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GALOIS CORRESPONDENCES FOR SUBFACTORS RELATED TO NORMAL SUBGROUPS

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ABSTRACT. For an outer action α of a finite group G on a factor M, it was proved that H is a normal subgroup of G if and only if there exists a finite group F and an outer action β of F on the crossed product algebra $M \rtimes_\alpha H$ with $M \rtimes_\alpha G \cong (M \rtimes_\alpha H) \rtimes_\beta F$. We generalize this to infinite group actions. For an outer action α of a discrete group, we obtain a Galois correspondence for crossed product algebras related to normal subgroups. When α satisfies a certain condition, we also obtain a Galois correspondence for fixed point algebras. Furthermore, for a minimal action α of a compact group G and a closed normal subgroup H, we prove $M^G = (M^H)^{\beta(G/H)}$ for a minimal action β of G/H on M^H.

1. Introduction

When a group acts on a von Neumann algebra, a *Galois correspondence* which says that there is a one-to-one correspondence between subgroups and subfactors is one of the important things in the theory of operator algebras. In [4], Izumi-Longo-Popa have recently obtained the general Galois correspondence of outer actions of discrete groups and minimal actions of compact groups on factors, which has been studied by several authors [2, 6, 7].

Let α be an action of a group G on a von Neumann algebra $A \subset \mathcal{B}(\mathcal{H})$. The *crossed product algebra* $A \rtimes G$ is the von Neumann algebra $(\pi_\alpha(A) \cup \{\lambda(g) | g \in G\})''$, where $\pi_\alpha(x)$ and $\lambda(g)$ are operators on the Hilbert space $L^2(G, \mathcal{H})$ defined by $(\pi_\alpha(x)\xi)(g) = \alpha_g^{-1}(x)\xi(g)$ and $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$, respectively. The *fixed point algebra* A^G is the von Neumann

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algebra $\{x \in A | \alpha_g(x) = x, \text{ for all } g \in G\}$. For a subgroup H of G, $A \rtimes H$ and A^H denote intermediate von Neumann algebras of $A \subset A \rtimes G$ and $A^G \subset A$, respectively.

Generally, let β be a mapping of a group G into the automorphism group and u a mapping of $G \times G$ into the unitary group of A. If u satisfies that $\beta_g \beta_h = \mathrm{Ad}u(g,h)\beta_{gh}$, $u(g,h)u(gh,k) = \beta_g(u(h,k))u(g,hk)$, and $u(1,h) = u(g,1) = 1$, then u is called a β-2-cocycle and (β, u) a *cocycle crossed action* of G on A. The *regular extension* $A \rtimes_{(\beta,u)} G$ of A by G is the von Neumann algebra $(\pi_\beta(A) \cup \{\lambda_u(g) | g \in G\})''$, where $\pi_\beta(x)$ and $\lambda_u(g)$ are operators on $L^2(G, \mathcal{H})$ defined by $(\pi_\beta(x)\xi)(g) = \beta_g^{-1}(x)\xi(g)$ and $(\lambda_u(g)\xi)(h) = u(h^{-1}, g)\xi(g^{-1}h)$, respectively.

On the other hand, when α is an outer action of a finite group G on a factor M, Teruya proved in [9] that H is a normal subgroup of G if and only if there exist a finite group F and an outer action β of F on the crossed product algebra $M \rtimes H$ with $M \rtimes G \cong (M \rtimes H) \rtimes_\beta F$. An important assumption here is the finiteness of groups. For an extension of it, we ask what happens without the finiteness of groups and it should be possible as well in the infinite case with the help of the general Galois correspondence in [4].

In this paper, we study an outer action of a discrete group and a minimal action of a compact group. Note that no assumptions are made on the finiteness of groups.

First, we consider an outer action α of a discrete group G on a factor M. We show that a subgroup H of G is normal if and only if there exist a discrete group F and an outer cocycle crossed action (β, u) of F on $M \rtimes H$ with $M \rtimes G \cong (M \rtimes H) \rtimes_{(\beta,u)} F$. In fact, we obtain a one-to-one correspondence between the set of all normal subgroups of G and that of all intermediate subfactors L of $M \subset M \rtimes G$ with $M \rtimes G \cong L \rtimes_{(\beta,u)} F$ for an outer cocyle crossed action (β, u) of some discrete group F on L. Moreover, if there exists a faithful normal conditional expectation of $(M^G)'$ onto M', then we get a one-to-one correspondence between subgroups of G and intermediate subfactors of $M^G \subset M$.

Next, we consider an action α of a compact group G on a factor M. α is called a minimal action if α is faithful and $(M^G)' \cap M = \mathbb{C} \cdot 1$. For a minimal action α of a compact group G on a factor M and a closed normal subgroup H of G, we prove that there exists a minimal action β of the quotient group G/H on a factor M^H with $M^G = (M^H)^{\beta(G/H)}$.

2. Intermediate subfactors of a crossed product inclusion

This section is concerned with a crossed product inclusion of an outer action of a discrete group on a factor. We investigate intermediate subfactors of a crossed product inclusion related to normal subgroups. The following gives a general Galois correspondence.

THEOREM 2.1. (Theorem 3.13 of [4]) *Let G be a discrete group and α an outer action of G on a factor M. Then the map $H \to M \rtimes H$ gives a one-to-one correspondence between the lattice of all subgroups of G and that of all intermediate subfactors of $M \subset M \rtimes G$.*

In this section, α denotes an outer action of a discrete group G on a factor M with the crossed product algebra $M \rtimes G$. It is well known that there exists a faithful normal conditional expectation E of $M \rtimes G$ onto M and an element $x \in M \rtimes G$ is expressed by $x = \sum x(g)\lambda(g)$, where $x(g) = E(x\lambda(g)^*)$ is the Fourier coefficient of x at g.

LEMMA 2.2. *For a normal subgroup H of G, there exists an outer cocycle crossed action (β, u) of G/H on $M \rtimes H$ such that the crossed product algebra $M \rtimes G$ is isomorphic to the regular extension $(M \rtimes H) \rtimes_{(\beta,u)} G/H$ of $M \rtimes H$ by G/H.*

PROOF. For each $\bar{g} \in G/H$, we define an automorphism $\beta_{\bar{g}}$ of $M \rtimes H$ by $\beta_{\bar{g}}(x) = \lambda(g)x\lambda(g)^*$. We also define a β-2-cocycle u by $u(\overline{g_1}, \overline{g_2}) = \lambda(g_1 g_2 k^{-1})$ ($\overline{g_1}, \overline{g_2} \in G/H$), where k is in the set G_0 of all representatives of G/H with $\overline{g_1}\ \overline{g_2} = \bar{k}$. It is straightforward to show that (β, u) is a cocycle crossed action of G/H on $M \rtimes H$ with the regular extension $(M \rtimes H) \rtimes_{(\beta,u)} G/H$ of $M \rtimes H$ by G/H.

For $\bar{g} \in G/H$, if there is a unitary $w \in M \rtimes H$ such that $\beta_{\bar{g}} = \mathrm{Ad}w$, then we have $w^*\lambda(g) \in (M \rtimes H)' \cap M \rtimes G$. Since α is outer, we get that $\lambda(g) \in M \rtimes H$ and $g \in H$ which implies that (β, u) is outer.

On the other hand, since $M \rtimes G = (M \rtimes H \cup \{\lambda(g) | g \in G_0\})''$ and $\{\lambda(g) | g \in G_0\}$ is the set of all implementing unitaries of β, we have $M \rtimes G \cong (M \rtimes H) \rtimes_{(\beta,u)} G/H$ from Theorem 7 in [3]. □

LEMMA 2.3. *For a subgroup H of G, if there exist a discrete group F and an outer action β of F on $M \rtimes H$ with $M \rtimes G \cong (M \rtimes H) \rtimes_\beta F$, then H is normal.*

PROOF. For a convenience, let L and E_L denote $M \rtimes H$ and a faithful normal conditional expectation of $M \rtimes G$ onto L, respectively.

Suppose that there exist a discrete group F and an outer action β of F on L with $M \rtimes G \cong L \rtimes_\beta F$, where the crossed product $L \rtimes_\beta F$ is the von Neumann algebra $(\pi_\beta(L) \cup \{\bar{\lambda}(f) | f \in F\})''$. When we identify $M \rtimes G$ with $L \rtimes_\beta F$, for any $g \in G \setminus H$, $\lambda(g) = \sum_{f \in F} y(f)\bar{\lambda}(f)$ with $y(f) = E_L(\lambda(g)\bar{\lambda}(f)^*)$.

For any $g \in G \setminus H$, since $\lambda(g) \notin L$, we can take an element $f_0 \in F$ such that $E_L(\lambda(g)\bar{\lambda}(f_0)^*) \neq 0$. It follows from $\lambda(g)^* M \lambda(g) = M$ and $\bar{\lambda}(f_0) L \bar{\lambda}(f_0)^* = L$ that $\bar{\lambda}(f_0)\lambda(g)^* y \lambda(g)\bar{\lambda}(f_0)^* \in L$ for any $y \in M$. So, the equality of

$$y\lambda(g)\bar{\lambda}(f_0)^* = \lambda(g)\bar{\lambda}(f_0)^*\bar{\lambda}(f_0)\lambda(g)^* y \lambda(g)\bar{\lambda}(f_0)^*$$

gives

$$yE_L(\lambda(g)\bar{\lambda}(f_0)^*) = E_L(\lambda(g)\bar{\lambda}(f_0)^*)\bar{\lambda}(f_0)\lambda(g)^* y \lambda(g)\bar{\lambda}(f_0)^*.$$

Thus we obtain

$$E_L(\lambda(g)\bar{\lambda}(f_0)^*)(\lambda(g)\bar{\lambda}(f_0)^*)^* \in M' \cap M \rtimes G = \mathbb{C} \cdot 1$$

which implies $\lambda(g)\bar{\lambda}(f_0)^* \in L$ and we conclude that for any $g \in G \setminus H$, $\lambda(g) L \lambda(g)^* = \lambda(g)\bar{\lambda}(f_0)^* L \bar{\lambda}(f_0)\lambda(g)^* = L$. Hence we have $\lambda(g)\lambda(h)\lambda(g)^* \in L$ for any $h \in H$ which implies that H is normal. □

We are now in a position to state and prove the main theorem.

THEOREM 2.4. *There is a one-to-one correspondence with the map $H \to M \rtimes H$ between the set of all normal subgroups H of G and that of all intermediate subfactors L of $M \subset M \rtimes G$ with $M \rtimes G \cong L \rtimes_{(\beta,u)} F$ for an outer cocycle crossed action (β, u) of some discrete group F on L.*

PROOF. Thanks to Lemma 2.2, for a normal subgroup H of G, there exists an outer cocycle action (β, u) of a discrete group G/H on a factor $M \rtimes H$ with $M \rtimes G \cong (M \rtimes H) \rtimes_{(\beta,u)} G/H$.

Conversely, let L be an intermediate factor of $M \subset M \rtimes G$ with $M \rtimes G \cong L \rtimes_{(\beta,u)} F$ for an outer cocycle crossed action (β, u) of some discrete group F on L. By Theorem 2.1, there exists a subgroup H of G with $L = M \rtimes H$ and we get two isomorphic inclusions $M \rtimes H \subset M \rtimes G$ and $M \rtimes H \subset (M \rtimes H) \rtimes_{(\beta,u)} F$. Now, let A be a von Neumann algebra $\mathcal{B}(l^2(G))$ and $\tilde{M}$ a properly infinite factor $M \otimes A$. By tensoring A on

$M \rtimes H \subset M \rtimes G$, we get $\tilde{M} \rtimes_{\tilde{\alpha}} H \subset \tilde{M} \rtimes_{\tilde{\alpha}} G$, where $\tilde{\alpha}$ is an outer action $\alpha \otimes id_A$ of G on $\tilde{M}$.

On the other hand, if we define $\tilde{\beta}$ and $\tilde{u}$ by $\tilde{\beta} = \beta \otimes id_A$ and $\tilde{u}(g,h) = u(g,h) \otimes 1 \quad (g,h \in F)$, then we get a cocycle crossed action $(\tilde{\beta}, \tilde{u})$ of F on $\tilde{M}$. By tensoring A on $M \rtimes H \subset (M \rtimes H) \rtimes_{(\beta,u)} F$, we get an inclusion $\tilde{M} \rtimes_{\tilde{\alpha}} H \subset (\tilde{M} \rtimes_{\tilde{\alpha}} H) \rtimes_{(\tilde{\beta},\tilde{u})} F$ of properly infinite factors and we note that the regular extension $(\tilde{M} \rtimes_{\tilde{\alpha}} H) \rtimes_{(\tilde{\beta},\tilde{u})} F$ is an ordinary crossed product algebra $(\tilde{M} \rtimes_{\tilde{\alpha}} H) \rtimes_{\bar{\beta}} F$, where $\bar{\beta}$ is an outer action given by a perturbation of $(\tilde{\beta}, \tilde{u})$ (see [8]). Thus we get two isomorphic inclusions $\tilde{M} \rtimes_{\tilde{\alpha}} H \subset \tilde{M} \rtimes_{\tilde{\alpha}} G$ and $\tilde{M} \rtimes_{\tilde{\alpha}} H \subset (\tilde{M} \rtimes_{\tilde{\alpha}} H) \rtimes_{\bar{\beta}} F$. Hence, it follows from Lemma 2.3 that H is a normal subgroup of G. □

As an application of our result, we obtain the following.

COROLLARY 2.5. *Let β be an outer action of a discrete group F on a factor N with a crossed product inclusion $N \subset N \rtimes_\beta F$. For a subgroup H of G, if $M \rtimes H \subset M \rtimes G$ is isomorphic to $N \subset N \rtimes_\beta F$, then H is normal.*

PROOF. Let H be a subgroup of G such that $M \rtimes H \subset M \rtimes G$ is isomorphic to $N \subset N \rtimes_\beta F$, where β is an outer action of a discrete group F on a factor N. If we let $\theta : M \rtimes H \to N$ be the restriction of a $*$-isomorphism from $M \rtimes G$ to $N \rtimes_\beta F$, then $\bar{\beta} = \theta^{-1} \circ \beta \circ \theta$ is an outer action of F on $M \rtimes H$ with the crossed product algebra $(M \rtimes H) \rtimes_{\bar{\beta}} F$ which is isomorphic to $N \rtimes_\beta F$. Thus we have $M \rtimes G \cong (M \rtimes H) \rtimes_\beta F$ and H is normal by Lemma 2.3. □

3. Intermediate subfactors of a fixed point inclusion

In this section, we deal with intermediate subfactors of a fixed point inclusion. We first study an outer action α of a discrete group G on a factor M with a faithful normal conditional expectation of $(M^G)'$ onto M'. To investigate the relation between the crossed product algebra and the fixed point algebra, recall that $M \rtimes G$ is the basic construction of $M^G \subset M$ when G is a finite group (see [6]). The following gives a simple proof of the generalization of this to an infinite group action.

PROPOSITION 3.1. *Let α be an outer action of a discrete group G on a factor M. If there exists a faithful normal conditional expectation of $(M^G)'$ onto M', then $M \rtimes G$ is the basic construction of $M^G \subset M$.*

PROOF. For an outer action α of a discrete group G on a factor M with a faithful normal conditional expectation of $(M^G)'$ onto M', note that an inclusion $M' \subset (M^G)'$ is isomorphic to a crossed product inclusion $M' \subset M' \rtimes_{\alpha'} G$, where α' is the induced outer action of G on M' (see [1]). Thus we get $J(M^G)'J \cong \bar{J}(M' \rtimes_{\alpha'} G)\bar{J}$, where J (resp. $\hat{J}$) is the canonical involution of $\mathcal{B}(L^2(M))$ (resp. $\mathcal{B}(l^2(G))$) and $\bar{J} = J \otimes \hat{J}$. It follows from $\bar{J}(M' \rtimes_{\alpha'} G)\bar{J} \cong M \rtimes_\alpha G$ that $J(M^G)'J \cong M \rtimes G$ which implies that $M \rtimes G$ is the basic construction of $M^G \subset M$. □

In the following, we establish a Galois correspondence for a fixed point inclusion of an outer action of a discrete group.

THEOREM 3.2. *Let α be an outer action of a discrete group G on a factor M with a faithful normal conditional expectation of $(M^G)'$ onto M'. The map $H \to M^H$ gives a one-to-one correspondence between the set of all subgroups of G and that of all intermediate subfactors of $M^G \subset M$.*

PROOF. Let α be an outer action of a discrete group G on a factor M with a faithful normal conditional expectation of $(M^G)'$ onto M'. For any subgroup H of G, M^H is an intermediate subfactor of $M^G \subset M$.

Let θ be an isomorphism from $M' \subset (M^G)'$ to $M' \subset M' \rtimes_{\alpha'} G$, where $M' \rtimes_{\alpha'} G = (\pi_{\alpha'}(M') \cup \{\lambda(g) | g \in G\})''$. For any intermediate subfactor K of $M^G \subset M$, $\theta(K')$ is an intermediate subfactor of $M' \subset M' \rtimes_{\alpha'} G$ and there exists a subgroup H of G with $\theta(K') = M' \rtimes_{\alpha'} H$ by Theorem 2.1. Since the correspondence in Theorem 2.1 is complete, we have $H = \{g \in G | \lambda(g) \in M' \rtimes_{\alpha'} H\}$ and so $(\theta^{-1}(M' \rtimes_{\alpha'} H))' = M^H$ which is equal to $K = M^H$. Therefore, the map $H \to M^H$ gives a one-to-one correspondence between all subgroups of G and all intermediate subfactors of $M^G \subset M$. □

Now, we consider a fixed point inclusion of a minimal action of a compact group and investigate intermediate subfactors related to closed normal subgroups. The following theorem illustrates that the fixed point algebra M^G can be described by an intermediate subfactor of $M^G \subset M$ related to a closed normal subgroup and a minimal action of a compact group.

THEOREM 3.3. *Let α be a minimal action of a compact group G on a factor M. If H is a closed normal subgroup of G, then there exists a minimal action β of a compact group G/H on a factor M^H with $M^G = (M^H)^{\beta(G/H)}$.*

PROOF. We know that for a closed normal subgroup H of G, the quotient group G/H is a compact group and M^H is an intermediate subfactor of $M^G \subset M$.

For any $\bar{g} \in G/H$, if we define $\alpha_{\bar{g}}^H$ by $\alpha_{\bar{g}}^H(x) = \alpha_g(x)$ for all $x \in M^H$, then $\alpha_{\bar{g}}^H$ is an automorphism of M^H. For $\overline{g_1}, \overline{g_2} \in G/H$ with $\overline{g_1} = \overline{g_2}$, it is easily verified that $\alpha_{\overline{g_1}}^H = \alpha_{\overline{g_2}}^H$ and $\alpha_{\overline{g_1}}^H \alpha_{\overline{g_2}}^H = \alpha_{\overline{g_1}\,\overline{g_2}}^H$. Thus α^H is an action of G/H on M^H.

On the other hand, for any $x \in M^G$ and $\bar{g} \in G/H$, we have $\alpha_{\bar{g}}^H(x) = \alpha_g(x) = x$ and so $x \in (M^H)^{\alpha^H(G/H)}$. Conversely, for any $x \in (M^H)^{\alpha^H(G/H)}$ and $g \in G$, $\alpha_g(x) = \alpha_{\bar{g}}^H(x) = x$ and so $x \in M^G$. Hence $M^G = (M^H)^{\alpha^H(G/H)}$ holds.

To complete the proof, we show that α^H is a minimal action. For any $\bar{g} \in G/H$ satisfying $\alpha_{\bar{g}}^H(x) = x$ for all $x \in M^H$, we have $\alpha_g(x) = x$ for all $x \in M^H$. Since the map $H \to M^H$ gives a complete Galois correspondence in Theorem 3.15 of [4], we have $g \in H$ which implies that α^H is faithful. Since $((M^H)^{\alpha^H(G/H)})' \cap M^H$ is contained in $(M^G)' \cap M = \mathbb{C} \cdot 1$, α^H is minimal. □

Let me point out the converse of Theorem 3.3 which is still open. It will be interesting to consider the assertion similar to Theorem 2.4 in the case of a fixed point inclusion of a minimal action of a compact group on a factor.

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