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A UNIFORM SPACE OF FUZZY IMPLICATION ALGEBRAS

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ABSTRACT. We discuss the filter generated by an arbitrary set in fuzzy implication algebra, and consider the uniformity of a fuzzy implication algebra by using filters.

1. Introduction

The concept of fuzzy implication algebras, which was introduced by W. M. Wu in [10], is the abstract concept of implication connectives of $[0,1]$-valued logics. In the same paper, he introduced the notion of the filter in a fuzzy implication algebra, and investigated their properties. Recently, many mathematical papers have been investigating the algebraic properties of fuzzy implication algebras([2, 3, 4]). In particular, D. Wu [11] introduced the concept of the commutativity in fuzzy implication algebras, and studied various properties. T. R. Zou [14] introduced the concept of P-filters and PFI-algebras, and obtained some important results.

On the other hand, G. J. Wang [6, 7, 8] established a new concept of the quasi-formal deductive system, and proved the soundness theorem and consistency theorem. In proof of the soundness theorem, he used the R_0-algebra, which is a new kind of algebraic systems for fuzzy logic, and so R_0-algebra is very important role of both classical logics and non-classical logics. D. W. Pei and G. J. Wang [5] proved the relation between R_0-algebra and fuzzy implication algebra, i.e., two kinds of algebraic systems are equivalent.

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In this paper, we discuss the filter generated by an arbitrary set in fuzzy implication algebra, and consider the uniformity of a fuzzy implication algebra by using filters.

DEFINITION 1.1 ([10]). A non-empty set X together with a binary operation $\rightarrow$ and a zero element 0 is said to be a *fuzzy implication algebra* if the following axioms are satisfied for all $x, y, z \in X$

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
(I3) $x \rightarrow x = 1$,
(I4) $x \rightarrow y = y \rightarrow x = 1$ imply $x = y$,
(I5) $0 \rightarrow x = 1$,

where $1 = 0 \rightarrow 0$. An order relation can be defined for all x and y in X to be $x \leq y$ if and only if $x \rightarrow y = 1$. It is clear that this order relation on X is a partial ordering.

In the sequel the binary operation " $\rightarrow$ " will be denoted by juxtaposition.

DEFINITION 1.2 ([10]). A subset F of a fuzzy implication algebra X is called a *filter* if it satisfies for all $x, y \in X$:

(F1) $1 \in F$,
(F2) $x \in F$ and $xy \in F$ imply $y \in F$.

Then we have the following proposition.

PROPOSITION 1.3. *Every filter F of a fuzzy implication algebra has the following property:*

$$x \leq y \text{ and } x \in F, \text{ then } y \in F.$$

PROOF. This proof is easy and so we omitted. □

LEMMA 1.4 ([10]). *Let X be a fuzzy implication algebra. Then for any $x, y, z \in X$, we have*

(1) $x \leq 1$,
(2) $1x = x$,
(3) $x \leq y$ *implies* $z \leq zy$ *and* $yz \leq xz$.

THEOREM 1.5. *Let F be a filter of a fuzzy implication algebra X. For any $x, y \in X$, define a relation " $\sim$ " on X by*

$$x \sim y \text{ if and only if } xy \in F \text{ and } yx \in F.$$

Then $\sim$ is a congruence relation on X.

PROOF. Since $1 \in F$, we have $xx = 1 \in F$ for all $x \in X$. This means that $\sim$ is reflexive. Let $x, y, z \in X$ be such that $x \sim y$ and $y \sim z$. Then $xy, yx \in F$ and $yz, zy \in F$. By (I2), we have $xy \leq (yz)(xz)$. By virtue of Proposition 1.3, we get $xz \in F$. By the same manner we can prove $zx \in F$. Thus we get $x \sim z$. This shows that $\sim$ is transitive. The symmetry of $\sim$ is immediated from the definition. Therefore $\sim$ is an equivalence relation on X.

Let $x, y, u, v \in X$ be such that $u \sim v$ and $x \sim y$. Then $uv, vu \in F$ and $xy, yx \in F$, and by (I2), we have

$$uv \leq (vx)(ux) \text{ and } vu \leq (ux)(vx).$$

In view of Proposition 1.3, $(vx)(ux) \in F$ and $(ux)(vx) \in F$, *i.e.*, $vx \sim ux$. In addition, we have

$$xy \leq (vx)(vy) \text{ and } yx \leq (vy)(vx).$$

Using Proposition 1.3, we get $(vx)(vy) \in F$ and $(vy)(vx) \in F$, *i.e.*, $vx \sim vy$. Since $\sim$ is an equivalence relation, we obtain $ux \sim vy$. Therefore $\sim$ is a congruence relation on X. □

DEFINITION 1.6 ([9]). Let M be any non-empty set and let U and V be any subsets of $M \times M$. Define
$U \circ V := \{(x, y) \in M \times M |$ for some $z \in M, (x, z) \in U$ and $(z, y) \in V\}$,
$U^{-1} := \{(x, y) \in M \times M | (y, x) \in U\}$,
$\triangle := \{(x, x) \in M \times M | x \in M\}$.

By a *uniformity* K on M we mean a non-empty collection K of subsets of $M \times M$ which satisfies the following conditions:
(U1) $\triangle \subset U$ for any $U \in K$,
(U2) if $U \in K$, then $U^{-1} \in K$,
(U3) if $U \in K$, then there exists a $V \in K$ such that $V \circ V \subset U$,
(U4) if $U, V \in K$, then $U \cap V \in K$,
(U5) if $U \in K$ and $U \subset V \subset M \times M$, then $V \in K$.
The pair (M, K) is called a *uniform space.*

2. Main Results

First we discuss the filter generated by a nonempty set in fuzzy implication algebras. The theorem below shows how we can make a filter beginning an arbitrary subset of fuzzy implication algebras.

THEOREM 2.1. *If A is a non-empty subset of a fuzzy implication algebra X, then the set*
(FG)
$\{x \in X | \exists a_i \in A, i = 1, \cdots, n,$ *such that* $a_1(a_2(\cdots(a_n x)\cdots)) = 1\}$

is the minimal filter containing A, which is called the *filter generated* by A.

PROOF. Let B be the set of (FG). Then we get $1 \in B$ since $x1 = 1$. Let $x, y \in X$ be such that $xy \in B$ and $x \in B$. Then there are $a_i, b_j \in A, i = 1, \cdots, m, j = 1, \cdots n$ such that

$$a_1(a_2(\cdots(a_m x)\cdots)) = 1 \text{ and } b_1(b_2(\cdots(b_n(xy))\cdots)) = 1,$$

and hence

$$x(b_1(b_2(\cdots(b_n y)\cdots)) = 1, \text{ or } x \leq b_1(b_2(\cdots(b_n y)\cdots)).$$

Leftly multiplying both sides of the above inequality by a_m, we have

$$a_m x \leq a_m(b_1(b_2(\cdots(b_n y)\cdots))).$$

Repeating the above argument m times we obtain

$$a_1(\cdots(a_m x)\cdots) \leq a_1(\cdots(a_m(b_1(\cdots(b_n y)\cdots)))\cdots),$$

and hence

$$a_1(\cdots(a_m(b_1(\cdots(b_n y)\cdots)))\cdots) = 1.$$

This means that $y \in B$. Summarizing the above facts B is a filter of X. Obviously, $A \subseteq B$. Let F be a filter containing A. In order to prove $B \subseteq F$ assume any $a \in B$. Then there are $c_1, \cdots c_l \in A$ such that $c_1(\cdots(c_l a)\cdots) = 1$. Since $1 \in F$, we have

$$c_1(\cdots(c_l a)\cdots) \in F.$$

Since F is a filter and $c_1 \in F$, it follows that

$$c_2(\cdots(c_l a)\cdots) \in F.$$

Repeating this argument n times we obtain $a \in F$, and hence $B \subseteq F$. Therefore B is the minimal filter containing A. □

THEOREM 2.2. *For each filter F of a fuzzy implication algebra X, define*

$$U_F := \{(x,y) \in X \times X | xy \in F \text{ and } yx \in F\}$$

and let

$$\mathcal{F}^* := \{U_F | F \text{ is a filter of } X\}.$$

Then $\mathcal{F}^*$ satisfies the conditions (U1)-(U4).

PROOF. Let $U_F \in \mathcal{F}^*$ and let $(x,x) \in \triangle$. Since $xx = 1 \in F$, we have $(x,x) \in U_F$. Thus (U1) holds.

Note that $(x,y) \in U_F$ if and only if $xy \in F$ and $yx \in F$ if and only if $(y,x) \in {U_F}^{-1}$ if and only if $(x,y) \in {U_F}^{-1}$. Hence ${U_F}^{-1} = U_F \in \mathcal{F}^*$, which shows (U2) is true.

To prove (U3), let $\Sigma(F) := \{F_\alpha | F_\alpha \subset F\}$ be the collection of filters contained in F. Clearly, $\Sigma(F)$ is not empty. Let G be the filter generated by $\bigcup_\alpha F_\alpha$. Then $U_G \in \mathcal{F}^*$. It is sufficient to show that $U_G \circ U_G \subset U_F$. If $(x,y) \in U_G \circ U_G$, then there exists $z \in X$ such that $(x,z) \in U_G$ and $(z,y) \in U_G$. It follows from Theorem 1.4 that $(x,y) \in U_G$, that is, $xy \in G$ and $yx \in G$. Since G is the minimal filter containing $\bigcup_\alpha F_\alpha$ and since $\bigcup_\alpha F_\alpha \subset F$, it follows that $G \subset F$. Hence $xy \in F$ and $yx \in F$, and thus $(x,y) \in U_F$. This proves $U_G \circ U_G \subset U_F$.

Finally we prove (U4). This will follow from the observation that $U_G \cap U_F = U_{G \cap F}$ for all $U_G, U_F \in \mathcal{F}^*$. Let $(x,y) \in U_G \cap U_F$. Then $(x,y) \in U_G$ and $(x,y) \in U_F$, which imply that $xy \in G$, $yx \in G$, $xy \in F$ and $yx \in F$. Hence $xy \in G \cap F$ and $yx \in G \cap F$, which shows $(x,y) \in U_{G \cap F}$. Similarly, we can show that $U_{G \cap F} \subset U_G \cap U_F$, whence $U_G \cap U_F = U_{G \cap F}$. This completes the proof. □

THEOREM 2.3. *Let X be a fuzzy implication algebra and let*

$$\mathcal{F} := \{U \subset X \times X | U \supset U_F \text{ for some } U_F \in \mathcal{F}^*\}.$$

Then $\mathcal{F}$ satisfies a uniformity on X and hence the pair $(X, \mathcal{F})$ is a uniform space .

PROOF. Using Theorem 2.2, we can show that $\mathcal{F}$ satisfies the conditions (U1)-(U4). To prove (U5), let $U \in \mathcal{F}$ and $U \subset V \subset X \times X$. Then there exists a $U_F \in \mathcal{F}^*$ such that $U_F \subset U \subset V$, which implies that $V \in \mathcal{F}$. This completes the proof. □

For $x \in X$ and $U \in \mathcal{F}$, we define

$$U[x] = \{y \in X | (x,y) \in U\}.$$

THEOREM 2.4. *Let X be a fuzzy implication algebra. For each $x \in X$, the collection $\mathcal{U}_x = \{U[x] | U \in \mathcal{F}\}$ forms a neighborhood base at x, making X a topological space.*

PROOF. First note that $x \in U[x]$ for each x. Second,

$$U_1[x] \cap U_2[x] = (U_1 \cap U_2)[x],$$

which means that the intersection of neighborhoods is a neighborhood. Finally, if $U[x] \in \mathcal{U}_x$ then there exists a $E \in \mathcal{F}$ such that $E \circ E \subset U$ by (U3). Then for any $y \in E[x], E[y] \subset U[x]$, so this property of neighborhoods is satisfied. □

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