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CHOW GROUPS OF COMPLETE REGULAR LOCAL RINGS III

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ABSTRACT. In this paper we will show that the followings ; (1) Let R be a regular local ring of dimension n. Then $A_{n-2}(R) = 0$. (2) Let R be a regular local ring of dimension n and I be an ideal in R of height 3 such that R/I is a Gorenstein ring. Then $[I] = 0$ in $A_{n-3}(R)$. (3) Let $R = V[[X_1, X_2, \cdots, X_5]]/(p + X_1^{t_1} + X_2^{t_2} + X_3^{t_3} + X_4^2 + X_5^2)$, where $p \neq 2$, t_1, t_2, t_3 are arbitrary positive integers and V is a complete discrete valuation ring with $(p) = m_V$. Assume that R/m is algebraically closed. Then all the Chow group for R is 0 except the last Chow group.

1. Introduction

We define the i-th Chow group $A_i(R)$ for a Noetherian Cohen -Macaulay ring R of dimension n by $Z_i(R)/Rat_i(R)$, where $Z_i(R)$ is the free abelian group generated by prime ideals in R of height(ht) $n - i$ and $Rat_i(R)$ is the subgroup of $Z_i(R)$ generated by the cycles of the form $\sum l(R_{P_i}/(q+(x))R_{P_i})[P_i]$, where q is a prime ideal of height $n-i-1$, $x \notin q$ and P_i ranges over the minimal associated prime ideals of $R/(q+(x))$ satisfying $\dim R/P = \dim R/(q+(x))$. When M is a finitely generated R-module, we have $[M] = \sum l(M_{P_i})[P_i]$, where P_i ranges over the minimal associated prime ideals of M satisfying $\dim R/P_i = \dim M$. From Claborn and Fossum [2], if R is a regular local ring, then the above definition is equivalent to the group $Z_i(R)/ < R/(x_1, x_2, \cdots, x_{n-i}) >$, where $< R/(x_1, x_2, \cdots, x_{n-i}) >$ is the subgroup of $Z_i(R)$ generated by $\sum l((R/(x_1, x_2, \cdots, x_{n-i}))_P)[P]$, and P ranges over the associated prime ideals of $R/(x_1, x_2, \cdots, x_{n-i})$, for all R-sequence $x_1, x_2, \cdots, x_{n-i}$. From this definition, when R is a regular local ring of dimension n, we

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have $A_0(R) = 0$, $A_{n-1}(R) = 0$, $A_n(R) = Z$(the ring of integers) and from [3], we get $A_1(R) = 0$. Because of the hardness of the problem, there are a few results on this problem. Even in a dimension 5 case, we can not characterize the Chow group completely. In Section 2, we shall discuss $A_{n-2}(R)$ for a complete regular local ring R of dimension n. In Section 3, we will show that when R is a regular local ring of dimension n and I is an ideal in R of height 3 such that R/I is a Gorenstein ring, $[I] = 0$ in $A_{n-3}(R)$ by using linkage class technique. In the dimension 5 case, to calculate $A_2(R)$ still remains open. In section 4, we shall discuss $A_2(R)$ for the ring $R = V[[X_1, X_2, \cdots, X_5]]/(p + X_1^{t_1} + X_2^{t_2} + X_3^{t_3} + X_4^2 + X_5^2)$, where $p \neq 2$, t_1, t_2, t_3 are arbitrary positive integers and V is a complete discrete valuation ring with $(p) = m_V$, and $A_{n-2}(R)$ for arbitrary complete regular local ring R. We also show that how the Chow group process is going on in the dimension 5 case.

2. Determination of $A_{n-2}(R)$

In calculating Chow group of complete regular local ring, Claborn and Fossum cleared unramified case by using generalized ideal class technique [2]. Thus only ramified case remains open. In a ramified regular local ring R of dimension n, we know $A_0(R) = 0$, $A_{n-1}(R) = 0$, $A_n(R) = Z$ from the definition; $A_1(R) = 0$ follows from [3, Theorem 1.1]. In order to characterize $A_{n-2}(R)$ we need the following Theorem due to W. Smoke.

THEOREM 2.1 (W. Smoke [8], Theorem 5.2). *Let R be a regular local ring of dimension n. Then the group $K_0(\underline{R}_2)$ is generated by the elements $[R/(x_1, x_2)]$, where $\underline{R}_2$ is the category of all finitely generated R-modules M such that $M_P = 0$ for all prime ideals of height less than 2, $K_0(\underline{R}_2)$ is the corresponding Grothendieck group and x_1, x_2 forms a regular sequence on R.*

THEOREM 2.2. *Let R be a regular local ring of dimension n. Then $A_{n-2}(R) = 0$.*

PROOF. In the proof of Theorem 2.1, W. Smoke first showed that every $[M]$ in $K_0(\underline{R}_2)$ is generated by $[R/I]$ and $[R/(x_1, x_2)]$, where x_1, x_2 in I form an R-sequence and I is a perfect ideal of codimension 2. Next, he showed that $[I/(x_1, x_2)]$ is generated by elements of the form $[R/(x_1, x_2)]$. We know that $[R/(x_1, x_2)] = 0$ in $A_{n-2}(R)$. Thus $[I/(x_1, x_2)] = 0$ in $A_{n-2}(R)$. Now from the following exact sequence

$$0 \to I/(x_1, x_2) \to R/(x_1, x_2) \to R/I \to 0$$

we conclude that $[R/I] = 0$ in $A_{n-2}(R)$. Hence $[M] = 0$ in $A_{n-2}(R)$ and therefore $A_{n-2}(R) = 0$ by the choice of M. □

3. Linkage class of codimension 3 Gorenstein ideals

In this section, we will discuss the linkage class of codimension 3 Gorenstein ideals.

The following Theorem (S. Lee and K. B. Hwang [6], Theorem 2.1) was already proved by using the lifting property of a Gorenstein ideal of codimension 3 and a lifting Theorem. In this paper, we are going to present a completely different approach which is an easier way than the previous approach to the Theorem-the idea of linkage of ideals for height 3 Gorenstein ideals. Our main tool is Watanabe's Theorem.

THEOREM 3.1. *Let R be a regular local ring of dimension n and I be an ideal in R of height 3 such that R/I is a Gorenstein ring. Then $[I] = 0$ in $A_{n-3}(R)$.*

For the proof of this theorem, we need the following definition (for the details about linkage class, refer to [7]).

DEFINITION 3.2. Let I and J be ideals of a Cohen-Macaulay ring R of same grade g. We say I is linked to J if there is an R-sequence $x_1, \cdots, x_g = \underline{x} \in I \cap J$ such that $J = \underline{x} : I$ (i.e., I and J are linked if $\mathrm{Hom}(R/I, R/\underline{x}) = J/\underline{x}$). The linkage class of a perfect ideal I is the set of all ideals which can be obtained from I by iterating the linkage procedure.

We now state the following theorem due to J. Watanabe.

THEOREM 3.3 (J. Watanabe [9], Theorem). *Let (R, m, k) be a regular local ring and I be an ideal in R of height 3 such that R/I is a Gorenstein ring. Then I is minimally generated by an odd number of elements.*

REMARK. In his proof, when I is a Gorenstein ideal minimally generated by n elements, J. Watanabe constructed an almost complete intersection ideal J of height 3 which is linked to I, and in turn J is linked to a Gorenstein ideal with $n-2$ generators (he used this to show that n must be odd). From this after a finite number of steps of linkage procedure, we get a complete intersection ideal J' which is in the same linkage class of a complete intersection ideal.

LEMMA 3.4. *Let I and J be perfect ideals of same grade g in a Gorenstein local ring R of dimension n. Suppose I is a Gorenstein ideal.*

(1) *If I is linked to J by an R-sequence $x_1, \cdots, x_g = \underline{x}$, then $[I] = 0$ in $A_{n-g}(R)$ if and only if $[J] = 0$ in $A_{n-g}(R)$.*
(2) *If I and J are in the same linkage class, then $[I] = 0$ in $A_{n-g}(R)$ if and only if $[J] = 0$ in $A_{n-g}(R)$.*

PROOF. (2) follows (1). Let's show (1). Since I is linked to J by an R-sequence $x_1, \cdots, x_g = \underline{x}$, we have $\mathrm{Hom}(R/I, R/\underline{x}) = J/\underline{x}$. By the definition of Gorenstein ideal, we have $\mathrm{Ext}^g_R(R/I, R/\underline{x}) \cong R/I$. On the other hand, $\mathrm{Ext}^g_R(R/I, R) = \mathrm{Hom}(R/I, R/\underline{x}) \cong J/\underline{x}$, and thus $[I] = 0$ in $A_{n-g}(R)$ if and only if $[J/\underline{x}] = 0$ in $A_{n-g}(R)$. Moreover, we have an exact sequence

$$0 \to J/\underline{x} \to R/\underline{x} \to R/J \to 0.$$

Since $[R/\underline{x}] = 0$ in $A_{n-g}(R)$, from the exact sequence, we have $[J] = 0$ in $A_{n-g}(R)$ if and only if $[J/\underline{x}] = 0$ in $A_{n-g}(R)$. Hence $[I] = 0$ in $A_{n-g}(R)$ if and only if $[J] = 0$ in $A_{n-g}(R)$. □

PROOF OF THEOREM 3.1. Since I is a Gorenstein ideal of grade 3, by our remark above, I is in the linkage class of a complete intersection ideal. Thus we can find a linkage relation between I and a complete intersection ideal by iterating the linkage procedure starting from I. By Lemma 3.4, we have the required result. □

In a dimension 5 case, from the Theorem 2.2 and the discussion at the introduction, we only remains to calculate $A_2(R)$.

4. Dimension of R is 5

Let $R = V[[X_1, X_2, \cdots, X_n]]/(p + X_1^2 + X_2^2 + \cdots + X_n^2)$, where $p \neq 2$ and V is a complete discrete valuation ring with $(p) = m_V$. Assume that R/m is algebraically closed. M. Levine showed $A_i(R) = 0$ for $0 \leq i < n$ for this ring by using K-theoretic techniques [4]. Later S.

P. Dutta showed this by using commutative algebra techniques [3]. In dimension 5, we have the following :

THEOREM 4.1. *Suppose that* $p \neq 2$, t_1, t_2, t_3 *are arbitrary positive integers and* V *is a complete discrete valuation ring with* $(p) = m_V$. *Let* $R = V[[X_1, X_2, \cdots, X_5]]/(p + X_1^{t_1} + X_2^{t_2} + X_3^{t_3} + X_4^2 + X_5^2)$. *Then* $A_i(R) = 0$ *for* $0 \leq i < 5$.

In the course of our proof of Theorem 4.1, we need the following result due to S. P. Dutta.

PROPOSITION 4.2 (S. P. Dutta [3], Proposition 2.2). *Let* (S, m, k), k *infinite, be a Cohen-Macaulay local ring of dimension* n *and let* f *be a non-zero-divisor in* m. *Let* I *be an ideal in* S *of height* i *such that* $[I] = 0$ *in* $A_{n-i}(S)$ *and let* f *be a non-zero-divisor on* S/I. *Then* $[I + fS/fS] = 0$ *in* $A_{n-1-i}(S/fS)$.

PROOF OF THEOREM 4.1. It suffices to show that $A_2(R) = 0$. Let P be a prime ideal of height 3 such that $[P] \in A_2(R)$. Since R/m is infinite and $htP \geq 2$, by applying non-singular transformation, if necessary, we may assume R/P is a module finite extension of $T = V[[x_1, x_2, x_3]]$([1], (23.5)). Since R/m is algebraically closed and $p \neq 2$, there is a unit j in R such that $j^2 = -1$. Write $a = p + X_1^{t_1} + X_2^{t_2} + X_3^{t_3}$, $Z = X_5 + jX_4$, $Z' = X_5 - jX_4$. Then $R = T[[Z, Z']]/(a + ZZ')$. Let's denote $T[[Z']]$ by S. Then $R = S[[Z]]/(a + ZZ')$. Let $Q = P \cap S$. Then $R/P(= \bar{A})$ is a module finite extension of $S/Q(= \bar{S})$. Hence the image z of Z in $\bar{A}$ satisfies a monic polynomial $F(Y) \in \bar{S}[Y]$. By the Henselian property, the coefficients of F, except that of the highest degree term, are in $m_{\bar{S}}$. Let $B = (S[Z]/(a + ZZ'))_{(m_S, Z)}$ and $J = P \cap B$. Then $\hat{B}$ (completion of B) $= R$ and $B/J \cong R/P$; when we denote by $F'(z)$ and $F''(z)$ the liftings of $F(z)$ in B and R respectively and denote both as $F(z)$, $B/(F(z)) = R/(F(z))$ and hence $JR = P$. If P contains any of a, z or z', then P contains z or z'. So $[P/zR] = 0$ in $A_2(R/zR)$ or $[P/z'R] = 0$ in $A_2(R/z'R)$. Hence $[P] = 0$ in $A_2(R)$. Suppose P does not contain any of a, z or z'. Note that $S[1/z'] = B[1/z']$. A primary decomposition of QB in B is of the form

$$QB = J \cap J_2 \cap \cdots \cap J_d.$$

Let Q_t denote the prime ideal corresponding to J_t for each $t = 2, \cdots, d$. By the definition of Chow group, we may assume that height of Q_t is 3 for all t. Then each J_t contains z'. Hence every prime ideal P_s which occurs in the primary decomposition of QR in R, except P contains z'. It follows that the corresponding primary component $[I_s] = 0$ in $A_2(R)$. Moreover, $[Q] = 0$ in $A_2(S)$ implies that $[Q[[Z]]] = 0$ in $A_3(S[[Z]])$ and hence $[QR] = 0$ in $A_2(R)$ by Proposition 4.2. Thus our required result follows from the following short exact sequence:

$$0 \to R/QR \to R/P \oplus R/\cap I_s \to R/(P + \cap I_s) \to 0. \qquad \square$$

REMARK. The arguments, in the proof of Theorem 4.1, do not work in general. The main difficulty comes from the fact that we do not know how to tackle the case when the Eisenstein polynomial is of the form $X_n^2 + aX_n + b$. It was shown in [3], that in several very important cases of Chow group problem, for arbitrary Eisenstein polynomials, we could get the desired result if we could handle the case of quadratic equations properly. The example below demonstrates this fact even for polynomials which are "very nice". Let $f = p + X_1^2 + \cdots + X_{n-2}^2 + X_{n-1}^4 + X_n^4$ and $R = V[[X_1, \cdots, X_n]]/(f)$ where V, p as above. Let $T = [[X_1, \cdots, X_{n-2}]]$ and $a = p + X_1^2 + \cdots + X_{n-2}^2$. As in the proof of Theorem 3.1, we can find a unit j in R such that $j^2 = -1$. Let $Z = X_{n-1}^2 + jX_n^2$, $Z' = X_{n-1}^2 - jX_n^2$ and $B = T[[Z, Z']]/(a + ZZ')$. Then $A_i(B) = 0$ for $i = 0, 1, \cdots, n-1$. Let $C = B[[X_{n-1}]]/(X_{n-1}^2 - (Z + Z')/2)$ and $D = C[[X_n]]/(X_n^2 - (Z - Z')/2j)$. Then $C = T[[Z, Z', X_{n-1}]]/(a + ZZ', X_{n-1}^2 - (Z+Z')/2) = T[[Z, X_{n-1}]]/(a + Z(2X_{n-1}^2 - Z))$ and $R = D$. Moreover, $B \hookrightarrow C$ and $C \hookrightarrow D$ are integral flat extension (for flatness see, Matsumura [5] Theorem23.1). As of the construction, proving $A_i(R) = 0$ for $i = 0, 1, \cdots, n-1$ is the same as proving $A_i(C) = 0$ for $i = 0, 1, \cdots, n-1$. Let $C' = T[[Z, X_{n-1}^2]]/(a + Z(2X_{n-1}^2 - Z)) = T[[Z, 2X_{n-1}^2 - Z]]/(a + Z(2X_{n-1}^2 - Z))$. Then $A_i(C') = 0$ for $i = 0, 1, \cdots, n-1$. This could be shown by using similar techniques as in Theorem. Now C is a quadratic extension of C' and D is a quadratic extension of C. Both involve the simplest quadratic equations, but still we do not know how to get the required results in the above cases.

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