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ON r-IDEALS IN INCLINE ALGEBRAS

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ABSTRACT. In this paper we show that if $\mathcal{K}$ is an incline with multiplicative identity and I is an r-ideal of $\mathcal{K}$ containing a unit u, then $I = \mathcal{K}$. Moreover, we show that in a non-zero incline $\mathcal{K}$ with multiplicative identity and zero element, every proper r-ideal in $\mathcal{K}$ is contained in a maximal r-ideal of $\mathcal{K}$.

Z. Q. Cao, K. H. Kim and F. W. Roush [3] introduced the notion of incline algebras in their book, *Incline algebra and applications*, and these concepts were studied by some authors ([2, 4, 5]). Inclines are a generalization of both Boolean and fuzzy algebras, and a special type of a semiring, and give a way to combine algebras and ordered structures to express the degree of intensity of binary relations. The present authors with Y. B. Jun [2] introduced the notion of quotient incline and obtained the structure of incline algebras, and also introduced the notion of prime and maximal ideals in an incline, and studied some relations between them in incline algebras. In this paper, as a continuation of [2], we show that if $\mathcal{K}$ is an incline with multiplicative identity and I is an r-ideal of $\mathcal{K}$ containing a unit u, then $I = \mathcal{K}$. Moreover, we show that in a non-zero incline $\mathcal{K}$ with multiplicative identity and zero element, every proper r-ideal in $\mathcal{K}$ is contained in a maximal r-ideal of $\mathcal{K}$.

DEFINITION 1 ([3]). An *incline* (*algebra*) is a set $\mathcal{K}$ with two binary operations denoted by " $+$ " and " $*$ " satisfying the following axioms: for all $x, y, z \in \mathcal{K}$,

(i) $x + y = y + x$,
(ii) $x + (y + z) = (x + y) + z$,
(iii) $x * (y * z) = (x * y) * z$,
(iv) $x * (y + z) = (x * y) + (x * z)$,

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(v) $(y+z)*x = (y*x)+(z*x)$,
(vi) $x+x=x$,
(vii) $x+(x*y)=x$,
(viii) $y+(x*y)=y$.
Furthermore, an incline algebra $\mathcal{K}$ is said to be *commutative* if $x*y = y*x$ for all $x, y \in \mathcal{K}$.

For convenience, we pronounce "+" (resp. "*") as *addition* (resp. *multiplication*). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x * x = x$ for all $x \in \mathcal{K}$ ([3, Proposition (1.1.1)]). Note that $x \leq y$ iff $x + y = y$ for all $x, y \in \mathcal{K}$. A *subincline* of an incline $\mathcal{K}$ is a non-empty subset M of $\mathcal{K}$ which is closed under addition and multiplication. An *ideal* in an incline $\mathcal{K}$ is a subincline $M \subseteq \mathcal{K}$ such that if $x \in M$ and $y \leq x$ then $y \in M$. An element 0 in an incline algebra $\mathcal{K}$ is a *zero element* if $x + 0 = x = 0 + x$ and $x * 0 = 0 * x = 0$, for any $x \in \mathcal{K}$.

DEFINITION 2 ([1]). Let $\mathcal{K}$ be an incline. A proper ideal P of $\mathcal{K}$ is said to be *prime* if for all $a, b \in \mathcal{K}, a * b \in P$ implies either $a \in P$ or $b \in P$. An ideal M in $\mathcal{K}$ is called a *maximal ideal* of $\mathcal{K}$ if $M \neq \mathcal{K}$ and for every ideal N with $M \subseteq N \subseteq \mathcal{K}$, either $N = M$ or $N = \mathcal{K}$.

EXAMPLE 3 ([1]). The ring $(Z_6, +, \cdot)$ has 4 ideals as follows:

$$I_1 = \langle 0 \rangle,\ I_2 = \langle 1 \rangle,\ I_3 = \langle 2 \rangle,\ I_4 = \langle 3 \rangle.$$

We define sum " + " and product " * " on $\mathcal{I} := \{I_i \mid i = 1, 2, 3, 4\}$ as follows:

$+$	I_1	I_2	I_3	I_4
I_1	I_1	I_2	I_3	I_4
I_2	I_2	I_2	I_2	I_2
I_3	I_3	I_2	I_3	I_2
I_4	I_4	I_2	I_2	I_4

Table 1

$*$	I_1	I_2	I_3	I_4
I_1	I_1	I_1	I_1	I_1
I_2	I_1	I_2	I_3	I_4
I_3	I_1	I_3	I_3	I_1
I_4	I_1	I_4	I_1	I_4

Table 2

Then $(\mathcal{I}, +, *)$ is an incline algebra and I_1 is the zero element of $\mathcal{I}$.

If we define the sets

$$L_1 := \{I_i \in \mathcal{I} | I_i \leq I_1\} = \{I_1\},$$
$$L_2 := \{I_i \in \mathcal{I} | I_i \leq I_2\} = \{I_1, I_2, I_3, I_4\},$$
$$L_3 := \{I_i \in \mathcal{I} | I_i \leq I_3\} = \{I_1, I_3\},$$

and

$$L_4 := \{I_i \in \mathcal{I} | I_i \leq I_4\} = \{I_1, I_4\},$$

then all L_i are ideals of $\mathcal{I}$ and especially L_3, L_4 are maximal ideals.

DEFINITION 4. An element $1_{\mathcal{K}}$ ($\neq$ zero element) in an incline algebra $\mathcal{K}$ is called a *multiplicative identity* if for any $x \in \mathcal{K}$, $x * 1_{\mathcal{K}} = 1_{\mathcal{K}} * x = x$.

Every incline $\mathcal{K}$ with zero element has two ideals, the improper ideal $\mathcal{K}$ and the trivial ideal $\{0\}$. For these ideals, the factor inclines are $\mathcal{K}/\mathcal{K}$, which has only one element, and $\mathcal{K}/\{0\}$, which is isomorphic to $\mathcal{K}$. A proper non-trivial ideal of an incline $\mathcal{K}$ is an ideal N of $\mathcal{K}$ such that $N \neq \mathcal{K}$ and $N \neq \{0\}$. An ideal I of an incline $\mathcal{K}$ is said to be an *r-ideal* if $x * y, y * x \in I$ for any $x \in I$ and $y \in \mathcal{K}$.

EXAMPLE 5. In Example 3, all L_i are r-ideals of an incline $\mathcal{I}$.

An element $u \in \mathcal{K}$ is called a *unit* if it has a multiplicative invertible element. An incline with $1_{\mathcal{K}} \neq 0$ and zero element in which every non-zero element is a unit is called a *field incline.*

THEOREM 6. *If $\mathcal{K}$ is an incline with multiplicative identity and I is an r-ideal of $\mathcal{K}$ containing a unit u, then $I = \mathcal{K}$.*

PROOF. If $k \in \mathcal{K}$, then by Definition 1-(viii) $u + k * u = u$, and hence $k * u \leq u \in I$. Since I is an ideal of $\mathcal{K}$, $k * u \in I$. Hence $k = k * 1_{\mathcal{K}} = k * (u * u') = (k * u) * u'$ for some $u' \in \mathcal{K}$. It follows from I is an r-ideal that $k \in I$. Hence $\mathcal{K} = I$, completing the proof. □

COROLLARY 7. *A field incline contains no proper non-trivial r-ideals.*

PROOF. Every non-zero element of a field incline $\mathcal{K}$ is a unit. By Theorem 6, any r-ideal of a field incline $\mathcal{K}$ is either $\{0\}$ or all of $\mathcal{K}$. □

THEOREM 8. *In a non-zero incline $\mathcal{K}$ with multiplicative identity and zero element, every proper r-ideal in $\mathcal{K}$ is contained in a maximal r-ideal of $\mathcal{K}$.*

PROOF. Let A be an r-ideal in $\mathcal{K}$ such that $A \neq \mathcal{K}$ and let Φ be the set of all r-ideals B in $\mathcal{K}$ such that $A \subseteq B \neq \mathcal{K}$. Since $A \in \Phi$, $\Phi \neq \emptyset$. Define a partial order $\leq$ on Φ by the set inclusion (i.e., $B_1 \leq B_2 \iff B_1 \subseteq B_2$). In order to apply Zorn's Lemma we must show that every chain $\mathcal{C} = \{C_i | i \in \Lambda\}$ of r-ideals in Φ has an upper bound in Φ. Set $C := \cup_{i \in \Lambda} C_i$. Let $a \leq b$ and $b \in C$. Then $b \in C_i$ for some i. Since C_i is an ideal of $\mathcal{K}$, $a \in C_i$ and so $a \in C$. Hence C is an ideal of $\mathcal{K}$. Clearly, C is an r-ideal of $\mathcal{K}$. Since $C_i \in \Phi$, $1_{\mathcal{K}} \notin C_i$ and so $\mathcal{K} \neq C$. Clearly, C is an upper bound of the chain $\mathcal{C}$. Thus, by Zorn's Lemma, Φ contains a maximal element. We know that a maximal element of Φ is obviously a maximal r-ideal of $\mathcal{K}$ containing A, proving the theorem. □

DEFINITION 9. Let A be a non-empty subset of the incline $\mathcal{K}$ with a zero element 0. Then the set

$$A^* := \{x \in \mathcal{K} | a * (a * x) = 0, \ \forall a \in A\}$$

is called the *annihilator* of A.

It is obvious that $0 \in A^*$. If $A = \{a\}$, we write $(a)^*$ in place of $\{a\}^*$.

REMARK. In an incline $\mathcal{K}$ with a zero element 0, $(0)^* = \mathcal{K}$.

PROPOSITION 10. *Let A and B be non-empty subsets of $\mathcal{K}$ with a zero element. If $A \subseteq B$, then $B^* \subseteq A^*$.*

PROOF. Let $x \in B^*$. Then $b * (b * x) = 0$ for all $b \in B$. Since $A \subseteq B$, we have $a * (a * x) = 0$ for all $a \in A$ and consequently $B^* \subseteq A^*$. □

THEOREM 11. *Let A be a non-empty subset of a commutative incline $\mathcal{K}$ with a zero element. Then A^* is an r-ideal of $\mathcal{K}$.*

PROOF. By the definition of A^*, A^* is a non-empty subincline of $\mathcal{K}$. Let $x \in A^*$ and $y \leq x$ for any $y \in \mathcal{K}$. Then $y + x = x$ and hence $0 = a * (a * x) = a * (a * (y + x)) = a * (a * y) + a * (a * x) = a * (a * y)$, which means that $y \in A^*$. Thus A^* is an ideal of $\mathcal{K}$. Next, if $x \in A^*$ and $y \in \mathcal{K}$, then $a * (a * (x * y)) = (a * (a * x)) * y = 0 * y = 0$ and hence $x * y \in A^*$. Similarly $y * x \in A^*$. Therefore A^* is an r-ideal of $\mathcal{K}$. □

By a *homomorphism* of inclines we shall mean a mapping f from an incline $\mathcal{K}$ into an incline $\mathcal{L}$ such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in \mathcal{K}$. Assume that inclines $\mathcal{K}$ and $\mathcal{L}$ have a zero element respectively. A homomorphism f is said to be *regular* if $f(0) = 0$. If the incline $\mathcal{L}$ is additively cancellative, i.e., $a + b = a + c$ implies $b = c$, then any homomorphism is regular.

PROPOSITION 12. *Let $f : \mathcal{K} \to \mathcal{L}$ be a regular homomorphism of inclines with a zero element, respectively, and let A be a non-empty subset of $\mathcal{K}$. Then $f(A^*) \subseteq (f(A))^*$.*

PROOF. Let $y \in f(A^*)$ and $b \in f(A)$. Then there exist $x \in A^*$ and $a \in A$ such that $f(x) = y$ and $f(a) = b$, respectively. It follows that

$$\begin{aligned} b * (b * y) &= f(a) * (f(a) * f(x)) \\ &= f(a * (a * x)) \\ &= f(0) \\ &= 0 \end{aligned}$$

so that $y \in (f(A))^*$. This completes the proof. □

LEMMA 13. *Let $f : \mathcal{K} \to \mathcal{L}$ be a homomorphism of inclines. If J is an r-ideal of $\mathcal{L}$, then $f^{-1}(J)$ is an r-ideal of $\mathcal{K}$.*

PROOF. Let $x, y \in f^{-1}(J)$. Then $f(x), f(y) \in J$. Since J is a subincline of $\mathcal{L}$, it follows that

$$f(x + y) = f(x) + f(y) \in J \quad \text{and} \quad f(x * y) = f(x) * f(y) \in J$$

so that $x + y, x * y \in f^{-1}(J)$. This shows that $f^{-1}(J)$ is a subincline of $\mathcal{K}$. Let $x, y \in \mathcal{K}$ be such that $x \leq y$ and $y \in f^{-1}(J)$. Then $x + y = y$, and so

$$f(x) + f(y) = f(x + y) = f(y), \quad \text{that is,} \quad f(x) \leq f(y),$$

and $f(y) \in J$. It follows that $f(x) \in J$ so that $x \in f^{-1}(J)$ is an ideal of $\mathcal{K}$. Now let $x \in f^{-1}(J)$ and $y \in \mathcal{K}$. Then $f(x) \in J$, and hence $f(x * y) = f(x) * f(y) \in J$ and $f(y * x) = f(y) * f(x) \in J$ since J is an r-ideal. Thus $x * y, y * x \in f^{-1}(J)$. Consequently, $f^{-1}(J)$ is an r-ideal of $\mathcal{K}$. □

THEOREM 14. *Let $\mathcal{K}$ be an incline with a zero element and $\mathcal{L}$ be a commutative incline with a zero element. Let $f : \mathcal{K} \to \mathcal{L}$ be a regular homomorphism. Then for every non-empty subset B of $\mathcal{L}$, $f^{-1}(B^*)$ is an r-ideal of $\mathcal{K}$ containing $(f^{-1}(B))^*$.*

PROOF. Let B be a non-empty subset of $\mathcal{L}$. Then by Theorem 11, B^* is an r-ideal of $\mathcal{L}$. It follows from Lemma 13 that $f^{-1}(B^*)$ is an r-ideal of $\mathcal{K}$. If $x \in (f^{-1}(B))^*$, then $f^{-1}(b) * (f^{-1}(b) * x) = 0$ for every $b \in B$, which implies that

$$\begin{aligned} 0 =& f(0) = f(f^{-1}(b) * (f^{-1}(b) * x)) \\ =& f(f^{-1}(b)) * (f(f^{-1}(b)) * f(x)) \\ =& b * (b * f(x)) \end{aligned}$$

for all $b \in B$. Hence $f(x) \in B^*$, i.e., $x \in f^{-1}(B^*)$. This completes the proof. □

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