where $A_i = H_{i-1} \times H_i$ and the subscripts i are taken modulo m. In this case, the polygonal product P is obtained by

$$P = (\cdots((P_0 *_{A_1} G_1) *_{A_2} G_2) \cdots) *_{A_m} G_m,$$

where $A_i = H_{i-1} \times H_i$ and the subscripts i are taken modulo m. Throughout this section, we assume that each G_i is central subgroup separable and $A_i \subset Z(G_i)$, where $A_i = H_{i-1} \times H_i$ is finitely generated abelian and $H_0 = H_m$.

In this section we show that P is $A_i x A_j$-separable for $x \in P$. We first show that the reduced polygonal product P_0 is $A_i x A_j$-separable for $x \in P_0$. For this purpose, we put $P_0 = E *_H F$, where $H = H_0 * H_2$ and

$$(4.1) \qquad E = A_1 *_{H_1} A_2 = (H_0 * H_2) \times H_1$$

$$(4.2) \qquad F = A_m *_{H_{m-1}} \cdots *_{H_3} A_3.$$

LEMMA 4.1. *Let E, H be as above. Let $U \leq A_1$. For each $S \triangleleft_f H$, there exists $L \triangleleft_f E$ such that $L \cap H = S$, $LU \cap LH_0 = L(U \cap H_0)$ and $LA_1 \cap LH = L(A_1 \cap H) = LH_0$.*

Proof. Let $\tilde{E} = E/S$. Then $\tilde{E} \cong \tilde{H} \times H_1$, where $\tilde{H} = H/S$. Since H_1 is finitely generated abelian, H_1 is subgroup separable [15]. Hence the finite extension, $\tilde{E} \cong \tilde{H} \times H_1$, of H_1 is subgroup separable. This implies that $\tilde{E}$ is $\tilde{U}$-separable and $\tilde{A}_1$-separable. Thus, there exists $\tilde{L} \triangleleft_f \tilde{E}$ such that $\tilde{h} \notin \tilde{L}\tilde{U}$ for all $\tilde{h} \in \tilde{H}_0 \backslash \tilde{U}$ and $\tilde{k} \notin \tilde{L}\tilde{A}_1$ for all $\tilde{k} \in \tilde{H} \backslash \tilde{A}_1$. Let L be the preimage of $\tilde{L}$ in E. Then $L \triangleleft_f E$ and $L \cap H = S$, $LU \cap LH_0 = L(U \cap H_0)$ and $LA_1 \cap LH = L(A_1 \cap H)$. □

LEMMA 4.2. *Let P_0, E, F be as above. Then P_0, E, F are H-finite.*

Proof. There is a natural homomorphism $\pi : P_0 \to H = H_0 * H_2$ by mapping $\pi(h) = h$ for $h \in H_0 \cup H_2$ and $\pi(k) = 1$ for $k \in H_j$ where $j \neq 0, 2$. For a given $U \triangleleft_f H$, let $N = \pi^{-1}(U)$. Then $N \triangleleft_f P_0$ and $N \cap H = U$. Hence P_0 is H-finite. Similarly, E and F are H-finite. □

LEMMA 4.3. *Let P_0, E, F be as above. Then P_0, E, F are H-separable.*

Proof. Since each A_i is finitely generated abelian, A_i is polycyclic. Then, as in the proof of Theorem 2.11 in [9], we can prove that E and F are H-separable. By Lemma 4.2, E and F are H-finite. This implies from Lemma 2.5 that $P_0 = E *_H F$ is H-separable. □

For the following three results, the cases when the H_i are cyclic were considered in Lemma 3.1, 3.3 and 3.4 in [10], respectively. Since their proofs in [10] can be easily applied to our cases, we omit the proofs of the next three lemmas.

LEMMA 4.4. *Let $F = G_1 *_{H_1} G_2 *_{H_2} \cdots *_{H_{m-1}} G_m$ $(m \geq 2)$, where $H_{i-1}, H_i \subset Z(G_i)$ and $H_{i-1} \cap H_i = 1$. Let $H = H_0 * H_m$. If $x \sim_F y$ for $x, y \in H$, then $x \sim_H y$.*

LEMMA 4.5. *Let $F = G_1 *_{H_1} G_2 *_{H_2} \cdots *_{H_{m-1}} G_m$ $(m \geq 2)$, where $H_{i-1}, H_i \subset Z(G_i)$ and $H_{i-1} \cap H_i = 1$. Let $H = H_0 * H_m$. If $[h, f] = 1$ for $1 \neq h \in H_0$ and $f \in F$, then $f \in G_1$ and $[c, f] = 1$ for all $c \in H_0$.*

LEMMA 4.6. *Let P be a polygonal product of central subgroup separable groups $G_1, G_2, \ldots, G_m$ $(m \geq 4)$, amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections. Let $1 \neq h_0 \in H_0$, $1 \neq h_1 \in H_1$ and $p \in P$.*
(1) If $[h_0, p] = 1$ then $[h, p] = 1$ for all $h \in H_0$; hence $Z_{G_1}(p) \cap H_0 = H_0$.
(2) If $h_0 h_1 \in Z_{G_1}(p)$ then $p \in G_1$; hence $Z_{G_1}(p) \cap (H_0 \times H_1) = H_0 \times H_1$.

LEMMA 4.7. *Let P be a polygonal product of central subgroup separable groups $G_1, G_2, \ldots, G_m$ $(m \geq 4)$, amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections. For any x, y in central subgroups of some vertex groups, if $x \sim_P y$ then $x = y$.*

Proof. WLOG, let $x \in Z(G_1)$. Let $E = G_1 *_{H_1} G_2$, $F = G_m *_{H_{m-1}} \cdots *_{H_3} G_3$ and $H = H_0 * H_2$. Then $P = E *_H F$.

(1) Suppose $x \sim_P h$ for some $h \in H$. We may assume that h is cyclically reduced in $H = H_0 * H_2$. By Theorem 2.1 (1), there exist cyclically reduced elements $h_i \in H$ such that $x \sim_E h_1 \sim_F h_2 \sim_E \cdots \sim_E h_r \sim_F h$. Since the h_i are cyclically reduced in $H = H_0 * H_2$, $\|h_i\| = 1$ or $\|h_i\| = 2n$. Consider $x \sim_E h_1$. Since $x \in G_1$ and $E = G_1 *_{H_1} G_2$, by Theorem 2.1, we must have $\|h_1\| = 1$. Hence $h_1 \in H_0$ or $h_1 \in H_2$. Thus, by Lemma 2.10, we have $x = h_1$. Hence $x = h_1 \in G_1 \cap H = H_0$. Now, consider $x = h_1 \sim_F h_2$. Then, as before, we have $\|h_2\| = 1$ and $x = h_2 \in H_0$. Inductively, we have $x = h$.

Since $x \sim_P y$, $y \sim_P h$. Then, as in above, $y \sim_{E(F)} h_1$ for some cyclically reduced $h_1 \in H$. Since $y \in Z(G_i)$, as before, $\|h_1\| = 1$, that is $h_1 \in H_0$ or $h_1 \in H_2$. Thus, by Lemma 2.10, we have $y = h_1$. Then, as above, $y \sim_P h$ implies $y = h$, proving that $x = h = y$.

(2) Suppose $x \not\sim_P h$ for any $h \in H$. Then x has the minimal length 1 in its conjugacy class in $P = E *_H F$. By Theorem 2.1, $x \sim_P y$ implies that $x, y \in E$ and $x \sim_E y$. If $x \in H_1 \subset Z(E)$, then $x = y$. If $x \notin H_1$, then x has the minimal length 1 in its conjugacy class in $E = G_1 *_{H_1} G_2$. Thus, by Theorem 2.1, $x, y \in G_1$ and $x \sim_{G_1} y$. Hence $x = y$. □

THEOREM 4.8. *Let P_0 be as above. Then P_0 is conjugacy separable.*

Proof. We note that the reduced polygonal product P_0, which is a polygonal product of finitely generated abelian groups $A_1, A_2, \ldots, A_m$, where $A_i = H_{i-1} \times H_i$, amalgamating subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections, is a graph product (see [8]) of the finitely generated abelian groups $H_1, H_2, \ldots, H_m$. Hence P_0 is c.s. by [5, Theorem 5.17]. □

THEOREM 4.9. *Let P_0 be as above. Then P_0 is $\{A_i\}$-d-separable.*

Proof. It suffices to show that P_0 is $\{A_1\}$-d-separable. Let $U, V \leq A_1$ and $g \notin UxV$, where $g, x \in P_0$. Let E, F, H be as in (4.1) and (4.2) above. Then $P_0 = E *_H F$. We shall find $L \lhd_f E$ and $M \lhd_f F$ with $L \cap H = M \cap H$ such that, in $\overline{P}_0 = \overline{E} *_{\overline{H}} \overline{F}$ where $\overline{E} = E/L$ and $\overline{F} = F/M$, $\|\overline{x}\| = \|x\|$, $\|\overline{g}\| = \|g\|$ and $\overline{g} \notin \overline{U}\overline{x}\overline{V}$. Since $\overline{P}_0$ is $\mathcal{RF}$ and $\overline{U}\overline{x}\overline{V}$ is finite, we can find $N \lhd_f G$ such that $g \notin NUxV$.

Case 1. $x \in E$. By Lemma 4.2 and 4.3, E and F are H-finite and H-separable. Since E is UxV-separable by Theorem 3.9, by Lemma 3.1, P_0 is UxV-separable.

Case 2. $x \in F \backslash H$.

(a) $g \in E$. Since $g \notin UxV$, clearly $x \notin UgV$. By Case 1, we can find $N \triangleleft_f P_0$ such that $x \notin NUgV$. Hence $g \notin NUxV$.

(b) $g \in F \backslash H$.

(1) Suppose $g \notin H_0xH_0$. By Lemma 4.3 and Theorem 3.9, there exists $M \triangleleft_f F$ such that $g, x \notin MH$ and $g \notin MH_0xH_0$. By Lemma 4.1, there exists $L \triangleleft_f E$ such that $L \cap H = M \cap H$ and $LA_1 \cap LH = L(A_1 \cap H)$. Let $\overline{P}_0 = E/L *_{\overline{H}} F/M$. We shall show that $\overline{g} \notin \overline{UxV}$. If $\overline{g} = \overline{uxv}$ for $u \in U$ and $v \in V$, then $\overline{u} \in \overline{H} \cap \overline{A}_1 = \overline{H \cap A_1} = \overline{H}_0$ and $\overline{v} \in \overline{H} \cap \overline{A}_1 = \overline{H}_0$. Hence $\overline{g} = \overline{uxv} \in \overline{H}_0 \overline{x} \overline{H}_0$, contradicting the choice of M. Hence $\overline{g} \notin \overline{UxV}$.

(2) Suppose $g = h_1xh_2$ for some $h_1, h_2 \in H_0$. Then $g = h_1xh_2 \notin (U \cap H_0)x(V \cap H_0)$ in F. By Lemma 4.3 and Theorem 3.9, there exists $M \triangleleft_f F$ such that $g, x \notin MH$ and $g \notin M(U \cap H_0)x(V \cap H_0)$. By Lemma 4.1, there exists $L_1 \triangleleft_f E$ such that $L_1 \cap H = M \cap H$, $L_1U \cap L_1H_0 = L_1(U \cap H_0)$, and $L_1A_1 \cap L_1H = L_1H_0$. Similarly, there exists $L_2 \triangleleft_f E$ such that $L_2 \cap H = M \cap H$, $L_2V \cap L_2H_0 = L_2(V \cap H_0)$, and $L_2A_1 \cap L_2H = L_2H_0$. Let $L = L_1 \cap L_2$. Then $L \triangleleft_f E$ and $L \cap H = M \cap H$. Moreover, $LU \cap LH_0 = L(U \cap H_0)$, $LV \cap LH_0 = L(V \cap H_0)$, and $LA_1 \cap LH = LH_0$. Let $\overline{P}_0 = E/L *_{\overline{H}} F/M$. Then, as in (1) above, $\overline{g} \notin \overline{UxV}$.

(c) $\|g\| \geq 2$. Since $E = H \times H_1$, every element in E can be written eh where $e \in H_1$ and $h \in H$. Thus every element in $P_0 = E *_H F$ can be written as an alternating product of e_i and f_i, where $1 \neq e_i \in H_1$ and $f_i \in F \backslash H$.

(1) $g = e_1f_1$, $g = f_1e_1$, or $g = e_1f_1e_2$, where $1 \neq e_i \in H_1$ and $f_1 \in F \backslash H$. We here consider $g = e_1f_1e_2$, since the others are similar. If $e_1 \notin UH$ (or $e_2 \notin HV$), then $e_1 \notin UH_0$ (or $e_2 \notin H_0V$). By Lemma 4.3, there exists $N_1 \triangleleft_f P_0$ such that $e_1, f_1, e_2 \notin N_1H$. By Theorem 3.9, there exists $L_1 \triangleleft_f E$ such that $e_1 \notin L_1UH_0$. By Lemma 4.1, there exists $L_2 \triangleleft_f E$ such that $L_2 \cap H = L_1 \cap N_1 \cap H$ and $L_2A_1 \cap L_2H = L_2H_0$. Let $L = L_1 \cap L_2 \cap N_1$. By Lemma 4.2, there exists $M_1 \triangleleft_f F$ such that $M_1 \cap H = L \cap H$. Let $M = M_1 \cap N_1$. Then $M \cap H = L \cap H$. In $\overline{P}_0 = E/L *_{\overline{H}} F/M$, $\|\overline{g}\| = \|g\| = 3$ and $\overline{e}_1 \notin \overline{UH_0}$. If $\overline{g} \in \overline{UxV}$, then $\overline{e}_1 = \overline{uh}$ for some $u \in U$ and $h \in H$. Thus $\overline{u}^{-1}\overline{e}_1 = \overline{h} \in \overline{A}_1 \cap \overline{H} = \overline{H}_0$. Hence $\overline{e}_1 \in \overline{UH_0}$, contradicting the choice of L_1. Therefore $\overline{g} \notin \overline{UxV}$.

Suppose $e_1 = uh_1$ and $e_2 = h_2v$ where $u \in U$, $v \in V$ and $h_1, h_2 \in H$. Since $g = e_1f_1e_2 \notin UxV$, $h_1f_1h_2 \notin UxV$. By (b) above, there exists $N \triangleleft_f P_0$ such that $h_1f_1h_2 \notin NUxV$. This implies that $g \notin NUxV$.

(2) $\|g\| \geq 4$ or $g = f_1e_1f_2$ where $e_1 \in H_1$ and $f_i \in F \backslash H$. Since P_0 is H-separable by Lemma 4.3, we can find $L \triangleleft_f E$ and $M \triangleleft_f F$ such that, in $\overline{P}_0 = E/L *_{\overline{H}} F/M$, $\|\overline{g}\| = \|g\|$. Then $\overline{g} \notin \overline{UxV}$.

Case 3. $\|x\| \geq 2$. By induction, we assume P_0 is U_1yV_1-separable for all $y \in P_0$ with $\|y\| < \|x\|$ and for any subgroup $U_1, V_1 \leq A_1$. Let $F_0 = \langle H_3, H_4, \ldots, H_{m-1}\rangle^F$. Then $F = F_0 \cdot H$ is a split extension of F_0 by a retract H. Hence every element in P_0 can be written as $(b_1)a_1b_2 \cdots a_{n-1}(b_n)h$, where $h \in H$, $1 \neq a_i \in H_1$ and $1 \neq b_i \in F_0$. Since the other cases are similar, we consider the case $x = b_1a_1 \cdots a_{n-1}b_nh$ where $h \in H$, $1 \neq a_i \in H_1$ and $1 \neq b_i \in F_0$.

(a) $\|g\| < \|x\|$. Clearly $x \notin UgV$. Since $\|g\| < \|x\|$, by induction, there exists $N \triangleleft_f G$ such that $x \notin NUgV$. Then $g \notin NUxV$.

(b) $\|g\| = \|x\|$.

Suppose $g = e_1f_1 \cdots e_nk$, where $k \in H$, $1 \neq e_i \in H_1$ and $1 \neq f_i \in F_0$. By Lemma 4.3, there exists $N_1 \triangleleft_f P_0$ such that $e_i, f_i, a_i, b_i \notin N_1H$. Let $\overline{P}_0 = E/(N_1 \cap E) *_{\overline{H}} F/(N_1 \cap F)$. Then $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Hence $\overline{g} \notin \overline{UxV}$.

Suppose $g = f_1e_1 \cdots f_nk$, where $k \in H$, $1 \neq e_i \in H_1$ and $1 \neq f_i \in F_0$.

(1) Suppose $f_1 \notin Ub_1H_0$. Since P_0 is H-separable (Lemma 4.3), there exists $N_1 \triangleleft_f P_0$ such that $a_i, b_i, e_i, f_i \notin N_1H$. By induction, there exists $N_2 \triangleleft_f P_0$ such that $f_1 \notin N_2Ub_1H_0$. By Lemma 4.1, there exists $L_1 \triangleleft_f E$ such that $L_1 \cap H = N_1 \cap N_2 \cap H$, $L_1A_1 \cap L_1H = L_1H_0$ and $L_1U \cap L_1H_0 = L_1(U \cap H_0)$. Let $M_1 \triangleleft_f F$ such that $L_1 \cap H = M_1 \cap H$. Let $L = L_1 \cap N_1 \cap N_2$ and $M = M_1 \cap N_1 \cap N_2$. Let $\overline{P}_0$ be as above. Then $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. If $\overline{g} = \overline{uxv}$ for $u \in U$ and $v \in V$, then $\overline{f}_1 = \overline{u}\overline{b}_1\overline{h}_1$ for $h_1 \in H$. Since $U \subset A_1$, $\overline{u} \in \overline{A}_1 \cap \overline{H} = \overline{H}_0$. Since $f_1, b_1 \in F_0$, $\overline{h}_1 = \overline{u}^{-1} \in \overline{H}_0$. Thus $\overline{f}_1 = \overline{u}\overline{b}_1\overline{h}_1 \in \overline{Ub_1H_0}$, which contradicts the choice of N_2. Hence $\overline{g} \notin \overline{UxV}$.

(2) Suppose $e_1f_2 \cdots e_{n-1}f_nk \notin H_0a_1b_2 \cdots a_{n-1}b_nhV$. By induction, we can find $N_2 \triangleleft_f P_0$ such that $e_1f_2 \cdots e_{n-1}f_nk \notin N_2H_0a_1b_2 \cdots a_{n-1}b_nhV$ Let N_1 and $\overline{P}_0$ be as in (1) above. If $\overline{g} = \overline{uxv}$ for $u \in U$ and $v \in V$, then $\overline{f}_1 = \overline{u}\overline{b}_1\overline{h}_1$ for $h_1 \in H$ and $\overline{e_1f_2 \cdots e_{n-1}f_nk} = \overline{h}_1^{-1}\overline{a_1b_2 \cdots a_{n-1}b_nh}\overline{v}$ for some $\overline{h}_1 \in \overline{H}$, $\overline{u} \in \overline{U}$ and $\overline{v} \in \overline{V}$. As (1) above, $\overline{h}_1 = \overline{u}^{-1} \in \overline{H}_0$. Hence $\overline{e_1f_2 \cdots e_{n-1}f_nk} \in \overline{H_0a_1b_2 \cdots a_{n-1}b_nhV}$, contradicting the choice of N_2. Hence $\overline{g} \notin \overline{UxV}$.

(3) Suppose $f_1 = ub_1h_1$ and $e_1f_2 \cdots e_{n-1}f_nk = h_2a_1b_2 \cdots a_{n-1}b_nhv$ for some $h_1, h_2 \in H_0$, $u \in U$ and $v \in V$. Then we have $b_1h_1h_2a_1b_2 \cdots a_{n-1}b_nh \notin Ub_1a_1 \cdots a_{n-1}b_nhV$. Hence $h_1h_2 \notin Z_{U \cap H_0}(b_1)Z_{V \cap H_0}(a_1 \cdots a_{n-1}b_nh)$. Let $U_0 = U \cap H_0$ and $V_0 = V \cap H_0$. By Lemma 4.6, $Z_{U_0}(b_1) = 1$ or $Z_{U_0}(b_1) = U_0$ and $Z_{V_0}(a_1 \cdots a_{n-1}b_nh) = 1$ or $Z_{V_0}(a_1 \cdots a_{n-1}b_nh) = V_0$.

(i) $Z_{U_0}(b_1) = 1$ and $Z_{V_0}(a_1 \cdots a_{n-1}b_nh) = 1$. By Lemma 4.7, $Z_{V_0}(a_1 \cdots a_{n-1}b_nh) = Z_{V_0}(a_1) \cap Z_{V_0}(b_2) \cap \cdots \cap Z_{V_0}(b_nh)$. Hence, by Lemma 4.6, either $Z_{V_0}(a_i) = 1$ or $Z_{V_0}(b_j) = 1$ or $Z_{V_0}(b_nh) = 1$ for some $1 \le i \le n$ or some $2 \le j \le n-1$. This implies from Lemma 4.6 that there exists $v_0 \in V_0$ such that $[v_0, a_i] \ne 1$ or $[v_0, b_j] \ne 1$ or $[v_0, b_nh] \ne 1$. Similarly, since $Z_{U_0}(b_1) = 1$, there exists $u_0 \in U_0$ such that $[u_0, b_1] \ne 1$. Since P_0 is $\mathcal{RF}$ by Theorem 2.6, there exists $N_1 \lhd_f P_0$ such that $h_1h_2 \notin N_1$, $[u_0, b_1] \notin N_1$ and $[v_0, a_i] \notin N_1$ or $[v_0, b_j] \notin N_1$ or $[v_0, b_nh] \notin N_1$. Let $N_2 \lhd_f P_0$ be such that $a_i, b_i \notin N_2H$. Let $N_1 \cap N_2 \cap H = S$. Since $S \lhd_f H$, by Lemma 4.1, there exists $L_1 \lhd_f E$ such that $L_1 \cap H = S$, $L_1U \cap L_1H_0 = L_1(U \cap H_0)$ and $L_1A \cap L_1H = L_1H_0$. Similarly, there exists $L_2 \lhd_f E$ such that $L_2 \cap H = S$, $L_2V \cap L_2H_0 = L_2(V \cap H_0)$ and $L_2A \cap L_2H = L_2H_0$. Let $K_1 = N_1 \cap N_2 \cap L_1 \cap L_2 \cap A_1$. Then $K_1 \lhd_f A_1$ and $\langle (K_1 \cap H_0) \times (K_1 \cap H_1) \rangle^{P_0} \subset N_1 \cap N_2$. Let $\overline{P}_0 = P_0 / \langle (K_1 \cap H_0) \times (K_1 \cap H_1) \rangle^{P_0}$. Then $\overline{P}_0$ is the polygonal product of $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_m$, amalgamating central subgroups $\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_m = \overline{H}_0$, with trivial intersections, where $\overline{A}_1 = A_1/(K_1 \cap H_0) \times (K_1 \cap H_1)$ is finite and $\overline{A}_2 = A_2/(K_1 \cap H_1)$, $\overline{A}_3 \cong A_3, \ldots, \overline{A}_{m-1} \cong A_{m-1}$ and $\overline{A}_m = A_m/(K_1 \cap H_0)$. By the choice of N_1, we have $[\overline{u}_0, \overline{b}_1] \ne 1$ and $[\overline{v}_0, \overline{a}_i] \ne 1$ or $[\overline{v}_0, \overline{b}_j] \ne 1$ or $[\overline{v}_0, \overline{b_nh}] \ne 1$. Hence, by Lemma 4.6, $Z_{\overline{U}_0}(\overline{b}_1) = 1$ and $Z_{\overline{V}_0}(\overline{a_1 \cdots a_{n-1}b_nh}) = 1$. By the choice of L_1 and L_2, we have $\overline{U} \cap \overline{H}_0 = \overline{U \cap H_0}$, $\overline{V} \cap \overline{H}_0 = \overline{V \cap H_0}$, and $\overline{A}_1 \cap \overline{H} = \overline{H}_0$. We shall show that $\overline{b_1h_1h_2a_1b_2 \cdots a_{n-1}b_nh} \notin \overline{Ub_1a_1 \cdots a_{n-1}b_nhV}$. Suppose $\overline{b_1h_1h_2a_1b_2 \cdots a_{n-1}b_nh} = \overline{u}_1\overline{b}_1a_1 \cdots a_{n-1}b_nh\overline{v}_1$ for some $\overline{u}_1 \in \overline{U}$ and $\overline{v}_1 \in \overline{V}$. Since $\overline{a}_i \in \overline{H}_1 \subset Z(\overline{E})$, there exists $\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_{n-1} \in \overline{H}$ such that $\overline{b}_1\overline{h_1h_2} = \overline{u}_1\overline{b}_1\overline{z}_1$, $\overline{a}_1 = \overline{z}_1^{-1}\overline{a}_1\overline{z}_1$, $\overline{b}_2 = \overline{z}_1^{-1}\overline{b}_2\overline{z}_2, \ldots$, $\overline{b_nh} = \overline{z}_{n-1}^{-1}\overline{b_nh}\overline{v}_1$. Also, since $b_i \in F_0$, we have $\overline{h_1h_2} = \overline{u}_1\overline{z}_1$, $\overline{z}_1 = \overline{z}_2, \ldots, \overline{z}_{n-2} = \overline{z}_{n-1}$ and $\overline{z}_{n-1} = \overline{h}\overline{v}_1\overline{h}^{-1}$. Since $h_1, h_2 \in H_0$ and $u \in A_1$, $\overline{z}_1 \in \overline{A}_1 \cap \overline{H} = \overline{H}_0$. Hence $\overline{z}_1 = \overline{z}_{n-1} \sim_{\overline{F}} \overline{v}_1$. Thus, by Lemma 4.7, $\overline{z}_1 = \overline{v}_1$. Now, since $\overline{U} \cap \overline{H}_0 = \overline{U \cap H_0}$ and $\overline{V} \cap \overline{H}_0 = \overline{V \cap H_0}$, we have $\overline{u}_1 \in \overline{U}_0$ and $\overline{v}_1 \in \overline{V}_0$. Consequently, $\overline{u}_1 \in Z_{\overline{U}_0}(\overline{b}_1) = 1$ and $\overline{v}_1 \in Z_{\overline{V}_0}(\overline{a_1 \cdots a_{n-1}b_nh}) = 1$. Hence $\overline{h_1h_2} = \overline{u}_1\overline{v}_1 = 1$, contradicting the choice of N_1. Thus $\overline{b_1h_1h_2a_1b_2 \cdots a_{n-1}b_nh} \notin \overline{Ub_1a_1 \cdots a_{n-1}b_nhV}$. Hence $\overline{g} \notin \overline{UxV}$ in $\overline{P}_0$. Since $\overline{P}_0$ is $\mathcal{RF}$ by Theorem 2.6 and $\overline{U}, \overline{V} \subset \overline{A}_1$ are finite, there exists $\overline{N} \lhd_f \overline{P}_0$ such that $\overline{g} \notin \overline{NUxV}$. Let N be the preimage of $\overline{N}$ in P_0. Then $N \lhd_f P_0$ and $g \notin NUxV$.

(ii) $Z_{U_0}(b_1) = U_0$ and $Z_{V_0}(a_1 \cdots a_{n-1}b_nh) = 1$. Then $h_1h_2 \notin U_0$. As in (i) above, either $Z_{V_0}(a_i) = 1$ or $Z_{V_0}(b_j) = 1$ or $Z_{V_0}(b_nh) = 1$ for some $1 \le i \le n$ or some $2 \le j \le n-1$. As in (i) above, there exists $N_1 \lhd_f P_0$ such that either $[v_0, a_i] \notin N_1$ or $[v_0, b_j] \notin N_1$ or $[v_0, b_nh] \notin N_1$ for some

$v_0 \in V_0$. By Lemma 4.3, we can $N_2 \lhd_f P_0$ be such that $a_i, b_i \notin N_2H$. Moreover, by Case 1 above, we can assume that $h_1h_2 \notin N_2U_0$. Let $K_1 = N_1 \cap N_2 \cap L_1 \cap L_2 \cap A_1$ and $\overline{P}_0 = P_0/\langle (K_1 \cap H_0) \times (K_1 \cap H_1)\rangle^{P_0}$ as in above. Then $\overline{P}_0$ is the polygonal product of $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_m$, amalgamating central subgroups $\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_m = \overline{H}_0$, with trivial intersections, where $\overline{A}_1$ is finite and $\overline{A}_3 \cong A_3, \ldots, \overline{A}_{m-1} \cong A_{m-1}$. As in (i), we have $Z_{\overline{U}_0}(\overline{b}_1) = \overline{U}_0$ and $Z_{\overline{V}_0}(\overline{a_1 \cdots a_{n-1}b_nh}) = 1$. We shall show that $\overline{b_1h_1h_2a_1b_2 \cdots a_{n-1}b_nh} \notin \overline{U}\overline{b}_1\overline{a}_1 \cdots \overline{a}_{n-1}\overline{b}_n\overline{h}\overline{V}$. If $\overline{b_1h_1h_2a_1b_2 \cdots a_{n-1}b_nh} = \overline{u}_1\overline{b}_1\overline{a}_1 \cdots \overline{a}_{n-1}\overline{b}_n\overline{h}\overline{v}_1$ for some $\overline{u}_1 \in \overline{U}$ and $\overline{v}_1 \in \overline{V}$ then, as in above, $\overline{u}_1 \in Z_{\overline{U}_0}(\overline{b}_1) = \overline{U}_0$ and $\overline{v}_1 \in Z_{\overline{V}_0}(\overline{a_1 \cdots a_{n-1}b_nh}) = 1$. Hence $\overline{h_1h_2} = \overline{u}_1\overline{v}_1 \in \overline{U}_0$, contradicting the choice of N_2. Therefore $\overline{b_1h_1h_2a_1b_2 \cdots a_{n-1}b_nh} \notin \overline{U}\overline{b}_1\overline{a}_1 \cdots \overline{a}_{n-1}\overline{b}_n\overline{h}\overline{V}$. Then as in (i) above, we can find $N \lhd_f P_0$ such that $g \notin NUxV$.

(iii) $Z_{U_0}(b_1) = 1$ and $Z_{V_0}(a_1 \cdots a_{n-1}b_nh) = V_0$.

(iv) $Z_{U_0}(b_1) = U_0$ and $Z_{V_0}(a_1 \cdots a_{n-1}b_nh) = V_0$.

The above two cases can be considered similarly.

(c) $\|g\| = \|x\| + 1$.

(1) Suppose $g = e_1f_1 \cdots e_nf_nk$, where $k \in H$, $1 \neq e_i \in H_1$ and $1 \neq f_i \in F_0$.

(i) Suppose $e_1 \notin UH$. Then $e_1 \notin UH_0$. By Case 1 above, $N_1 \lhd_f P_0$ such that $e_1 \notin N_1UH_0$. Since P_0 is H-separable (Lemma 4.3), there exists $N_2 \lhd_f P_0$ such that $a_i, b_i, e_i, f_i \notin N_2H$. By Lemma 4.1, there exists $L_1 \lhd_f E$ such that $L_1 \cap H = N_1 \cap N_2 \cap H$ and $L_1A_1 \cap L_1H = L_1H_0$. Let $M_1 \lhd_f F$ such that $L_1 \cap H = M_1 \cap H$. Let $L = L_1 \cap N_1 \cap N_2$ and $M = M_1 \cap N_1 \cap N_2$. Let $\overline{P}_0 = E/L *_{\overline{H}} F/M$ be as above. Then $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$ and $\overline{e}_1 \notin \overline{UH}_0$. If $\overline{g} = \overline{uxv}$ for $u \in U$ and $v \in V$, then $\overline{e}_1 = \overline{uh_1}$ for $h_1 \in H$. Since $U \subset A_1$, $\overline{u}^{-1}\overline{e}_1 = \overline{h}_1 \in \overline{A}_1 \cap \overline{H} = \overline{H}_0$. Hence $\overline{e}_1 \in \overline{UH}_0$, contradicting the choice of N_1. Hence $\overline{g} \notin \overline{UxV}$.

(ii) Suppose $e_1 \in UH$. Let $e_1 = uh_1$ for some $u \in U$ and $h_1 \in H$. Then $h_1f_1e_2 \cdots f_nk \notin UxV$. Hence $f_1'e_2' \cdots f_n'h_1k \notin UxV$, where $f_i' = h_1f_ih_1^{-1} \in F_0$ and $e_i' = h_1e_ih_1^{-1} = e_i \in H_1$. Thus, by (b) above, we can find $\overline{P}_0$ such that $\overline{f_1'e_2' \cdots f_n'h_1k} \notin \overline{UxV}$. Hence $\overline{g} \notin \overline{UxV}$.

(2) Suppose $g = f_1e_1 \cdots f_ne_nk$, where $k \in H$, $1 \neq e_i \in H_1$ and $1 \neq f_i \in F_0$.

(i) Suppose $e_nk \notin HV$. Since $e_n \in H_1 \subset Z(E)$, $e_n \notin HV$. Hence $e_n \notin H_0V$. As in (i) of (1) above, $N_1 \lhd_f P_0$ such that $e_n \notin N_1H_0V$. Let $\overline{P}_0$ be as in (i) of (1) above such that $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$ and $\overline{e}_n \notin \overline{H}_0\overline{V}$. If $\overline{g} = \overline{uxv}$ for $u \in U$ and $v \in V$, then $\overline{e}_n\overline{k} = \overline{h}_1\overline{h}\overline{v}$ for

$h_1 \in H$. Since $V \subset A_1$, $\overline{e}_n\overline{v}^{-1} = \overline{k}^{-1}\overline{h}_1\overline{h} \in \overline{A}_1 \cap \overline{H} = \overline{H}_0$. Hence $\overline{e}_n \in \overline{H}_0\overline{V}$, contradicting the choice of N_1. Hence $\overline{g} \notin \overline{UxV}$.

(ii) Suppose $e_n k \in HV$. Let $e_n k = h_1 v$ for some $v \in V$ and $h_1 \in H$. Then $f_1e_2\cdots f_nh_1 \notin UxV$. By (b) above, we can find $\overline{P}_0$ such that $\overline{f_1e_2\cdots f_nh_1} \notin \overline{UxV}$. Hence $\overline{g} \notin \overline{UxV}$.

(d) $\|g\| \geq \|x\| + 2$.

(1) Suppose $g = f_1e_1\cdots f_ne_nf_{n+1}k$ or $\|g\| > n+2$, where $k \in H$, $1 \neq e_i \in H_1$ and $1 \neq f_i \in F_0$. Let $\overline{P}_0$ be as above such that $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$. Then $\overline{g} \notin \overline{UxV}$.

(2) Suppose $g = e_1f_1\cdots e_nf_ne_{n+1}k$, where $k \in H$, $1 \neq e_i \in H_1$ and $1 \neq f_i \in F_0$. If $e_1 \notin UH$ or $e_{n+1}k = ke_{n+1} \notin HV$ then, as in (c) above, we can find $\overline{P}_0$ such that $\|\overline{g}\| = \|g\|$, $\|\overline{x}\| = \|x\|$ and $\overline{e}_1 \notin \overline{UH}_0$ or $\overline{e}_{n+1}\overline{k} \notin \overline{H}_0\overline{V}$. Then $\overline{g} \notin \overline{UxV}$ as before. So we suppose $e_1 = u_1h_1$ and $e_{n+1}k = h_2v_1$, where $u_1 \in U$, $v_1 \in V$ and $h_1, h_2 \in H$. Then we have $h_1f_1e_2\cdots f_nh_2 \notin UxV$. Hence $f_1'e_2'\cdots f_n'h_1h_2 \notin UxV$, where $f_i' = h_1f_ih_1^{-1} \in F_0$ and $e_i' = h_1e_ih_1^{-1} = e_i \in H_1$. Thus by (b) above, we can find $\overline{P}_0$ such that $\overline{f_1'e_2'\cdots f_n'h_1h_2} \notin \overline{UxV}$. Hence $\overline{g} \notin \overline{UxV}$. □

LEMMA 4.10. *Let $F = A_m *_{H_{m-1}} \cdots *_{H_3} A_3$ and $H = H_0 * H_2$ be as in (4.2).*
(1) Let $V \subset A_3$. For each $S \lhd_f H$, there exists $M \lhd_f F$ such that $M \cap H \subset S$, $MV \cap MH_2 = M(V \cap H_2)$ and $MA_3 \cap MH = M(A_3 \cap H) = MH_2$.
(2) Let $V \subset A_i$ where $i \neq 3, m$. For each $S \lhd_f H$, there exists $M \lhd_f F$ such that $M \cap H = S$, $MV \cap MH_2 = M = M(V \cap H_2)$ and $MA_i \cap MH = M = M(A_i \cap H)$.

Proof. **(1)** Let $\tilde{F} = \tilde{A}_m *_{\tilde{H}_{m-1}} \cdots *_{\tilde{H}_3} \tilde{A}_3$, where $\tilde{A}_m = A_m/(H_0 \cap S)$, $\tilde{A}_3 = A_3/(H_2 \cap S)$ and $\tilde{A}_j \cong A_m$ for $3 < j < m$. Let $\pi_1 : F \to \tilde{F}$ be a natural homomorphism. Then $\pi_1(S) = \tilde{S} \lhd_f \tilde{H}$. Since $\tilde{V}$ is a central subgroup of $\tilde{A}_3$, by Theorem 3.9, there exists $\tilde{M}_1 \lhd_f \tilde{F}$ such that $\tilde{h} \notin \tilde{M}_1\tilde{V}$ for all $\tilde{h} \in \tilde{H}_2 \backslash \tilde{V}$. Let $M_1 = \pi_1^{-1}(\tilde{M}_1)$. Then $M_1 \lhd_f F$. Let $\pi : F \to H = H_0 * H_2$ be a natural homomorphism by mapping $\pi(h) = h$ for $h \in H_0 \cup H_2$ and $\pi(k) = 1$ for $k \in H_j$ where $j \neq 0, 2$. Let $M_2 = \pi^{-1}(S \cap M_1)$. Then $M_2 \lhd_f F$ and $M_2 \cap H = S \cap M_1$. Let $M = M_1 \cap M_2$. Then $M \lhd_f F$ and $M \cap H \subset M_2 \cap H \subset S$. Since $M \subset M_1$, we have $\tilde{h} \notin \tilde{M}\tilde{V}$ for all $\tilde{h} \in \tilde{H}_2 \backslash \tilde{V}$.

We shall show $MV \cap MH_2 = M(V \cap H_2)$. Let $m_1v = m_2h$ where $m_1, m_2 \in M$, $v \in V$ and $h \in H_2$. Then $\tilde{m}_1\tilde{v} = \tilde{m}_2\tilde{h}$, hence $\tilde{h} \in \tilde{M}\tilde{V}$. This implies $\tilde{h} \in \tilde{V}$ by the choice of $\tilde{M}_1$. Let $\tilde{h} = \tilde{v}_1$, where $v_1 \in V$. Then $hv_1^{-1} \in \text{Ker } \pi_1 \cap A_3 = \langle H_0 \cap S, H_2 \cap S \rangle^F \cap A_3 = H_2 \cap S$. Since $h \in H_2$,

$v_1 \in H_2 \cap V$. Also, since $hv_1^{-1} \in \operatorname{Ker} \pi_1 \subset M_1$, $hv_1^{-1} \in M_1 \cap S \subset M_2$. Hence $hv_1^{-1} = m_3 \in M$ and $v_1 \in V \cap H_2$. Therefore $m_2h = m_2m_3v_1 \in M(V \cap H_2)$. This shows that $MV \cap MH_2 \subset M(V \cap H_2)$. Thus $MV \cap MH_2 = M(V \cap H_2)$.

Clearly $M(A_3 \cap H) = MH_2$. We shall show $MA_3 \cap MH = M(A_3 \cap H)$. Let $m_1h = m_2a$ where $m_1, m_2 \in M$, $a \in A_3$ and $h \in H$. Since $H_3 \subset \ker \pi \subset M_2$ and $A_3 = H_2 \times H_3$, $M_2A_3 \cap M_2H = M_2H_2 \cap M_2H = M_2H_2$. Then $m_1h = m_2a = m_3h_1$ where $m_3 \in M_2$ and $h_1 \in H_2$. Thus $m_3^{-1}m_1 = h_1h^{-1} \in M_2 \cap H = M_1 \cap S$. Hence $m_3 \in M$ and $m_1h = m_2a = m_3h_1 \in MH_2 = M(A_3 \cap H)$. This shows that $MA_3 \cap MH \subset M(A_3 \cap H)$. Hence $MA_3 \cap MH = M(A_3 \cap H)$.

(2) Let π be as above. Let $M = \pi^{-1}(S)$. Then $M \cap H = S$. Since $V \subset A_i \subset \ker \pi \subset M$ for $i \neq 3, m$, $MV = M$ and $MA_i = M$. Hence $MV \cap MH_2 = M = M(V \cap H_2)$ and $MA_i \cap MH = M = M(A_i \cap H)$ for $i \neq 3, m$. □

THEOREM 4.11. *Let P_0 be as above. Then P_0 is $\{A_i, A_j\}$-d-separable.*

Proof. It suffices to show that P_0 is $\{A_1, A_i\}$-d-separable. Let $U \leq A_1$, $V \subset A_i$ and $g \notin UxV$ where $g, x \in P_0$. By Theorem 4.9, we may assume $i \neq 1$. We have to consider the following cases, (1) $V \subset A_2$ or $V \subset A_m$, (2) $V \subset A_3$, and (3) $V \subset A_i$ for $i \neq 1, 2, 3, m$. The case (1) is similar to Theorem 4.9 and the case (3) is easier than the case (2). Hence we shall consider the case (2).

So we let $U \leq A_1$, $V \subset A_3$ and $g \notin UxV$ where $g, x \in P_0$. Let E, F, H be as in (4.1) and (4.2) above. Then $P_0 = E *_H F$. We shall find $L \lhd_f E$ and $M \lhd_f F$ with $L \cap H = M \cap H$ such that, in $\overline{P_0} = \overline{E} *_{\overline{H}} \overline{F}$ where $\overline{E} = E/L$ and $\overline{F} = F/M$, $\|\overline{x}\| = \|x\|$, $\|\overline{g}\| = \|g\|$ and $\overline{g} \notin \overline{U}\overline{x}\overline{V}$. Since $\overline{P_0}$ is $\mathcal{RF}$ and $\overline{U}\overline{x}\overline{V}$ is finite, we can find $N \lhd_f G$ such that $g \notin NUxV$.

Case 1. $x \in E$.

(a) $g \in E$. Since $g \notin UxV$ and $V \subset F$, $g \notin Ux(V \cap H) = Ux(V \cap H_2)$. By Theorem 3.9, there exists $L_1 \lhd_f E$ such that $g \notin L_1Ux(V \cap H_2)$. By Lemma 4.10, there exists $M \lhd_f F$ such that $M \cap H \subset L_1 \cap H$, $MV \cap MH_2 = M(V \cap H_2)$ and $MA_3 \cap MH = MH_2$. By Lemma 4.2, there exists $L_2 \lhd_f E$ such that $L_2 \cap H = M \cap H$. Let $L = L_1 \cap L_2$. Then $L \cap H = M \cap H$. Let $\overline{P_0} = \overline{E} *_{\overline{H}} \overline{F}$ be as above. If $\overline{g} = \overline{uxv} \in \overline{U}\overline{x}\overline{V}$, then $\overline{v} \in \overline{H} \cap \overline{V} \subset \overline{H} \cap \overline{A}_3 = \overline{H}_2$. Hence $\overline{v} \in \overline{H}_2 \cap \overline{V} = \overline{V \cap H_2}$. Thus $\overline{g} = \overline{uxv} \in \overline{U}\overline{x}(\overline{V \cap H_2})$, contradicting the choice of L_1. Thus $\overline{g} \notin \overline{U}\overline{x}\overline{V}$.

(b) $g \in F \backslash H$.

(i) $x \in H$. Then $g \notin (U \cap H_0)xV$ in F. By Theorem 3.9, there exists $M \lhd_f F$ such that $g \notin M(U \cap H_0)xV$ in F. By Lemma 4.1, there exists $L \lhd_f E$ such that $L \cap H = M \cap H$, $LU \cap LH_0 = L(U \cap H_0)$ and $LA_1 \cap LH = LH_0$. Let $\overline{P}_0 = \overline{E} *_{\overline{H}} \overline{F}$ be as above. If $\overline{g} = \overline{uxv} \in \overline{U}\overline{x}\overline{V}$, then $\overline{u} \in \overline{H} \cap \overline{U} \subset \overline{H} \cap \overline{A}_1 = \overline{H}_0$. Hence $\overline{u} \in \overline{H}_0 \cap \overline{U} = \overline{U \cap H_0}$. Thus $\overline{g} = \overline{uxv} \in (\overline{U \cap H_0})\overline{x}\overline{V}$, contradicting the choice of M. Thus $\overline{g} \notin \overline{U}\overline{x}\overline{V}$.

(ii) $x \in E \backslash H$. Since $E = H_1 \times H$, we can put $x = a_1 h$, where $1 \neq a_1 \in H_1$ and $h \in H$. If $x \in UH$, then $a_1 \in UH$. Let $a_1 = u_1 h_1$ where $u_1 \in U$ and $h_1 \in H$. Then $g \notin UxV = Uh_1hV$. By above (i), we can find $\overline{P}_0$ such that $\overline{g} \notin \overline{Uh_1hV} = \overline{U}\overline{x}\overline{V}$. Suppose $x \notin UH$. Then $a_1 \notin UH_0$. By Theorem 3.9, there exists $L_1 \lhd_f E$ such that $a_1 \notin L_1UH_0$. Since P_0 is H-separable by Lemma 4.3, there exists $N_1 \lhd_f P_0$ such that $x, g \notin N_1H$. By Lemma 4.1, there exists $L_2 \lhd_f E$ such that $L_2 \cap H = L_1 \cap N_1 \cap H$, $L_2U \cap L_2H_0 = L_2(U \cap H_0)$ and $L_2A_1 \cap L_2H = L_2H_0$. Let $M_1 \lhd_f F$ such that $M_1 \cap H = L_2 \cap H$. Let $L = L_1 \cap L_2 \cap N_1$ and $M = M_1 \cap N_1$. In $\overline{P}_0 = \overline{E} *_{\overline{H}} \overline{F}$, $\overline{a}_1 \notin \overline{UH_0}$, $\overline{U} \cap \overline{H}_0 = \overline{U \cap H_0}$ and $\overline{A}_1 \cap \overline{H} = \overline{H}_0$. If $\overline{g} = \overline{uxv} \in \overline{U}\overline{x}\overline{V}$, then $\overline{ux} \in \overline{H}$. Hence $\overline{ua}_1 \in \overline{H} \cap \overline{A}_1 = \overline{H}_0$. Hence $\overline{a}_1 \in \overline{UH_0}$, contradicting the choice of L_1. Thus $\overline{g} \notin \overline{U}\overline{x}\overline{V}$.

(c) $\|g\| \geq 2$. If $\|g\| \geq 3$ or if $g = f_1 e_1$ where $f_1 \in F \backslash H$ and $e_1 \in E \backslash H$, then we can find $\overline{P}_0 = \overline{E} *_{\overline{H}} \overline{F}$ such that $\|\overline{g}\| = \|g\|$. Then $\overline{g} \notin \overline{U}\overline{x}\overline{V}$. Thus, we suppose $g = e_1 f_1$ where $1 \neq e_1 \in H_1$ and $f_1 \in F \backslash H$.

(i) $x \in H$. If $e_1 \notin UH_0$, as before, we can find $L \lhd_f E$ such that $e_1 \notin LUH_0$, $LU \cap LH_0 = L(U \cap H_0)$ and $LA_1 \cap LH = LH_0$. Let $M \lhd_f F$ such that $M \cap H = L \cap H$ and $f_1 \notin MH$. In $\overline{P}_0 = \overline{E} *_{\overline{H}} \overline{F}$, if $\overline{g} = \overline{e_1 f_1} = \overline{uxv} \in \overline{U}\overline{x}\overline{V}$, then $\overline{e}_1 = \overline{uxh}$ and $\overline{f}_1 = \overline{hv}$ for some $h \in H$. Hence $\overline{u}^{-1}\overline{e}_1 = \overline{xh} \in \overline{H} \cap \overline{A}_1 = \overline{H}_0$. Hence $\overline{e}_1 \in \overline{UH_0}$, contradicting the choice of L. Thus $\overline{g} \notin \overline{U}\overline{x}\overline{V}$. If $e_1 = uh_0 \in UH_0$, then $h_0 f_1 \notin UxV$. Since $h_0 f_1 \in F$, by (b) above, we can find $\overline{P}_0 = \overline{E} *_{\overline{H}} \overline{F}$ such that $\overline{h_0 f_1} \notin \overline{UxV}$. Hence $\overline{g} \notin \overline{U}\overline{x}\overline{V}$.

(ii) $x \in E \backslash H$. As in (ii) of (b) above, let $x = a_1 h$, where $1 \neq a_1 \in H_1$ and $h \in H$. If $f_1 \notin H_0hV$, there exists $M \lhd_f F$ such that $f_1 \notin MH_0hV$ and $f_1 \notin MH$. Let $L \lhd_f E$ such that $L \cap H = M \cap H$, $LU \cap LH_0 = L(U \cap H_0)$ and $LA_1 \cap LH = LH_0$. In $\overline{P}_0 = \overline{E} *_{\overline{H}} \overline{F}$, if $\overline{g} \in \overline{U}\overline{x}\overline{V}$ then $\overline{e_1 f_1} = \overline{uxv}$ for some $u \in U$ and $v \in V$. Hence $\overline{e}_1 = \overline{ua_1 z}$ and $\overline{f_1} = \overline{z}^{-1}\overline{hv}$ for some $z \in H$. Thus $\overline{a}_1^{-1}\overline{u}^{-1}\overline{e}_1 = \overline{z} \in \overline{H} \cap \overline{A}_1 = \overline{H}_0$. This implies $\overline{f_1} \in \overline{H_0 hV}$, contradicting the choice of M. Thus $\overline{g} \notin \overline{U}\overline{x}\overline{V}$. If $f_1 = h_0 hv \in H_0 hV$, then $e_1 h_0 h \notin UxV$. Since $e_1 h_0 h \in E$, by (a) above, we can find $\overline{P}_0 = \overline{E} *_{\overline{H}} \overline{F}$ such that $\overline{e_1 h_0 h} \notin \overline{UxV}$. Thus $\overline{g} \notin \overline{U}\overline{x}\overline{V}$.

Case 2. $x \in F \backslash H$.

(a) $g \in E$. Since $g \notin UxV$, clearly $x \notin UgV$. By Case 1, we can find $N \lhd_f P_0$ such that $x \notin NUgV$. Hence $g \notin NUxV$.

(b) $g \in F \backslash H$.

(1) Suppose $g \notin H_0xV$. By Lemma 4.3 and Theorem 3.9, there exists $M \lhd_f F$ such that $g, x \notin MH$ and $g \notin MH_0xV$. By Lemma 4.1, there exists $L \lhd_f E$ such that $L \cap H = M \cap H$ and $LA_1 \cap LH = LH_0$. Let $\overline{P}_0 = E/L *_{\overline{H}} F/M$ as above. Then $\overline{g} \notin \overline{H_0xV}$, $\overline{g}, \overline{x} \in \overline{F} \backslash \overline{H}$, and $\overline{H} \cap \overline{A}_1 = \overline{H}_0$. If $\overline{g} = \overline{uxv}$ for $u \in U$ and $v \in V$, then $\overline{u} \in \overline{H} \cap \overline{A}_1 = \overline{H}_0$. Hence $\overline{g} = \overline{uxv} \in \overline{H}_0\overline{x}\overline{V}$, contradicting the choice of M. Hence $\overline{g} \notin \overline{UxV}$.

(2) Suppose $g = h_1xv$ for some $h_1 \in H_0$ and $v \in V$. Then $g = h_1xv \notin (U \cap H_0)xV$ in F. By Lemma 4.3 and Theorem 3.9, there exists $M \lhd_f F$ such that $g, x \notin MH$ and $g \notin M(U \cap H_0)xV$. By Lemma 4.1, there exists $L \lhd_f E$ such that $L \cap H = M \cap H$, $LU \cap LH_0 = L(U \cap H_0)$, and $LA_1 \cap LH = LH_0$. Let $\overline{P}_0 = E/L *_{\overline{H}} F/M$. Then, as in (1) above, $\overline{g} \notin \overline{UxV}$.

(c) $\|g\| \geq 2$. This case is similar to (c) of Case 2 in the proof of Theorem 4.9.

Case 3. $\|x\| \geq 2$. By induction, we assume P_0 is U_1yV_1-separable for all $y \in P_0$ with $\|y\| < \|x\|$ and for any subgroup $U_1 \leq A_1$ and any $V_1 \leq A_3$. Let $F_0 = \langle H_3, H_4, \ldots, H_{m-1} \rangle^{P_0}$. Then $F = F_0 \cdot H$ is a split extension of F_0 by a retract H. Hence every element in P_0 can be written as $(b_1)a_1b_2 \cdots a_{n-1}(b_n)h$, where $h \in H$, $1 \neq a_i \in H_1$ and $1 \neq b_i \in F_0$. Since the other cases are similar, we here consider a relatively complicate case, $x = a_1b_1 \cdots a_nb_nh$ where $h \in H$, $1 \neq a_i \in H_1$ and $1 \neq b_i \in F_0$.

(a) $\|g\| < \|x\|$. Clearly $x \notin UgV$. Since $\|g\| < \|x\|$, by induction, there exists $N \lhd_f G$ such that $x \notin NUgV$. Then $g \notin NUxV$.

(b) $\|g\| = \|x\|$.

Suppose $g = f_1e_1 \cdots f_ne_nk$, where $k \in H$, $1 \neq e_i \in H_1$ and $1 \neq f_i \in F_0$. By Lemma 4.3, there exists $N_1 \lhd_f P_0$ such that $e_i, f_i, a_i, b_i \notin N_1H$. Let $\overline{P}_0 = E/(N_1 \cap E) *_{\overline{H}} F/(N_1 \cap F)$. Then $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Hence $\overline{g} \notin \overline{UxV}$.

Suppose $g = e_1f_1 \cdots e_nf_nk$, where $k \in H$, $1 \neq e_i \in H_1$ and $1 \neq f_i \in F_0$.

(1) Suppose $e_1 \notin Ua_1H_0$. Since P_0 is H-separable (Lemma 4.3), there exists $N_1 \lhd_f P_0$ such that $a_i, b_i, e_i, f_i \notin N_1H$. By Theorem 3.9, there exists $L_1 \lhd_f E$ such that $e_1 \notin L_1Ua_1H_0$. By Lemma 4.1, there exists $L_2 \lhd_f E$ such that $L_2 \cap H = N_1 \cap L_1 \cap H$, $L_2A_1 \cap L_2H = L_2H_0$ and $L_2U \cap L_2H_0 = L_2(U \cap H_0)$. Let $M_1 \lhd_f F$ such that $M_1 \cap H = L_1 \cap L_2 \cap N_1 \cap H$. Let $L = N_1 \cap L_1 \cap L_2$ and $M = N_1 \cap M_1$. Let $\overline{P}_0$ be as above. Then $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. If $\overline{g} = \overline{uxv}$ for $u \in U$ and $v \in V$, then $\overline{e}_1 = \overline{ua}_1\overline{h}_1$ for $h_1 \in H$. Since $u, e_1, a_1 \in A_1$, $\overline{h}_1 \in \overline{A}_1 \cap \overline{H} = \overline{H}_0$. Thus $\overline{e}_1 \in \overline{Ua_1H_0}$, which contradicts the choice of L_1. Hence $\overline{g} \notin \overline{UxV}$.

(2) Suppose $f_1e_2\cdots e_nf_nk \notin H_0b_1a_2\cdots a_nb_nhV$. By induction, there exists $N_2\lhd_f P_0$ such that $f_1e_2\cdots e_nf_nk \notin N_2H_0b_1a_2\cdots a_nb_nhV$. Let N_1 and $\overline{P}_0$ be as in (1) above. If $\overline{g} = \overline{uxv}$ for $u \in U$ and $v \in V$, then $\overline{e}_1 = \overline{u}\overline{a}_1\overline{h}_1$ for $h_1 \in H$ and $\overline{f_1e_2\cdots e_nf_nk} = \overline{h}_1^{-1}\overline{b_1a_2\cdots a_nb_nh}\overline{v}$ for some $\overline{h}_1 \in \overline{H}$, $\overline{u} \in \overline{U}$ and $\overline{v} \in \overline{V}$. As (1) above, $\overline{h}_1 \in \overline{H}_0$. Hence $\overline{f_1e_2\cdots e_nf_nk} \in \overline{H}_0\overline{b_1a_2\cdots a_nb_nhV}$, contradicting the choice of N_2. Hence $\overline{g} \notin \overline{UxV}$.

(3) Suppose $e_1 = ua_1h_1$ and $f_1e_2\cdots e_nf_nk = h_2b_1a_2\cdots a_nb_nhv$ for some $h_1, h_2 \in H_0$, $u \in U$ and $v \in V$. Then we have $a_1h_1h_2b_1a_2\cdots a_nb_nh \notin Ua_1b_1\cdots a_nb_nhV$. Hence $h_1h_2 \notin Z_{U\cap H_0}(a_1)Z_{V\cap H_0}(b_1\cdots a_nb_nh)$. Since $U \subset A_1$ and $V \subset A_3$, $Z_{U\cap H_0}(a_1) = U \cap H_0$ and $V \cap H_0 = 1$. Let $U_0 = U \cap H_0$. Hence $h_1h_2 \notin U$. As in Case 3 of Theorem 4.9, we can find $N\lhd_f P_0$ such that $h_1h_2 \notin NU$ and $a_i, b_i \notin NH$. By Lemma 4.1, there exists $L\lhd_f E$ such that $L \cap H = S$, $LU \cap LH_0 = L(U \cap H_0)$, and $LA_1 \cap LH = LH_0$. Similarly, by Lemma 4.10, there exists $M\lhd_f F$ such that $M \cap H = S$, $MV \cap MH_2 = M(V \cap H_2)$, and $MA_3 \cap MH = MH_2$. Let

$$S = \langle (N \cap L \cap H_0) \times (N \cap L \cap H_1), (N \cap M \cap H_2) \times (N \cap M \cap H_3)\rangle^{P_0}.$$

Then $S \subset N$ and $\overline{P}_0 = P_0/S$ is the polygonal product of $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_m$ amalgamating central subgroups $\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_m = \overline{H}_0$ with trivial intersections, where $\overline{A}_1, \overline{A}_3$ are finite and $\overline{A}_j \cong A_j$ for $j \neq 1, 2, 3, m$. Now, by the choice of L, we have $\overline{U} \cap \overline{H}_0 = \overline{U \cap H_0}$. Clearly, since $V \subset A_3$, $\overline{V} \cap \overline{H}_0 = 1$. If $\overline{g} \in \overline{UxV}$, then $\overline{a_1h_1h_2b_1a_2\cdots a_nb_nh} = \overline{u}_1\overline{a_1b_1\cdots a_nb_nh}\overline{v}_1$ for some $\overline{u}_1 \in \overline{U}$ and $\overline{v}_1 \in \overline{V}$. Thus, as in (i) of Case 3 in the proof of Theorem 4.9, we have $\overline{h_1h_2} = \overline{u}_1\overline{v}_1$, where $\overline{u}_1 \in Z_{\overline{U}_0}(\overline{a}_1) = \overline{U}_0$ and $\overline{v}_1 \in Z_{\overline{V}\cap\overline{H}_0}(\overline{b_1\cdots a_nb_nh}) = 1$. Thus $\overline{h_1h_2} \in \overline{U}$, contradicting the choice of N. Hence $\overline{g} \notin \overline{UxV}$ in $\overline{P}_0$. Since $\overline{P}_0$ is $\mathcal{RF}$ and $\overline{U}, \overline{V}$ are finite, there exists $\overline{N}\lhd_f\overline{P}_0$ such that $\overline{g} \notin \overline{NUxV}$. Let N be the preimage of $\overline{N}$ in P_0. Then $N\lhd_f P_0$ and $g \notin NUxV$.

(c) $\|g\| \geq \|x\| + 1$.

Let $\overline{P}_0$ be as above such that $\|\overline{g}\| = \|g\|$ and $\|\overline{x}\| = \|x\|$. Then $\overline{g} \notin \overline{UxV}$. □

LEMMA 4.12. *Let P be a polygonal product of central subgroup separable groups $G_1, G_2, \ldots, G_m$ $(m \geq 4)$, amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections. Let $A_i = H_{i-1} \times H_i$. Then P is A_i-finite for all i and $H_0 * H_2$-finite.*

Proof. Let P_0 be the reduced polygonal product of abelian groups $A_1, A_2, \ldots, A_m$, where $A_i = H_{i-1} \times H_i$, amalgamating subgroups $H_1, H_2, \ldots, H_m = H_0$. Let $H = H_0 * H_2$. Then $P_0 = E *_H F$ as in (4.1) and

(4.2). Since E and F are tree products of central subgroup separable groups A_i, E and F are A_i-finite by Corollary 2.8. By Lemma 4.2, E and F are H-finite. Thus, by Lemma 2.4, $P_0 = E *_H F$ is A_i-finite for all i. Let $P_i = (\cdots((P_0 *_{A_1} G_1) *_{A_2} G_2)\cdots) *_{A_i} G_i$. Then $P = P_{m-1} *_{A_m} G_m$. By induction, we assume that P_{m-1} is A_i-finite for all i. Thus P_{m-1} and G_m are A_m-finite. Hence, by Lemma 2.4, $P = P_{m-1} *_{A_m} G_m$ is A_i-finite for all i.

Now we shall show that P is H-finite. P_0 is A_1-finite by above and G_1 is A_1-finite by Corollary 2.8. Since P_0 is H-finite by Lemma 4.2, $P_1 = P_0 *_{A_1} G_1$ is H-finite by Lemma 2.4. Inductively, suppose P_{m-1} is H-finite. Then $P = P_{m-1} *_{A_m} G_m$ is H-finite by Lemma 2.4 again, since P_{m-1} and G_m are A_m-finite. □

LEMMA 4.13. *Let P be a polygonal product of central subgroup separable groups $G_1, G_2, \ldots, G_m$ ($m \geq 4$), amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections. Let $A_i = H_{i-1} \times H_i$. Then P is A_i-separable for all i and $H_0 * H_2$-separable.*

Proof. Let P_i be as in above. Since E and F are tree products of central subgroup separable groups A_i, E and F are A_i-separable by Corollary 2.8. Also E and F are H-finite (Lemma 4.2) and H-separable (Lemma 4.3). Then, by Lemma 2.5, $P_0 = E *_H F$ is A_i-separable for all i. Inductively, suppose P_{m-1} is A_i-separable for all i. Now P_{m-1} are A_m-finite by Lemma 4.12 and A_m-separable by assumption. Also G_m is A_m-finite and A_m-separable by Corollary 2.8. This follows from Lemma 2.5 that $P = P_{m-1} *_{A_m} G_m$ is A_i-separable for all i.

Now P_0 and G_1 are A_1-finite and A_1-separable. Since P_0 is H-separable (Lemma 4.3), by Lemma 2.5, $P_1 = P_0 *_{A_1} G_1$ is H-separable. As before, inductively, we can see that $P = P_{m-1} *_{A_m} G_m$ is H-separable. □

LEMMA 4.14. *Let G be a central subgroup separable group and H_1, $H_2 \leq Z(G)$ such that $H_1 \cap H_2 = 1$. For each $U \lhd_f H_1 \times H_2$, there exists $N \lhd_f G$ such that $N \cap H_1 = U \cap H_1$ and $N \cap H_2 = H_2$.*

Proof. Since $U \cap H_1 \lhd_f H_1$, let $h_0 = 1, h_1, \ldots, h_s$ be coset representatives of $U \cap H_1$ in H_1. Then $h_i \notin (U \cap H_1)H_2 \subset Z(G)$ for $i \neq 0$. Since G is central subgroup separable, there exists $M \lhd_f G$ such that $h_i \notin M(U \cap H_1)H_2$ for all $i \neq 0$. Let $N = M(U \cap H_1)H_2$. Then $N \lhd_f G$, $N \cap H_2 = H_2$ and $U \cap H_1 \subset N \cap H_1$. To show $N \cap H_1 \subset U \cap H_1$, let $x \in N \cap H_1$. Then $x = h_i u$ for some i and some $u \in U \cap H_1$. Since $x, u \in N$, $h_i \in N = M(U \cap H_1)H_2$. By the choice of M, this implies

$h_i = 1$. Hence $x = u \in U \cap H_1$. This shows that $N \cap H_1 \subset U \cap H_1$. Hence $N \cap H_1 = U \cap H_1$. □

LEMMA 4.15. *Let P be a polygonal product of central subgroup separable groups $A, G_2, \ldots, G_m$ $(m \geq 4)$, amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections, where $A = H_0 \times H_1$. For every $U \triangleleft_f A$, there exists $N \triangleleft_f P$ such that $N \cap A = U$ and if $Nx \sim_{P/N} Ny$ for any $x, y \in A$ then $Nx = Ny$.*

Proof. Let $U \triangleleft_f A$ be given. Consider $V = (U \cap H_0) \times H_{m-1} \triangleleft_f H_0 \times H_{m-1}$. By Lemma 4.14, there exists $M \triangleleft_f G_m$ such that $M \cap H_0 = V \cap H_0 = U \cap H_0$ and $M \cap H_{m-1} = H_{m-1}$. Similarly, by considering $(U \cap H_1) \times H_2 \triangleleft_f H_1 \times H_2$ in G_2, there exists $L \triangleleft_f G_2$ such that $L \cap H_1 = U \cap H_1$ and $L \cap H_2 = H_2$. Let $\overline{G}_m = G_m/M$, $\overline{A} = A/U$, $\overline{G}_2 = G_2/L$ and $\overline{P} = \overline{G}_m *_{\overline{H}_0} \overline{A} *_{\overline{H}_1} \overline{G}_2$. Then $\overline{P} \cong P/\langle U, G_3, \ldots, G_{m-1}\rangle^P$. Hence there exists a natural homomorphism $\pi : P \to \overline{P}$ by mapping $\pi(y) = 1$ for $y \in U \cup G_i$ $(i \neq 2, m)$. Since $\overline{G}_m, \overline{A}, \overline{G}_2$ are finite, $\overline{P}$ is trivially a tree product of central subgroup separable groups amalgamating central subgroups. Hence $\overline{P}$ is conjugacy separable by Theorem 2.2 and, by Lemma 2.10, $\overline{x} \not\sim_{\overline{P}} \overline{y}$ for all $\overline{x} \neq \overline{y} \in \overline{A}$. Thus, there exists $\overline{N} \triangleleft_f \overline{P}$ such that $\overline{N} \cap \overline{A} = 1$ and $\overline{N}\overline{x} \not\sim_{\overline{P}/\overline{N}} \overline{N}\overline{y}$ for all $\overline{x} \neq \overline{y} \in \overline{A}$. Let $N = \pi^{-1}(\overline{N})$. Then $N \triangleleft_f P$, $N \cap A = U$ and if $Nx \sim_{P/N} Ny$ for any $x, y \in A$ then $Nx = Ny$. □

LEMMA 4.16. *Let P be a polygonal product of central subgroup separable groups $A, G_2, \ldots, G_m$ $(m \geq 4)$, amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections, where $A = H_0 \times H_1$. Let $A_i = H_{i-1} \times H_i$ and $S \leq A_i$. For every $U \triangleleft_f A$, there exists $N \triangleleft_f P$ such that $N \cap A = U$ and $NA \cap NS = N(A \cap S)$.*

Proof. If $S \subset A$ then the lemma holds trivially by Lemma 4.12. Hence we consider the following cases:

Case 1. $S \subset G_2$ (or $S \subset G_m$).

Let $E = A *_{H_1} G_2$, $F = G_m *_{H_{m-1}} G_{m-1} *_{H_{m-2}} \cdots *_{H_3} G_3$ and $H = H_0 * H_2$. Then $P = E *_H F$. By Lemma 2.9, there exists $L \triangleleft_f E$ such that such that $L \cap A = U$ and $LA \cap LS = L(A \cap S)$. Since P is H-finite (Lemma 4.12), F is H-finite. There exists $M \triangleleft_f F$ such that $M \cap H = L \cap H$. Let $\overline{G} = \overline{E} *_{\overline{H}} \overline{F}$ where $\overline{E} = E/L$ and $\overline{F} = F/M$. Since $\overline{A}, \overline{S}$ are finite and $\overline{P}$ is $\mathcal{RF}$, there exists $\overline{N} \triangleleft_f \overline{P}$ such that $\overline{N} \cap \overline{A} = 1$ and $\overline{a} \notin \overline{NS}$ for all $\overline{a} \in \overline{A} \backslash \overline{S}$. Let N be the preimage of $\overline{N}$ in P. Then clearly $N \triangleleft_f P$ and $N \cap A = U$. To show $NA \cap NS \subset N(A \cap S)$, let $n_1 a = n_2 s$, where $a \in A$, $s \in S$ and $n_1, n_2 \in N$. Then $\overline{n}_1 \overline{a} = \overline{n}_2 \overline{s}$.

Hence $\overline{a} \in \overline{NS}$. By the choice of $\overline{N}$, $\overline{a} \in \overline{S}$. Thus, for some $l \in L$, $a = ls \in LA \cap LS = L(A \cap S)$. This implies $a = l_1 d$ for some $l_1 \in L$ and $d \in A \cap S$. Hence $ad^{-1} = l_1 \in A \cap L = U \subset L \subset N$. Thus $Na = Nd \in N(A \cap S)$. This shows $NA \cap NS \subset N(A \cap S)$. Therefore $NA \cap NS = N(A \cap S)$.

Case 2. $S \subset G_i$ for $i \neq 2, m$.

As in the proof of Lemma 4.15, let $\overline{P} = \overline{G}_m *_{\overline{H}_0} \overline{A} *_{\overline{H}_1} \overline{G}_2$, where $\overline{A} = A/U$, $\overline{G}_m = G_m/M$ and $\overline{G}_2 = G_2/L$ are finite. Then $\overline{S} = 1$. Since $\overline{P}$ is $\mathcal{RF}$ and $\overline{A}$ is finite, there exists $\overline{N} \lhd_f \overline{P}$ such that $\overline{N} \cap \overline{A} = 1$. Let N be the preimage of $\overline{N}$ in P. Then $N \lhd_f P$, $N \cap A = U$ and $NA \cap NS = NA \cap N = N = N(A \cap S)$. □

THEOREM 4.17. *Let P be a polygonal product of central subgroup separable groups $G_1, G_2, \ldots, G_m$ $(m \geq 4)$, amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections. Let $A_i = H_{i-1} \times H_i$. Then P is $\{A_i, A_j\}$-d-separable for all i, j.*

Proof. The reduced polygonal product P_0, which is a polygonal product of abelian groups $A_1, A_2, \ldots, A_m$, where $A_i = H_{i-1} \times H_i$, amalgamating subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections, is $\{A_i, A_j\}$-d-separable by Theorem 4.11. Let $P_i = (\cdots((P_0 *_{A_1} G_1) *_{A_2} G_2) \cdots) *_{A_i} G_i$. Then $P = P_m = P_{m-1} *_{A_m} G_m$. By induction, we assume that P_{m-1} is $\{A_i, A_j\}$-d-separable for all i, j. To prove the theorem, we apply Theorem 3.5 to $P = P_{m-1} *_{A_m} G_m$.

D1. By induction, P_{m-1} is $\{A_m, A_m\}$-d-separable, hence $\{A_m\}$-d-separable and G_m is $\{A_m\}$-d-separable by Theorem 3.9.

D2 and D3. By Lemma 4.7 and Lemma 4.15, P_{m-1} has the property D2 and D3, respectively. Since $A_m \subset Z(G_m)$, G_m has the property D2. To see that G_m has the property D3, let $U \lhd_f A_m$. Since G_m is A_m-finite, there exists $M \lhd_f G_m$ such that $M \cap A_m = U$. In $\overline{G}_m = G/M$, if $\overline{x} \sim_{\overline{G}_m} \overline{y}$ for $x, y \in A_m$, then $\overline{x} = \overline{y}$, since $x, y \in A_m \subset Z(G_m)$. Thus G_m has the property D3.

D4′. P_{m-1} has the property D4′ by Lemma 4.16.

By induction, P_{m-1} is $\{A_m, A_i, A_j\}$-d-separable. Hence, by Theorem 3.5, $P = P_{m-1} *_{A_m} G_m$ is $\{A_m, A_i, A_j\}$-d-separable. Thus P is $\{A_i, A_j\}$-d-separable. □

COROLLARY 4.18. *Let P be a polygonal product of central subgroup separable groups $G_1, G_2, \ldots, G_m$ $(m \geq 4)$, amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections. Let $A_i = H_{i-1} \times H_i$. Then, for each i and j, P is $A_i x A_j$-separable for $x \in P$.*

5. Conjugacy separability

In this section we use the criterion in [11] to show that polygonal products of polycyclic-by-finite groups amalgamating central edge groups with trivial intersections are conjugacy separable.

DEFINITION 5.1. Let G be a group and let H be a subgroup of G. We say G is *H-conjugacy separable* if, for each $x \in G$ such that $\{x\}^G \cap H = \emptyset$, there exists $N \triangleleft_f G$ such that, in $\overline{G} = G/N$, $\{\overline{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$.

We note that if $H \subset Z(G)$ then G is H-conjugacy separable if and only if G is H-separable. Thus, if G is central subgroup separable and if $H \subset Z(G)$ is finitely generated, then G is H-conjugacy separable. More generally, we have:

THEOREM 5.2. [11, Thoerem 3.14] *Let G be a tree product of n central subgroup separable groups amalgamating central edge groups. Let H be a finitely generated central subgroup of a vertex group of G. Then G is H-conjugacy separable.*

LEMMA 5.3. *Let P be the polygonal product of central subgroup separable groups $A, G_2, \ldots, G_m$ $(m \geq 4)$, amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections, where $A = H_0 \times H_1$. Then P is A-conjugacy separable.*

Proof. Let $x \in P$ and $\{x\}^P \cap A = \emptyset$. Let $E = A *_{H_1} G_2$, $F = G_m *_{H_{m-1}} \cdots *_{H_3} G_3$, and $H = H_0 * H_2$. Then $P = E *_H F$. Clearly $x \notin A$. We may assume that x has minimal length in its conjugacy class in $P = E *_H F$.

Case 1. Suppose $x \in E$. Thus $\{x\}^E \cap A = \emptyset$. Moreover, we can assume that x has minimal length in its conjugacy class in $E = A *_{H_1} G_2$.

(a) $\|x\| \geq 2$ in E. Since x has minimal length in its conjugacy class in E, x is cyclically reduced in E, say, $x = a_1 b_1 \cdots a_n b_n$, where $a_i \in A \backslash H_1$ and $b_i \in G_2 \backslash H_1$. Now $A = H_0 \times H_1$. Thus, we can assume $1 \neq a_i \in H_0$. Since G_2 is central subgroup separable, there exists $L_2 \triangleleft_f G_2$ such that $b_i \notin L_2 H_1$ for all i. Similarly, since $a_i \in G_m \backslash H_{m-1}$, there exists $L_m \triangleleft_f G_m$ such that $a_i \notin L_m H_{m-1}$ for all i. Now $(L_2 H_1 \cap H_2) \times H_3 \triangleleft_f H_2 \times H_3$. By Lemma 4.14, there exists $L_3 \triangleleft_f G_3$ such that $L_3 \cap H_2 = L_2 H_1 \cap H_2$ and $L_3 \cap H_3 = H_3$. Let $\overline{G}_m = G_m / L_m H_{m-1}$, $\overline{A} = A/(L_m H_{m-1} \cap H_0) H_1$, $\overline{G}_2 = G_2 / L_2 H_1$, and $\overline{G}_3 = G_3 / L_3$. Let

$$\overline{P} = \overline{G}_m *_{\overline{H}_0} \overline{A} *_{\overline{H}_1} \overline{G}_2 *_{\overline{H}_2} \overline{G}_3.$$

Since $\overline{H}_1 = 1$, $\overline{A} = \overline{H}_0 \subset \overline{G}_m$. Hence $\overline{P} = \overline{G}_m * \overline{G}_2 *_{\overline{H}_2} \overline{G}_3$. Then there exists a natural homomorphism $\pi : P \to \overline{P}$ by mapping $\pi(y) = 1$ for $y \in G_i$ $(i \neq 2, 3, m)$. Note that $1 \neq \overline{a}_i \in \overline{H}_0 \subset \overline{G}_m$ and $1 \neq \overline{b}_i \in \overline{G}_2$. Thus, in $\overline{P} = \overline{G}_m * \overline{B}$, where $\overline{B} = \overline{G}_2 *_{\overline{H}_2} \overline{G}_3$, we have $\|\overline{x}\| = \|x\| = 2n$. Since $\overline{A} = \overline{H}_0 \subset \overline{G}_m$, by Theorem 2.1, $\overline{x} \not\sim_{\overline{P}} \overline{a}$ for all $\overline{a} \in \overline{A}$. Now $\overline{P}$ is conjugacy separable by Theorem 2.2, since $\overline{G}_m, \overline{G}_2, \overline{G}_3$ are finite. Hence, there exists $\overline{N} \triangleleft_f \overline{P}$ such that $\overline{N}\overline{x} \not\sim_{\overline{P}/\overline{N}} \overline{N}\overline{a}$ for all (finitely many) $\overline{a} \in \overline{A}$. Let $N = \pi^{-1}(\overline{N})$. Then $N \triangleleft_f P$ and $Nx \not\sim_{P/N} Na$ for all $a \in A$. Hence $\{Nx\}^{P/N} \cap NA/N = \emptyset$.

(b) $x = b_1 \in G_2 \backslash H_1$. Let $L_2 \triangleleft_f G_2$ such that $b_1 \notin L_2 H_1$. Let $\overline{G}_2 = G_2/L_2 H_1$ and $\overline{G}_3 = G_3/L_3$ be as above. Consider the natural homomorphism $\pi_1 : P \to \overline{G}_2 *_{\overline{H}_2} \overline{G}_3$ by mapping $\pi(y) = 1$ for $y \in G_i$ $(i \neq 2, 3)$. Then $\overline{x} \neq 1$ and $\overline{a} = 1$ for all $a \in A$. Since $\overline{P}$ is $\mathcal{RF}$, there exists $\overline{N} \triangleleft_f \overline{P}$ such that $\overline{x} \notin \overline{N}$. Let $N = \pi_1^{-1}(\overline{N})$. Then $N \triangleleft_f P$ and $Nx \neq N$ and $Na = N$ for all $a \in A$. Thus $\{Nx\}^{P/N} \cap NA/N = \emptyset$.

Case 2. Suppose $x \in F \backslash H$. Since $\{x\}^P \cap H_0 = \emptyset$, $\{x\}^F \cap H_0 = \emptyset$. Let $\overline{G}_2 = G_2/H_1$ and $\overline{G}_i = G_i$ for $i \neq 2$. Let $\overline{P} = \overline{G}_2 *_{\overline{H}_2} \overline{G}_3 *_{\overline{H}_3} \cdots *_{\overline{H}_{m-1}} \overline{G}_m$. Then there exists a natural homomorphism $\pi : P \to \overline{P}$ by mapping $\pi(y) = 1$ for $y \in H_1$. Clearly $\overline{A} = \overline{H}_0 \subset \overline{G}_m$. Consider $\overline{P} = \overline{G}_2 *_{\overline{H}_2} \overline{F}$, where $\overline{F} = \overline{G}_3 *_{\overline{H}_3} \cdots *_{\overline{H}_{m-1}} \overline{G}_m$. Clearly $\{\overline{x}\}^{\overline{F}} \cap \overline{H}_0 = \emptyset$, since $\overline{F} = F$. We shall show that $\{\overline{x}\}^{\overline{P}} \cap \overline{H}_0 = \emptyset$. Suppose $\overline{x} \sim_{\overline{P}} \overline{h}$ for $\overline{h} \in \overline{H}_0$.

(1) If $\overline{x}$ has the minimal length 1 in its conjugacy class in $\overline{P} = \overline{G}_2 *_{\overline{H}_2} \overline{F}$ then, by Theorem 2.1, $\overline{x}, \overline{h} \in \overline{F}$ and $\overline{x} \sim_{\overline{F}} \overline{h}$, contradicting the fact that $\{\overline{x}\}^{\overline{F}} \cap \overline{H}_0 = \emptyset$.

(2) If $\overline{x}$ has the minimal length 0 in its conjugacy class in $\overline{P}$, say $\overline{x} \sim_{\overline{F}} \overline{z}$ for $\overline{z} \in \overline{H}_2$, then there exists $\overline{z}_i \in \overline{H}_2$ such that $\overline{x} \sim_{\overline{F}} \overline{z}_1 \sim_{\overline{G}_2} \overline{z}_2 \sim_{\overline{F}} \cdots \sim_{\overline{G}_2} \overline{z}_r = \overline{z}$ by Theorem 2.1. Since $H_2 \subset Z(G_2)$, $\overline{z}_i \sim_{\overline{G}_2} \overline{z}_{i+1}$ implies $\overline{z}_i = \overline{z}_{i+1}$. Thus we have $\overline{x} \sim_{\overline{F}} \overline{z}$. Also, since $\overline{x} \sim_{\overline{P}} \overline{h}$, $\overline{h} \sim_{\overline{P}} \overline{z}$. Then, as above, $\overline{h} \sim_{\overline{F}} \overline{z}$. Hence $\overline{x} \sim_{\overline{F}} \overline{h}$ which contradicts $\{\overline{x}\}^{\overline{F}} \cap \overline{H}_0 = \emptyset$. Consequently, $\{\overline{x}\}^{\overline{P}} \cap \overline{H}_0 = \emptyset$. Now clearly $\overline{P}$ is a tree product of central subgroup separable groups amalgamating central subgroups. Hence, by Theorem 5.2, there exists $\overline{N} \triangleleft_f \overline{P}$ such that $\{\overline{N}\overline{x}\}^{\overline{P}/\overline{N}} \cap \overline{N}\overline{H}_0/\overline{N} = \emptyset$. Since $\overline{A} = \overline{H}_0$, we have $\{\overline{N}\overline{x}\}^{\overline{P}/\overline{N}} \cap \overline{N}\overline{A}/\overline{N} = \emptyset$. Let $N = \pi_1^{-1}(\overline{N})$. Then $N \triangleleft_f P$ and $\{Nx\}^{P/N} \cap NA/N = \emptyset$.

Case 3. Suppose $x \notin E \cup F$. Since x has minimal length in its conjugacy class in $P = E *_H F$, x is cyclically reduced. Let $x =$

$e_1f_1\cdots e_nf_n$, say, where $e_i \in E\backslash H$ and $f_i \in F\backslash H$. Since P are H-separable (Lemma 4.13), there exists $M\triangleleft_f P$ such that $e_i \notin MH$ and $f_i \notin MH$. Let $\overline{P} = E/(M\cap E) *_{\overline{H}} E/(M\cap F)$. Then $\|\overline{x}\| = \|x\| = 2n$. Thus $\{\overline{x}\}^{\overline{P}} \cap \overline{A} = \emptyset$ by Theorem 2.1. Since $\overline{A}$ is finite and $\overline{P}$ is conjugacy separable (Theorem 2.2), as in (a) of Case 1, we can find $N\triangleleft_f P$ such that $\{Nx\}^{P/N} \cap NA/N = \emptyset$. □

Using the following criterion, we shall show the conjugacy separability of our polygonal products of central subgroup separable groups, with trivial intersections (Theorem 5.5).

THEOREM 5.4. [11] *Let $G = A *_H B$ and $H \subset Z(A)$. Suppose A and B satisfy the following:*

C1. *Both A and B are conjugacy separable.*

C2. *A, B are H-finite.*

C3. *B is H-conjugacy separable.*

C4. *G is HxH-separable for $x \in G$.*

C5. *For every $M_1\triangleleft_f B$, there exists $M\triangleleft_f B$ such that $M \subset M_1$ and, in $\overline{B} = B/M$, we have $\overline{h} \not\sim_{\overline{B}} \overline{k}$ for all $\overline{h} \neq \overline{k}$ in $\overline{H}$.*

Then G is conjugacy separable.

THEOREM 5.5. *Let P be a polygonal product of central subgroup separable and conjugacy separable groups $G_1, G_2, \ldots, G_m$ ($m \geq 4$), amalgamating central subgroups $H_1, H_2, \ldots, H_m = H_0$, with trivial intersections. Then P is conjugacy separable.*

Proof. By Theorem 4.8, the reduced polygonal product P_0 is conjugacy separable. Let $P_i = (\cdots((P_0 *_{A_1} G_1) *_{A_2} G_2)\cdots) *_{A_i} G_i$. Then $P = P_m = P_{m-1} *_{A_m} G_m$. By induction on m, we can assume P_{m-1} to be conjugacy separable. To prove the theorem, we apply Theorem 5.4 to $P = P_{m-1} *_{A_m} G_m$.

C1. By assumption, P_{m-1}, G_m are conjugacy separable.

C2. P_{m-1} is A_m-finite by Lemma 4.12 and G_m is A_m-finite by Corollary 2.8.

C3. Since $A_m \subset Z(G_m)$ and G_m is A_m-separable, G_m is A_m-conjugacy separable. By Lemma 5.3, P_{m-1} is A_m-conjugacy separable.

C4. By Corollary 4.18, P is A_mxA_m-separable for $x \in P$.

C5. Let $M_1\triangleleft_f P_{m-1}$ be given. Then $U = M_1 \cap A_m \triangleleft_f A_m$. By Lemma 4.15, there exists $M_2\triangleleft_f P_{m-1}$ such that $M_2 \cap A_m = U$ and, in $\tilde{P}_{m-1} = P_{m-1}/M_2$, if $\tilde{x} \sim_{\tilde{P}_{m-1}} \tilde{y}$ for $x, y \in A_m$ then $\tilde{x} = \tilde{y}$. Let $M = M_1 \cap M_2$. Then $M\triangleleft_f P_{m-1}$. Let $\overline{P}_{m-1} = P_{m-1}/M$. Suppose

$\overline{x} \sim_{\overline{P}_{m-1}} \overline{y}$ for $x, y \in A_m$. Then $\tilde{x} \sim_{\tilde{P}_{m-1}} \tilde{y}$. Hence $\tilde{x} = \tilde{y}$ by the choice of M_2. Thus $y^{-1}x \in M_2 \cap A_m = U = M_1 \cap A_m$. Hence $y^{-1}x \in M$, that is $\overline{x} = \overline{y}$.

Consequently, by Theorem 5.4, $P = P_{m-1} *_{A_m} G_m$ is conjugacy separable. □

Since polycyclic-by-finite groups are conjugacy separable [4] and subgroup separable ([15] or [18, p. 148]), we have the following:

THEOREM 5.6. *Let P be a polygonal product of polycyclic-by-finite groups $G_1, G_2, \ldots, G_m$ ($m \geq 4$), amalgamating central subgroups H_1, H_2, …, $H_m = H_0$, with trivial intersections. Then P is conjugacy separable.*

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