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ON MAXIMAL SUBSET-SUM-DISTINCT SEQUENCES

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ABSTRACT. Since P. Erdös introduced the concept of "subset-sum-distinctness", lots of mathematicians have been interested in a "dense" set having distinct subset sums. In this paper, we establish a couple of theorems on maximal subset-sum-distinct sequence with respect to the set inclusion.

1. Introduction

A subset-sum-distinct set is defined as a set of positive integers such that no two finite subsets have the same sum. In this case we say briefly that it is "SSD" or it is an "SSD-set". If an increasing sequence of positive integers $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ is SSD, we say that $\mathbf{a}$ is a subset-sum-distinct sequence or briefly "SSD-sequence".

DEFINITION 1.1. An SSD-sequence $\mathbf{a}$ is called *maximal* if any set of positive integers containing $\mathbf{a}$ properly cannot be an SSD-sequence.

Stimulated by P. Erdös' open question ([6, p. 114, problem C8]), finite dense SSD-sets have been considered by many mathematicians (see [1], [2], [3], [4], [5, pp. 59–60], [7]). In the next section, we change our point of view to maximal SSD-sequences which are as dense as possible in the sense that no properly larger SSD-sequences exist.

As preliminary knowledge, we observe that SSD-sets are closed under a number of operations:

(i) (Subsets) A subset of an SSD-set is again an SSD-set; hence SSD-sets are also closed under intersection.
(ii) (Dilation) If $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ is SSD, then so is $k\ \mathbf{a} = \{ka_n\}_{n=1}^{\infty}$ where k is a positive integer.

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(iii) (Contractions) Let $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ be SSD and $B = \{b_1, b_2, \cdots, b_k\}$ a finite subset of $\mathbf{a}$. Remove every element of B from $\mathbf{a}$ and then adjoin $b_1 + b_2 + \cdots + b_k$.

(iv) (Greedy closure) For a given SSD-set A, let b be the smallest positive integer such that $b \notin A$ and A with b adjoined is SSD. Call this new set A'; let $A' = A$ if there is no such b. Repeat this procedure a countably infinite number of times. We call the resulting sequence the greedy closure of A. For example, the greedy closure of $\{3, 5, 21\}$ is $\{1, 3, 5, 10, 21, 41, 82, 164, \cdots\}$. Note that every greedy closure of a given SSD-set is a maximal SSD-sequence.

(v) (Translations) If $\{a_1, a_2, \cdots, a_k\}$ is SSD and the integer $K > a_1 + a_2 + \cdots + a_k$, then $\{K + a_1, K + a_2, \cdots, K + a_k\}$ is SSD also. A simple proof appears as Lemma 2.1 in the next section.

(vi) (Large dilation adjunctions) If $A = \{a_1, a_2, \cdots, a_k\}$ and B are SSD, then so is A with $K \cdot B = \{Kb : b \in B\}$ adjoined, provided K is an integer satisfying $K > a_1 + a_2 + \cdots + a_k$.

2. Main Theorems

We start with a lemma that will be used in the proof of Theorem 2.4.

LEMMA 2.1. *If $\{b_1, b_2, b_3, \cdots, b_m\}$ is SSD and $K > b_1 + b_2 + \cdots + b_m$, then so is the set*

$$A := \{K + b_1, K + b_2, K + b_3, \cdots, K + b_m\}.$$

PROOF. Suppose that A is not SSD. Then there are two distinct subsets I, J of $\{1, 2, 3, \cdots, m\}$ such that

$$\sum_{i \in I}(K + b_i) = \sum_{j \in J}(K + b_j).$$

Since $\{b_1, b_2, \cdots, b_m\}$ is SSD, we have $|I| \neq |J|$. So, we may assume that $|J| > |I|$. But then we have

$$K \leq (|J| - |I|)K = \sum_{i \in I} b_i - \sum_{j \in J} b_j \leq b_1 + b_2 + \cdots + b_m < K,$$

a contradiction. □

The following theorem gives a delicate sufficient condition for an SSD-sequence to be maximal.

THEOREM 2.2. *For an SSD-sequence* $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, *suppose there exists* N *such that* $a_{N+1} \leq 2a_N + 1$ *and*

$$\frac{a_{j+1}}{a_j} \leq 3, \quad j \geq N \tag{2.1}$$

and for any integer n *with* $1 \leq n \leq a_N$ *we have, for some* $I, J \subseteq \{1, 2, 3, \cdots, N\}$,

$$n = \sum_{i \in I} a_i - \sum_{j \in J} a_j. \tag{2.2}$$

Then $\mathbf{a}$ *is maximal. Moreover the constant 3 in* (2.1) *is the best possible in the sense that if 3 is replaced by any* $\beta > 3$, *then the theorem fails to hold.*

PROOF. For the maximality of $\mathbf{a}$, it suffices to show that any positive integer n can be written as in (2.2) with $I, J \subseteq \{1, 2, 3, \cdots\}$. We show that, for any positive integer k,

$$\text{For all } n \text{ with } 1 \leq n \leq \sum_{j=0}^{k-1} a_{N+j}, \qquad n = \sum_{i \in I} a_i - \sum_{j \in J} a_j \tag{2.3}$$

for some $I, J \subseteq \{1, 2, 3, \cdots, N + k - 1\}$. We show first that, for any positive integer k,

$$a_{N+k} \leq 2 \sum_{j=0}^{k-1} a_{N+j} + 1 \tag{2.4}$$

by using induction on k. Note that it is true for $k = 1$ by assumption of the theorem. Suppose (2.4) is true for fixed k. Then

$$\begin{aligned} a_{N+k+1} &\leq 3a_{N+k} \qquad \text{by} \qquad (2.1) \\ &= 2a_{N+k} + a_{N+k} \\ &\leq 2a_{N+k} + 2\sum_{j=0}^{k-1} a_{N+j} + 1 \\ &= 2\sum_{j=0}^{k} a_{N+j} + 1 \end{aligned}$$

which proves (2.4) for all k. Let us prove (2.3) by induction on k. If $k=1$, then we have (2.2). Suppose (2.3) is true for fixed k and $1 \le n \le \sum_{j=0}^{k} a_{N+j}$. If $1 \le n \le \sum_{j=0}^{k-1} a_{N+j}$, then we can take the same I, J as in (2.3). If $n \ge a_{N+k}$, then $0 \le n - a_{N+k} \le \sum_{j=0}^{k-1} a_{N+j}$. Hence

$$n - a_{N+k} = \sum_{i \in I} a_i - \sum_{j \in J} a_j$$

for some $I, J \subseteq \{1, 2, 3, \cdots, N+k-1\}$ by the induction hypothesis. Thus

$$n = \sum_{i \in I_1} a_i - \sum_{j \in J_1} a_j \quad \text{where} \quad I_1 = I \cup \{N+k\},\ J_1 = J.$$

Now assume that $\sum_{j=0}^{k-1} a_{N+j} < n < a_{N+k}$. Then

$$1 \le a_{N+k} - n \le a_{N+k} - \sum_{j=0}^{k-1} a_{N+j} - 1 \le \sum_{j=0}^{k-1} a_{N+j}$$

where the last inequality comes from (2.4). Hence, by the induction hypothesis,

$$a_{N+k} - n = \sum_{i \in I} a_i - \sum_{j \in J} a_j$$

for some $I, J \subseteq \{1, 2, 3, \cdots, N+k-1\}$. Thus

$$n = \sum_{i \in I_2} a_i - \sum_{j \in J_2} a_j \quad \text{where} \quad I_2 = J \cup \{N+k\},\ J_2 = I.$$

For the last statement of the theorem, let $\beta > 3$, and choose a positive integer m large enough so that $(3^{m+1}+1)/3^m < \beta$. Then $\mathbf{a} = \{1, 3, 3^2, 3^3, \cdots, 3^m, 3^{m+1}+1, 3^{m+2}, 3^{m+3}, \cdots\}$ satisfies all the conditions of the theorem with $N=1$ if the constant 3 in (2.1) is replaced by β. But $\mathbf{a}$ is not maximal since $\mathbf{a} \cup \{(3^{m+1}+1)/2\}$ forms an SSD-set. □

COROLLARY 2.3. *If $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ is an SSD-sequence such that*

$$a_1 = 1 \qquad \text{and} \qquad \frac{a_{j+1}}{a_j} \le 3, \quad j \ge 1,$$

then $\mathbf{a}$ is maximal. In particular, $\{1, 3, 3^2, 3^3, \cdots\}$ is a maximal SSD-sequence.

PROOF. Apply Theorem 2.2 with $N = 1$. □

As the following theorem shows, it is quite curious that there is no positive lower bound on the reciprocal sums of maximal SSD-sequences.

THEOREM 2.4. *For any $\epsilon > 0$, there exists a maximal SSD-sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \frac{1}{a_n} < \epsilon$.*

PROOF. Let $\epsilon > 0$ be given. First choose an integer m large enough so that

$$\frac{2}{2^m} + \frac{m}{2^m} < \epsilon. \tag{2.5}$$

Then define a sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ by

$$a_n = \begin{cases} 2^m & \text{if } n = 1, \\ 2^m + 2^{n-2} & \text{if } 2 \leq n \leq m+1, \\ 2^{n-1} & \text{if } n \geq m+2. \end{cases}$$

We show that $\mathbf{a}$ is a maximal SSD-sequence whose reciprocal sum is less than ϵ. For a contradiction, suppose $\mathbf{a}$ is not SSD. Split $\mathbf{a}$ into two parts X and Y where

$$\begin{aligned} X &:= \{a_1, a_{m+2}, a_{m+3}, a_{m+4}, \cdots\} = \{2^m, 2^{m+1}, 2^{m+2}, 2^{m+3}, \cdots\}, \\ Y &:= \{a_2, a_3, a_4, \cdots, a_{m+1}\} \\ &= \{2^m + 1, 2^m + 2, 2^m + 2^2, \cdots, 2^m + 2^{m-1}\}. \end{aligned}$$

Obviously, X is SSD and, by Lemma 2.1, Y is SSD also. Hence there exist $X_1, X_2 \subseteq X$ and $W_1, W_2 \subseteq \{1, 2, 2^2, \cdots, 2^{m-1}\}$, all disjoint from each other, such that

$$\sum_{x_1 \in X_1} x_1 + \sum_{w_1 \in W_1} (2^m + w_1) = \sum_{x_2 \in X_2} x_2 + \sum_{w_2 \in W_2} (2^m + w_2).$$

Since 2^m divides all the elements of X, we obtain

$$\sum_{w_1 \in W_1} w_1 - \sum_{w_2 \in W_2} w_2 \equiv 0 \pmod{2^m}. \tag{2.6}$$

But this is impossible because the absolute value of the left side of (2.6) is greater than 0 and less than 2^m. Thus we obtain the SSD property of **a**. For this choice of **a**, we have

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{x \in X} \frac{1}{x} + \sum_{y \in Y} \frac{1}{y} \leq \frac{2}{2^m} + \frac{m}{2^m} < \epsilon$$

by (2.5). It remains to prove the maximality of **a**. It is enough to show that, for any positive integer n, there exist two subsets U, V of **a** such that

$$n = \sum_{u \in U} u - \sum_{v \in V} v. \tag{2.7}$$

Let $n = n_1 2^m + n_2$ where n_1, n_2 are non-negative integers with $0 \leq n_2 \leq 2^m - 1$. Clearly we can find $A \subseteq X$ and $B \subseteq Y$ such that

$$\sum_{b \in B} b = |B| 2^m + n_2 \qquad \text{and} \qquad \sum_{a \in A} a = |n_1 - |B|| 2^m.$$

Thus we obtain (2.7) if we take

$$U = \begin{cases} A \cup B & \text{if} \quad n_1 \geq |B|, \\ B & \text{if} \quad n_1 < |B|, \end{cases} \quad \text{and} \quad V = \begin{cases} \phi & \text{if} \quad n_1 \geq |B|, \\ A & \text{if} \quad n_1 < |B|. \end{cases} \qquad \square$$

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