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JORDAN DERIVATIONS AND JORDAN LEFT DERIVATIONS OF BANACH ALGEBRAS

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ABSTRACT. In this paper we obtain some results concerning Jordan derivations and Jordan left derivations mapping into the Jacobson radical. Our main result is the following: Let d be a Jordan derivation (resp. Jordan left derivation) of a complex Banach algebra A. If $d^2(x) = 0$ for all $x \in A$, then we have $d(A) \subseteq rad(A)$

1. Introduction

Let A be an associative algebra over the complex field $\mathbb{C}$. A linear mapping $d : A \to A$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x,\ y \in A$ and a *Jordan derivation* (resp. *Jordan left derivation*) if $d(x * y) = d(x) * y + x * d(y)$ for all $x,\ y \in A$ (resp. $d(x^2) = 2xd(x)$ for all $x \in A$), where $a * b$ denotes the Jordan product $ab + ba$. In this paper we write $[x, y]$ for the commutator $xy - yx$.

Obviously every derivation is a Jordan derivation, but the converse is not true in general except the case when the algebra is semiprime [2, Theorem 1]. If $r(x) = 0$ $(x \in A)$, then x is called *quasinilpotent*, where $r(\cdot)$ denotes the spectral radius, and henceforth, $Q(A)$ will denote the set of all quasinilpotent elements of A.

Our research is based on the Singer-Wermer theorem [9] which states that every bounded derivation of a commutative Banach algebra has its range in the Jacobson radical.

In 1988, Thomas [10] generalized the Singer-Wermer theorem by dropping the boundedness of a derivation. This generalization was known as the Singer-Wermer conjecture.

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The Kleinecke-Shirokov theorem ([4], [7]) which is a classical theorem of a Banach algebra theory asserts that if the elements a, b in a Banach algebra A satisfy $[a,[a,b]]=0$, then $[a,b]$ is quasinilpotent. The assumption $[a,[a,b]]=0$ can be reformulated as $d^2(b)=0$, and the conclusion as $d(b)$ is quasinilpotent, where $d=[\cdot,\ a]$ is the inner derivation implemented by a. Furthermore, Mathieu and Murphy [5, Theorem 2.1] proved that the Kleinecke-Shirokov theorem holds for arbitrary bounded derivations (not necessarily inner) and the validity of Mathieu and Murphy's result for any derivation was given by Thomas [11, Theorem 2.9] (this result implies the Singer-Wermer conjecture) [11, p. 152]. Since every derivation d on A such that $d(x)\in Q(A)$ for all $x\in A$ maps A into $rad(A)$ (see [5], [12]), the Thomas' result suggests a global version such as that $d(A)\subseteq rad(A)$ if $d^2(x)=0$ for all $x\in A$.

The purpose of this paper is to obtain the results which modify the global version of the above Thomas' result to any Jordan derivation (resp. Jordan left derivation) on Banach algebras.

2. Jordan derivations on Banach algebras

The following lemma will play a crucial role to prove the results concerning Jordan derivations on Banach algebras in this section.

LEMMA 2.1. *Let d be a Jordan derivation of a Banach algebra A such that $d^2(x)\in Q(A)$ for all $x\in A$. Then d leaves each primitive ideal of A invariant.*

PROOF. Suppose that $d^2(x)\in Q(A)$ for all $x\in A$. By setting $d_1(1)=0$, d can be extended to a Jordan derivation d_1 on the unitization A_1 of A, and it is clear that $d_1^2(x)\in Q(A_1)$ for all $x\in A_1$, and hence, without loss of generality we can assume that A is unital. Let P be a primitive ideal of A. Then A has an irreducible representation $\pi : A\to L(X)$ on a Banach space X with kernel P, where $L(X)$ is the algebra of all linear mappings on X. An application of [8, Lemma 3.1] yields that $d^n(x^n)-n!(d(x))^n\in P$ for all $x\in P$ and $n\in\mathbb{N}$.

Therefore, for all $x\in P$ and $k\in\mathbb{N}$, we obtain that

$$\begin{aligned} r(d(x)+P) &= r(d(x)^{2k}+P)^{1/2k} \\ &= ((2k)!)^{-1/2k}\ r(d^{2k}(x^{2k})+P)^{1/2k} \\ &\le ((2k)!)^{-1/2k}\ r(d^{2k}(x^{2k}))^{1/2k}=0 \end{aligned}$$

whence $d(x) + P \in Q(A/P)$ for all $x \in P$.

We claim that $d(P) \subseteq P$. Let $x_0 \in P$ and suppose that $d(x_0) \notin P$. We first note that the normed division algebra $\mathcal{D} = \{T \in L(X) : aT\xi = T(a\xi),\ a \in A,\ \xi \in X\} \cong \mathbb{C}1$ ([1, Corollary 25.5]).

If $\pi(d(x_0)) \in \mathbb{C}1$, then $a\pi(d(x_0))\xi = \pi(d(x_0))(a\xi)$ for all $a \in A$ and $\xi \in X$, and hence $ad(x_0)\xi = d(x_0)a\xi$ for all $a \in A$ and $\xi \in X$. This shows that $ad(x_0) - d(x_0)a \in P$ for all $a \in A$, that is, $d(x_0) + P \in Z(A/P)$, where $Z(A/P)$ is the center of a primitive Banach algebra A/P. Since $Z(A/P) \cong \mathbb{C}1$ and $d(x_0) + P$ is quasinilpotent, it follows that $d(x_0) + P = P$ which means that $d(x_0) \in P$. This contradiction implies that $\pi(d(x_0)) \notin \mathbb{C}1$.

Now $\{1, \pi(d(x_0))\}$ is linearly independent whence there exists $\xi_0 \in X$ such that $\{\xi_0, \pi(d(x_0))\xi_0\}$, that is, $\{\xi_0, d(x_0)\xi_0\}$ is linearly independent. Using the Jacobson density theorem, we choose a $y \in A$ such that $y\xi_0 = \xi_0$ and $yd(x_0)\xi_0 = \xi_0 - d(x_0)\xi_0$. Then $(d(x_0) * y)\xi_0 = d(x_0)y\xi_0 + yd(x_0)\xi_0 = \xi_0$, so we obtain $d(x_0) * y \notin Q(A)$ and the relation $\mathrm{Sp}(A,\ x) = \bigcup_P \mathrm{Sp}(A/P,\ x + P)$, where $\mathrm{Sp}(B,\ \cdot)$ denotes the spectrum with respect to the unital algebra B and the union is taken over all primitive ideals of A, shows that $d(x_0) * y + P \notin Q(A/P)$. However $d(x_0) * y + P \in Q(A/P)$ since $d(x_0) * y + P = (d(x_0 * y) - x_0 * d(y)) + P \in d(P) + P$ $(y \in A)$. This contradiction gives us that $d(P) \subseteq P$, and completes the proof of the theorem. □

By using Lemma 2.1, we have the following main results.

THEOREM 2.2. *Let d be a Jordan derivation of a Banach algebra A such that $d^2(x) = 0$ for all $x \in A$. Then d maps A into $rad(A)$.*

PROOF. By the assumption, it is immediate that $d^2(x) \in Q(A)$ for all $x \in A$, and so Lemma 2.1 shows that d leaves each primitive ideal P of A invariant whence it follows that a Jordan derivation d induces a Jordan derivation $\tilde{d}$ of the primitive Banach algebra A/P defined by $\tilde{d}(x + P) = d(x) + P$ $(x \in A)$.

Observe that $\tilde{d}$ is a derivation [2, Theorem 1] by the primeness of A/P. Since the hypothesis $d^2(x) = 0$ for all $x \in A$ gives $\tilde{d}^2 = 0$ on A/P, it follows from Posner's first theorem [6, Theorem 1] that $\tilde{d} = 0$ on A/P.

We thus see that $d(A) \subseteq P$. Since this holds for any primitive ideal P of A, we conclude that $d(A) \subseteq rad(A)$. □

THEOREM 2.3. *Let d be a Jordan derivation of a Banach algebra A*

such that $d(A) \subseteq Z(A)$, where $Z(A)$ is a center of A. Then d maps A into $rad(A)$.

PROOF. Suppose that $d(A) \subseteq Z(A)$. The restriction $d|_{Z(A)}$ is a derivation of a commutative Banach algebra $Z(A)$, and hence Thomas' theorem [10] implies that $d(Z(A)) \subseteq rad(Z(A)) = Z(A) \cap rad(A)$. From the assumption, we see that $d^2(A) \subseteq rad(A)$, and an application of Lemma 2.1 with [1, Proposition 25.1 (i)] yields that d leaves each primitive ideal P of A invariant. Hence a Jordan derivation d induces a Jordan derivation $\tilde{d}$ of the Banach algebra A/P. Now the induced Jordan derivation $\tilde{d}$ of the prime algebra A/P becomes a derivation, and satisfies $\tilde{d}^2 = 0$ because of $d^2(A) \subseteq rad(A)$, and thus the remainder follows the same fashion as in the latter half of the proof of Theorem 2.2. So, the theorem follows. □

REMARK. The fact that $d^2(x) \in Q(A)$ for all $x \in A$ does not, in general, imply that $d(A) \subseteq rad(A)$. For, in the case when A is semisimple, if $a \in A$ is not in the center such that $a^2 = 0$, $d(x) = [x,\ a]$, then $d^2(x)^2 = 0$ for all $x \in A$ (and hence $d^2(x) \in Q(A)$ for all $x \in A$) in spite of $d \neq 0$.

3. Jordan left derivations on Banach algebras

To prove the main result of this section, we need the following lemmas.

LEMMA 3.1. *Let d be a Jordan left derivation of an algebra A. Then for all $x,\ y \in A$:*

1°. $d(xy + yx) = 2xd(y) + 2yd(x)$,

2°. $d(xyx) = x^2d(y) + 3xyd(x) - yxd(x)$.

PROOF. A special case of Proposition 1.1 in [3]. □

LEMMA 3.2. *Let d be a Jordan left derivation of a noncommutative prime algebra A. Then we have $d = 0$.*

PROOF. A special case of Corollary 1.3 in [3]. □

Now we have the main result in this section:

THEOREM 3.3. *Let d be a Jordan left derivation of a Banach algebra A such that $d^2(x) = 0$ for all $x \in A$. Then d maps A into $rad(A)$.*

PROOF. As in the proof of Lemma 2.1, by setting $d_1(1) = 0$, d can be extended to a Jordan left derivation d_1 on the unitization A_1 of A, and it is clear that $d_1^2(x) = 0$ for all $x \in A_1$, and hence, without loss of generality we can assume that A is unital.

We first intend to prove that if $d^2(x) = 0$ for all $x \in A$, then $d(x)$ is quasinilpotent for all $x \in A$.

Suppose that

$$d^2(x) = 0 \qquad \text{for all } x \in A. \tag{1}$$

Let us replace x by x^2 in (1). Then we obtain

$$d(xd(x)) = 0 \qquad \text{for all } x \in A. \tag{2}$$

Using Lemma 3.1 and (2), we have

$$d([d(x),\ x]) = 2d(x)^2 \qquad \text{for all } x \in A. \tag{3}$$

In fact,

$$\begin{aligned} d([d(x),\ x]) &= d(d(x)x + xd(x)) - 2d(xd(x)) \\ &= 2d(x)^2 + 2xd^2(x) - 2d(xd(x)) \\ &= 2d(x)^2. \end{aligned}$$

The linearization of (3) gives

$$d([d(x),\ y] + [d(y),\ x]) = 2d(x)d(y) + 2d(y)d(x) \qquad \text{for all } x,\ y \in A.$$

Substituting x^2 for y in the above relation, and using Lemma 3.1 and (3),

$$\begin{aligned} 0 &= d([d(x),\ x^2] + [d(x^2),\ x]) - 2d(x)d(x^2) - 2d(x^2)d(x) \\ &= d([d(x,\ x]x + x[d(x,\ x]) + 2d(x[d(x),\ x]) \\ &\quad - 4d(x)xd(x) - 4xd(x)^2 \\ &= 2[d(x),\ x]d(x) + 2xd([d(x),\ x]) + 2d(x[d(x),\ x]) \\ &\quad - 4d(x)xd(x) - 4xd(x)^2 \\ &= \ - 2d(x)xd(x) - 2xd(x)^2 + 2d(x[d(x),\ x]) \end{aligned}$$

whence we see that

$$d(x[d(x),\ x]) = d(x)xd(x) + xd(x)^2 \qquad \text{for all } \ x \in A. \tag{4}$$

Now using (3), (4) and Lemma 3.1, we obtain

$$d([[d(x),\ x],\ x]) = 0 \qquad \text{for all } \ x \in A. \tag{5}$$

Indeed,

$$\begin{aligned} d([[d(x),\ x],\ x]) &= d([d(x),\ x]x - x[d(x),\ x]) \\ &= d([d(x),\ x]x + x[d(x),\ x]) - 2d(x[d(x),\ x]) \\ &= 2[d(x),\ x]d(x) + 2xd([d(x),\ x]) - 2d(x[d(x),\ x]) \\ &= 2[d(x),\ x]d(x) + 2xd(x)^2 - 2d(x)xd(x) \\ &= 2[d(x),\ x]d(x) - 2[d(x),\ x]d(x) = 0. \end{aligned}$$

Also according to (5) and Lemma 3.1, we have

$$\begin{aligned} 0 &= d([[d(x),\ x],\ x]) \\ &= d(d(x)x^2 + x^2d(x)) - 2d(xd(x)x) \\ &= 4d(x)xd(x) + 2x^2d^2(x) - 2x^2d^2(x) - 6xd(x)^2 + 2xd(x)^2 \\ &= 4[d(x),\ x]d(x), \end{aligned}$$

that is,

$$[d(x),\ x]d(x) = 0 \qquad \text{for all } \ x \in A. \tag{6}$$

From (6) and Lemma 3.1, it follows that

$$\begin{aligned} 0 &= d([d(x),\ x]d(x)) \\ &= d(d(x)xd(x) - d(xd(x)^2) \\ &= d(x)^3 - d(xd(x)^2), \end{aligned}$$

and hence we get

$$d(xd(x)^2) = d(x)^3 \qquad \text{for all } \ x \in A. \tag{7}$$

Applying Lemma 3.1 and (7), we have for all $x \in A$,

$$\begin{aligned} d([d(x)^2,\ x]) &= d(d(x)^2x - xd(x)^2) \\ &= d(d(x)^2x + xd(x)^2) - 2d(xd(x)^2) \\ &= 2d(x)^3 + 2xd(d(x)^2) - 2d(xd(x)^2) \\ &= 2d(x)^3 + 4xd(x)d^2(x) - 2d(x)^3 = 0. \end{aligned}$$

On the other hand, using (3), (6) and Lemma 3.1, we see that for all $x \in A$,

$$\begin{aligned} 0 &= d([d(x)^2,\ x]) \\ &= d([d(x),\ x]d(x) + d(x)[d(x),\ x]) \\ &= 2[d(x),\ x]d(x) + 2d(x)d([d(x),\ x]) \\ &= 4d(x)^3 \end{aligned}$$

whence we arrive at the fact that $d(x)^3 = 0$ for all $x \in A$. This means that for all $x \in A$, $d(x)$ is nilpotent, and so $d(x)$ is quasinilpotent for all $x \in A$.

Now let P be any primitive ideal of A. Note that A/P is a primitive algebra, and hence is semisimple. Let $y \in A$ and $x \in P$. Then by Lemma 3.1, we observe that

$$2yd(x) = d(xy + yx) - 2xd(y) \in d(P) + P.$$

This shows that $d(P) + P/P$ is a left ideal of A/P. Since $d(x) \in Q(A)$ for all $x \in A$, we see that $d(x) + P \in Q(A/P)$ for all $x \in P$ by using the relation $\mathrm{Sp}(A,\ x) = \bigcup_P \mathrm{Sp}(A/P,\ x + P)$ as in the proof of Lemma 2.1.

Hence we conclude that $d(P) + P/P$ is a quasinilpotent left ideal of A/P, which is contained in $rad(A/P)$ by [1, Proposition 25.1 (ii)]. Semisimplicity forces $d(P) \subseteq P$. Therefore d induces a Jordan left derivation $\tilde{d}$ on the Banach algebra A/P.

Assume that A/P is commutative. Then $\tilde{d}$ is a derivation, and so it follows from [10] that $\tilde{d} = 0$. In the case when A/P is noncommutative, since A/P is prime, we see that $\tilde{d} = 0$ by Lemma 3.2. Thus, in any case $\tilde{d} = 0$, that is, $d(A) \subseteq P$ for any primitive ideal P of A. This implies that $d(A) \subseteq rad(A)$. The proof of the theorem is complete. □

References

[1] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, 1973.

[2] M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), 1003–1006.

[3] M. Brešar and J. Vukman, *On left derivations and related mappings*, Proc. Amer. Math. Soc. **110** (1990), 7–16.

[4] D. C. Kleinecke, *On operator commutators*, Proc. Amer. Math. Soc. **8** (1957), 535–536.

[5] M. Mathieu and G. J. Murphy, *Derivations mapping into the radical*, Arch. Math. **57** (1991), 469–474.

[6] E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[7] F. V. Shirokov, *Proof of a conjective of Kaplansky*, Uspekhi Mat. Nauk **11** (1956), 167–168.

[8] A. M. Sinclair, *Jordan homomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209–214.

[9] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260–264.

[10] M. P. Thomas, *The image of a derivation is contained in the radical*, Ann. of Math. **128** (1988), 435–460.

[11] ———, *Primitive ideals and derivations on non-commutative Banach algebras*, Pacific J. Math. **159** (1993), 139–152.

[12] Yu. V. Turovskiĭ and V. S. Shul'man, *Conditions for massiveness of the range of the derivation of a Banach algebra and associated differential operators*, Math. Notes **42** (1987), 669–674.

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