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SIMPLE LIE ALGEBRAS WHICH GENERALIZE KPS'S LIE ALGEBRAS

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ABSTRACT. In this paper we generalize the Lie algebras of KPS's in [4], which have no toral elements. However our generalized Lie algebras have toral elements. Moreover our Lie algebras are not isomorphic to the Witt algebra $W(n)$ with a toral element.

1. Introduction

Let F be a field of characteristic zero and let $\lambda_{i,j} \in F$. The skew polynomial ring $R(\lambda) = F[x_1, \cdots, x_n]$ with relations $x_i x_j = \lambda_{i,j} x_j x_i$ has been called a quasi-polynomial ring in [1]. (For more details, please refer to [4].) The corresponding Laurent polynomial ring

$$F[x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}] = P(\lambda) = S$$

obtained by inverting the x_i's, was studied by McConnell and Pettit [6]. In particular, it has been shown [6, Proposition 1.3] that S is simple if and only if the center of S is F if and only if there does not exist $m = (m_1, \cdots, m_n) \in Z^n$ with m_i's not all zero such that for all j, $1 \leq j \leq n$,

$$(\lambda_{1,j})^{m_1} \cdots (\lambda_{n,j})^{m_n} = 1.$$

Corresponding to S there is a matrix $\lambda = (\lambda_{i,j})$ with $\lambda_{j,i} = \lambda_{i,j}^{-1}$ for $i \neq j$ and $\lambda_{i,i} = 1$. Kirkman, Processi and Small showed in KPS's [4] that if S is a simple ring, then the Lie algebra $\mathrm{ad}(S)$ of inner derivations on S is a simple Lie algebra of KPS [4].

In this paper, we will define the Lie bracket on S by the algebra multiplication:

$$[X, Y] = XY - YX \text{ for } X, Y \in S.$$

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Note that this Lie algebra is isomorphic to ad(S) [4, p. 3770]. We will show that the Lie algebra M' in Section 2 is simple (please refer to Theorem 1 in Section 2) and the Lie algebra M' is not isomorphic to any Cartan type subalgebra of the Witt algebra $W^+(n)$ [9] (please refer to Proposition 2 in Section 3).

2. Main results

Let L_s be the Lie algebra with basis

$$B = \{x_1^{i_1} \cdots x_n^{i_n} | x_1^{i_1} \cdots x_n^{i_n} \in S\}$$

with the Lie bracket on basis element as follows:

$$\begin{aligned} & [x_1^{i_1} \cdots x_n^{i_n}, x_1^{j_1} \cdots x_n^{j_n}] \\ = \;& x_1^{i_1} \cdots x_n^{i_n} x_1^{j_1} \cdots x_n^{j_n} - x_1^{j_1} \cdots x_n^{j_n} x_1^{i_1} \cdots x_n^{i_n} \end{aligned}$$

for any $x_1^{j_1} \cdots x_n^{j_n}, x_1^{i_1} \cdots x_n^{i_n} \in B$.

Then we can easily see the Lie algebra $\mathrm{ad}(S) \cong [L_s, L_s] = L_s'$.

Let $M_n(S) = (M_n(S), +, \cdot)$ be the matrix ring such that $\{(a_{i,j}) \in M_n(S) | a_{i,j} \in S, 1 \le i, j \le n\}$. Let us introduce the Lie bracket $[,]$ on $M_n(S)$ as follows:

$$[m_1, m_2] = m_1 \cdot m_2 - m_2 \cdot m_1 \quad \text{for all } m_1, m_2 \in M_n(S).$$

It is easy to see that $(M_n(S), +, \cdot, [,])$ is a well-defined Lie algebra. Let us denote $M_n(S)$ as the usual F-algebra and $M_n(S)_{[,]}$ as the corresponding Lie algebra to $M_n(S)$. Since $M_n(S)$ has the non-trivial center, we consider the derived algebra $M_n(S)' = [M_n(S), M_n(S)]$, for removing the center for the simplicity of $M_n(S)'$. Let us denote $(e_{i,k})$ by the n by n matrix whose i, k-entry is 1 and other entries are zeros. Similarly let us denote $(ae_{i,k})$ by the matrix whose i, k-entry is a and other entries are 0. Thus we have the basis of the Lie algebra of $M' := M_n(S)'$ as follows: $sl_n(F) \subset M'$, $(Xe_{i,i}) \in M'$ for $i \in \{1, \cdots, n\}$, where X is a non-scalar monomial in S, and $(Xe_{i,j}) \in M'$ for any monomial $X \in S$ with $i \neq j$.

For a given element l of the Lie algebra L, we define the centralizer $C_L(l) = \{x \in L | [x, l] = 0\}$. Then $C_L(l)$ is a subalgebra of L.

For $i \neq j$ and $a \in S$, we have

$$[(ae_{i,j}), (e_{i,i}) - (e_{j,j})] \neq 0.$$

This shows that $(ae_{i,j})$ is not in the $C_{M_n(S)_{[,]}}(M_n(S)_{[,]}) = Z(M_n(S)_{[,]})$. Note that for any $l \in sl_n(F)$, $l \notin Z(M_n(S)_{[,]})$. For any $a \in S$,

$$(ae_{i,j}) \cdot l - l \cdot (ae_{i,j}) = 0 \text{ if and only if } a = 1.$$

Thus we have proved the following lemma.

LEMMA 1. *The Lie algebra M' has the following standard basis*

$$\begin{aligned}&\{(ae_{i,j})|1 \le i \le j \le n, a \text{ is a non-zero monomial in } S\}\\ \cup\ &\{(ae_{i,i})|1 \le i \le n, 1 \ne a \text{ is a non-zero monomial in } S\}\\ \cup\ &\{(e_{i,i}) - (e_{j,j})|1 \le i \ne j \le n\}.\end{aligned}$$

LEMMA 2. *In the skew-polynomial ring S, we have $C_S(x_1) = F[x_1]$.*

PROOF. For any monomial $x_1^{i_1} \cdots x_n^{i_n} \in S$, we have

$$\begin{aligned}&[x_1, x_1^{i_1} \cdots x_n^{i_n}]\\ =\ & x_1^{i_1+1} \cdots x_n^{i_n} - x_1^{i_1} \cdots x_n^{i_n} x_1\\ =\ & (1 - \lambda_{2,1}^{i_2} \cdots \lambda_{n,1}^{i_n}) x_1^{i_1+1} \cdots x_n^{i_n} = 0\end{aligned}$$

if and only if $1 - \lambda_{2,1}^{i_2} \cdots \lambda_{n,1}^{i_n} = 0$. But by the definition of S there is no such relation. Therefore, $i_2 = \cdots = i_n = 0$. □

LEMMA 3. *For any non-trivial ideal I of M', there is an element $(a_{i,j}) \in I$ such that $a_{p,r}$ is not a scalar for any $p, r \in \{1, \cdots, n\}$.*

PROOF. Let $(a_{i,j})$ be an n by n matrix of scalars. Since $sl_n(F)$ is simple, there is an element $(a_{i,j}) \in I$ such that $a_{n-1,n-1} = 1$, $a_{n,n} = -1$, and other entries are zero. Take $(b_{i,j}) \in M'$ such that $a_{n-1,n} = x_1$ and all other entries are zero. Then $[(a_{i,j}), (b_{i,j})]$ is the required one. □

THEOREM 1. *The Lie algebra M' is a simple Lie algebra.*

PROOF. To prove this theorem let us prove the following step first.

Step I. If I is an ideal containing an element $(a_{i,j})$ where $a_{n,n}$ is an non-zero and other terms are zero, then $I = M'$.
Proof of Step I. Note that $M_1' \cong S'$ where S' is the derived Lie algebra of S with the skew-polynomial multiplication of S. Since $M_1'(S)$ is a

simple Lie algebra, $(a_{n,n}e_{n,n}) \in I$ for any non-scalar $a_{n,n} \in S$. For any monomial $X \in S$, we have the following:

$$[(Xe_{n,n}), (e_{n-1,n})] = -(Xe_{n-1,n}) \in I.$$

Furthermore, we have

$$[(X^{-1}e_{n-1,n}), (Xe_{n,n})] = (X^{-1}Xe_{n-1,n}) \in I.$$

Therefore, $sl_n(F) \subset I$. Also,

$$(1) \qquad [(Xe_{n-1,n}), (e_{n,n-1})] = (Xe_{n-1,n-1}) + (-Xe_{n,n}) \in I.$$

From the above argument, we have $(Xe_{i,i}) + (-Xe_{j,j}) \in I$, $1 \le i, j \le n$ for any $X \in S$. For any $a_{i,j} \in S$ with $i < j$, and for any $(Xe_{i,i}) - (Xe_{k,k}) \in I, (k \neq i, j)$ we have

$$[(a_{i,j}e_{i,j}), (Xe_{i,i}) - (Xe_{k,k})] = -(Xa_{i,j}e_{i,j}) \in I.$$

This is an arbitrary basis element in M', thus $I = M'$. So Step I is proved.

Using Step I, let us prove the theorem.

For any non-zero element $(a_{r,s}) \in I$, we assume there are entries $a_{i,j}$ with $i > j$. By the above step, we can assume there is another non-zero entry, $a_{k,l} \neq 0$, with $i = k$ and $j < l$ or $k > i$ (if not, the theorem is proved.)

Case I. Assume there are non-zero entries in the i-th row and all other entries are zero. If $a_{i,j}$ is the first non-zero entry, then there is a non-zero entry $a_{i,l}$. The i, j entry of

$$(2) \qquad [(a_{i,j}e_{i,j}) + (a_{i,l}e_{i,l}) + \cdots + (a_{i,n}e_{i,n}), (Xe_{i,i})]$$

is zero and the i, i-entry of (2) is $a_{i,i}X - Xa_{i,i} \neq 0$ by the choice of appropriate $X \in S$. This contradicts the fact that $a_{i,j}$ is the first non-zero entry of the element in this ideal.

Case II. Assume the t, p-entry $a_{t,p}$ is non-zero . Then the i, j-entry of the matrix

$$(3) \qquad [(a_{i,j}e_{i,j}) + \cdots + (a_{n,n}e_{n,n}), (Xe_{t,t})]$$

is zero. But the t, j-entry of (3) is $-Xa_{t,p}$ which is non-zero where $t > i$. This contradicts the choice of $a_{i,j}$. Therefore, we have proved the theorem. □

Clearly we have the following corollary.

Corollary 1. ([4, Theorem. 1.3, Theorem. 3.1]). $M_1(S)' \cong \mathrm{ad}(S)$.

The set of toral elements of $M_n(S)$ is the same as $sl_n(F)$. G. Benkart proposed that the Lie algebra L which has no toral element (or ad-diagonal) may be embedded in an extension algebra of L, which is simple. We have the Lie algebra $M(S)'$ which is the toral extension of ad(S) i.e. the Lie algebra ad(S) has no toral element but the Lie algebra M' has toral elements.

Consider the Lie algebra $SM_n(F)$ with basis

$$B_1 = \{\lambda(a_{i,j}) | \lambda \in S, \lambda \text{ is a monomial }, (a_{i,j}) \in M_n(F)\},$$

and Lie bracket

$$[\lambda(a_{i,j}), \mu(b_{i,j})] = \lambda\mu(a_{i,j})(b_{i,j}) - \mu\lambda(b_{i,j})(a_{i,j})$$

for any $\lambda(a_{i,j}), \mu(b_{i,j}) \in B_1$. Then the Lie algebra

$$[SM_n(F), SM_n(F)] = SM_n(F)' := SM_n'$$

has a basis

$$\begin{aligned} & B_2 \\ = \ & \{x_1^{i_1} \cdots x_n^{i_n}(e_{i,j}) | x_1^{i_1} \cdots x_n^{i_n} \in S, (e_{i,j}) \in M_n(F), \\ & \qquad \text{for } i \neq j \in \{1, \cdots, n\}\} \\ & \cup \{x_1^{i_1} \cdots x_n^{i_n}(e_{i,i}) | x_1^{i_1} \cdots x_n^{i_n} \in S \text{ at least one of } i_1, \cdots, i_n \\ & \qquad \text{is non-zero}, (e_{i,i}) \in M_n(F), \text{ for } i \in \{1, \cdots, n\}\}. \end{aligned} \tag{4}$$

PROPOSITION 1. *Two Lie algebras M' and SM_n' are isomorphic.*

PROOF. Define an F linear map $\theta : M' \to SM_n'$, which is the identity map on $sl_n(F)$. Extend this map linearly to M' by $\theta((\lambda e_{i,j})) = \lambda(e_{i,j})$ for any $(\lambda e_{i,j})$. Then this map is one to one and onto. Thus, we have proved the proposition. □

Let S be the quasi-polynomial ring in Introduction. S and $M_n(F)$ are simple F-algebras, then $S \otimes_F M_n(F)$ is a simple F-algebra [10]. Since S is a simple F-algebra, $M_n(S)$ is a simple F-algebra [6]. Let $(e_{i,j})$ be the n by n matrix such that the entry of the i-th row and j-the column is 1 and the other entries are zero. Let us define F-linear map $\theta : S \otimes M_n(F) \to M_n(S)$ by $\theta(f \otimes (e_{i,j})) = f(e_{i,j})$ where $f \in S$, and$(e_{i,j}) \in M_n(S)$. Then clearly θ is the isomorphism between them. So we have that $S \otimes M_n(F) \cong M_n(S)$.

PROPOSITION 2. *The dimension of the maximal torus of $SM_n(F)'$ with respect to the B_2 in (4) is $n-1$.*

PROOF. The Lie algebra $SM_n(F)'$ is Z^n-graded Lie algebra such that the Lie gradation is compatible with the Z^n-gradation of F-algebra S. It is straightforward that the maximal torus with respect to the basis B_2 in (4) is $\{(e_{ii}) - (e_{jj}) | 1 \le i \ne j \le n\}$. Therefore, we have proved the proposition. □

COROLLARY 2. *The dimension of the maximal torus of $SM_1(F)' = S'$ with respect to the B_2 in (4) is 0.*

PROOF. It is straightforward by Proposition 2. □

3. The relations between $M_n(S)'$ and the Witt algebra

Let us compare the Lie algebras which are defined in this paper and the well-known Witt algebra $W^+(n)$ and its subalgebras (please refer to the Rudakov's paper [9] for more details on the Witt algebra and its subalgebras, i.e. Cartan type Lie algebras.) Let us introduce the Witt algebra $W^+(n)$. The Witt algebra $W^+(n)$ has the standard basis

$$W_B = \{x_1^{i_1} \cdots x_n^{i_n} \partial_s | i_1, \cdots, i_n \in N, 1 \le s \le n\}$$

with the Lie bracket on basis elements as follows:

$$\begin{aligned}&[x_1^{i_1} \cdots x_n^{i_n} \partial_u, x_1^{j_1} \cdots x_n^{j_n} \partial_t] \\ &= j_u x_1^{i_1+j_1} \cdots x_n^{i_n+j_n} x_u^{-1} \partial_t - i_t x_1^{i_1+j_1} \cdots x_n^{i_n+j_n} x_t^{-1} \partial_u.\end{aligned}$$

The Witt algebra $W^+(n)$ is simple. The subspace of $W^+(n)$ which is spanned by all ad-diagonals with respect to the standard basis W_B has dimension n and the basis is $\{x_s \partial_s | 1 \le s \le n\}$ [8]. All the simple subalgebras of $W^+(n)$ are classified as the special type Lie algebras $S^+(n)$, the Hamiltonian type Lie algebra $H^+(n)$, and kontact type Lie algebras $K^+(n)$. It is well known that all those Cartan type Lie algebras have ad-diagonal elements [9].

The Lie algebras $M_n(S)'$ in this paper has a simple subalgebra $\{(a, b) | a, b \in Z\}$ which has no ad-diagonal. Thus we have proved the following proposition.

PROPOSITION 3. *The Lie algebra $M_n(S)'$ is not isomorphic to the simple subalgebra of $W^+(n)$.*

Let us introduce few interesting problems which are related to the Lie algebras in this paper.

Problem 1. Find all the Lie automorphisms of $M_n(S)'$.

Problem 2. Find all the Lie derivations of $M_n(S)'$.

Problem 3. Find some subalgebra of $M_n(S)'$ which are invariant under all automorphisms of $M_n(S)'$.

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