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ON G-INVARIANT MINIMAL HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN S^5

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ABSTRACT. Let $G = O(2) \times O(2) \times O(2)$ and let M^4 be a closed G-invariant minimal hypersurface with constant scalar curvature in S^5. If M^4 has 2 distinct principal curvatures at some point, then $S = 4$. Moreover, if $S > 4$, then M^4 does not have simple principal curvatures everywhere.

Introduction

Let M^n be a closed minimally immersed hypersurface in the unit sphere S^{n+1}, and h its second fundamental form. Denote by R and S its scalar curvature and the square norm of h, respectively. It is well known that $S = n(n-1) - R$ from the structure equations of both M^n and S^{n+1}. In particular, S is constant if and only if M has constant scalar curvature. In 1968, J. Simons [8] observed that if $S \leq n$ everywhere and S is constant, then $S \in \{0, n\}$. Clearly, M^n is an equatorial sphere if $S = 0$. And when $S = n$, M^n is indeed a product of spheres, due to the works of Chern, do Carmo, and Kobayashi [3] and Lawson [5].

We are interested in the following conjecture posed by Chern [9].

CHERN CONJECTURE. *For any $n \geq 3$, the set R_n of the real numbers each of which can be realized as the constant scalar curvature of a closed minimally immersed hypersurface in S^{n+1} is discrete.*

C. K. Peng and C. L. Terng [7] proved

THEOREM ([Peng and Terng, 1983]). *Let M^n be a closed minimally immersed hypersurface with constant scalar curvature in S^{n+1}. If $S > n$, then $S > n + 1/(12n)$.*

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S. Chang [2] proved the following theorem by showing that $S = 3$ if $S \geq 3$ and M^3 has multiple principal curvatures at some point.

THEOREM ([Chang, 1993]). *A closed minimally immersed hypersurface with constant scalar curvature in S^4 is either an equatorial 3-sphere, a product of spheres, or a Cartan's minimal hypersurface. In particular, $R_n = \{0, 3, 6\}$.*

H. Yang and Q. M. Cheng [10] proved

THEOREM ([Yang and Cheng, 1998]). *Let M^n be a closed minimally immersed hypersurface with constant scalar curvature in S^{n+1}. If $S > n$, then $S \geq n + n/3$.*

Let $G \simeq O(k) \times O(p) \times O(q) \subset O(k+p+q)$ and set $k+p+q = n+2$. Then W. Y. Hsiang [4] investigated G-*invariant, minimal hypersurfaces,* M^n in S^{n+1}, by studying their generating curves, M^n/G, in the orbit space S^{n+1}/G and proved

THEOREM ([Hsiang, 1987]). *For each dimension $n \geq 3$, there exist infinitely many, mutually noncongruent closed G-invariant minimal hypersurfaces in S^{n+1}, where $G \simeq O(k) \times O(k) \times O(q)$ and $k = 2$ or 3.*

We studied G-invariant minimal hypersurfaces, in stead of minimal ones, with constant scalar curvatures in S^5. In this paper, we shall prove the following theorem:

THEOREM. *Let M^4 be a closed G-invariant minimal hypersurface with constant scalar curvature in S^5, where $G = O(2) \times O(2) \times O(2)$.*

(1) *If M^4 has 2 distinct principal curvatures at some point, then $S = 4$.*
(2) *If $S > 4$, then M^4 does not have simple principal curvatures everywhere.*

1. Preliminaries

Let M^n be a manifold of dimension n immersed in a Riemannian manifold N^{n+1} of dimension $n+1$. Let $\overline{\nabla}$ and $\langle\, , \rangle$ be the connection and metric tensor respectively of N^{n+1} and let $\bar{\mathcal{R}}$ be the curvature tensor with respect to the connection $\overline{\nabla}$ on N^{n+1}. Choose a local orthonormal frame field $e_1, \ldots, e_{n+1}$ in N^{n+1} such that after restriction to M^n, the $e_1, \ldots, e_n$ are tangent to M^n. Denote the dual coframe by $\{\omega_A\}$. Here we will always use $i, j, k, \ldots,$ for indices running over $\{1, 2, \ldots, n\}$ and

$A, B, C, \dots,$ over $\{1, 2, \dots, n+1\}$. As usual, the *second fundamental form* h and the *mean curvature* H of M^n in N^{n+1} are respectively defined by

$$h(v, w) = \langle \overline{\nabla}_v w,\, e_{n+1} \rangle \quad \text{and} \quad H = \sum_i h(e_i,\, e_i).$$

And the *scalar curvature* $\bar{R}$ of N^{n+1} is defined by

$$\bar{R} = \sum_{A,B} \langle \bar{\mathcal{R}}(e_A,\, e_B) e_B,\, e_A \rangle.$$

Then the structure equations of N^{n+1} are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$
$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D,$$

where $K_{ABCD} = \langle \bar{\mathcal{R}}(e_A,\, e_B) e_D,\, e_C \rangle$. When N^{n+1} is the unit sphere S^{n+1}, we have

$$K_{ABCD} = \delta_{AC}\, \delta_{BD} - \delta_{AD}\, \delta_{BC}.$$

Next, we restrict all tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n. Then

$$\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0.$$

By Cartan's lemma, we can write

$$\omega_{(n+1)i} = -\sum_j h_{ij}\, \omega_j.$$

Here,

$$\begin{aligned} h_{ij} &= -\omega_{(n+1)i}(e_j) = -\langle \overline{\nabla}_{e_j} e_{n+1},\, e_i \rangle \\ &= \langle \overline{\nabla}_{e_j} e_i,\, e_{n+1} \rangle = h(e_j, e_i) = h(e_i, e_j). \end{aligned}$$

Second, from

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = \sum_l \omega_{il} \wedge \omega_{lj} - \frac{1}{2} \sum_{l,m} R_{ijlm}\, \omega_l \wedge \omega_m,$$

we find the curvature tensor of M^n is

$$R_{ijlm} = K_{ijlm} + h_{il}\,h_{jm} - h_{im}\,h_{jl}. \tag{1.1}$$

If M^n is a piece of minimally immersed hypersurface in the unit sphere S^{n+1} and R is the scalar curvature of M^n, then we have

$$R = n(n-1) - S, \tag{1.2}$$

where $S = \sum_{i,j} h_{ij}^2$ is the *square norm* of h.

Given a symmetric 2-tensor $T = \sum_{i,j} T_{ij}\,\omega_i\,\omega_j$ on M^n, we also define its covariant derivatives, denoted by $\nabla T, \nabla^2 T$ and $\nabla^3 T$, etc. with components $T_{ij,k}$, $T_{ij,kl}$ and $T_{ij,klp}$, respectively, as follows:

$$\begin{aligned}
\sum_k T_{ij,k}\,\omega_k &= d\,T_{ij} + \sum_s T_{sj}\,\omega_{si} + \sum_s T_{is}\,\omega_{sj},\\
\sum_l T_{ij,kl}\,\omega_l &= d\,T_{ij,k} + \sum_s T_{sj,k}\,\omega_{si}\\
&\quad + \sum_s T_{is,k}\,\omega_{sj} + \sum_s T_{ij,s}\,\omega_{sk},\\
\sum_p T_{ij,klp}\,\omega_p &= d\,T_{ij,kl} + \sum_s T_{sj,kl}\,\omega_{si} + \sum_s T_{is,kl}\,\omega_{sj}\\
&\quad + \sum_s T_{ij,sl}\,\omega_{sk} + \sum_s T_{ij,ks}\,\omega_{sl}.
\end{aligned} \tag{1.3}$$

In general, the resulting tensors are no longer symmetric, and the rule to switch sub-index obeys the Ricci formula as follows:

$$\begin{aligned}
T_{ij,kl} - T_{ij,lk} &= \sum_s T_{sj}\,R_{sikl} + \sum_s T_{is}\,R_{sjkl},\\
T_{ij,klp} - T_{ij,kpl} &= \sum_s T_{sj,k}\,R_{silp} + \sum_s T_{is,k}\,R_{sjlp}\\
&\quad + \sum_s T_{ij,s}\,R_{sklp},\\
T_{ij,klpm} - T_{ij,klmp} &= \sum_s T_{sj,kl}\,R_{sipm} + \sum_s T_{is,kl}\,R_{sjpm}\\
&\quad + \sum_s T_{ij,sl}\,R_{skpm} + \sum_s T_{ij,ks}\,R_{slpm}.
\end{aligned} \tag{1.4}$$

For the sake of simplicity, we always omit the comma (,) between indices in the special case $T = \sum_{i,j} h_{ij}\,\omega_i\,\omega_j$ with $N^{n+1} = S^{n+1}$.

Since $\sum_{C,D} K_{(n+1)iCD}\,\omega_C \wedge \omega_D = 0$ on M^n when $N^{n+1} = S^{n+1}$, we find

$$d(\sum_j h_{ij}\,\omega_j) = \sum_{j,l} h_{jl}\,\omega_l \wedge \omega_{ji}.$$

Therefore,

$$\sum_{j,l} h_{ijl}\,\omega_l \wedge \omega_j = \sum_j (dh_{ij} + \sum_l h_{lj}\,\omega_{li} + \sum_l h_{il}\,\omega_{lj}) \wedge \omega_j = 0;$$

i.e., h_{ijl} is symmetric in all indices.

Moreover, in the case that M^n is minimal, we have

$$\begin{aligned}
\sum_l h_{ijll} &= \sum_l h_{lijl} \\
&= \sum_l \{h_{lilj} + \sum_m (h_{mi} R_{mljl} + h_{lm} R_{mijl})\} \\
&= (n-1)h_{ij} + \sum_{l,m} \Big\{ - h_{mi} h_{ml} h_{lj} + h_{lm}(\delta_{mj}\delta_{il} - \delta_{ml}\delta_{ij} \\
&\qquad + h_{mj} h_{il} - h_{ml} h_{ij})\Big\} \\
&= nh_{ij} - \sum_{l,m} h_{lm} h_{ml} h_{ij} = (n-S)h_{ij}.
\end{aligned} \tag{1.5}$$

It follows that

$$\frac{1}{2}\,\Delta S = (n-S)S + \sum_{i,j,l} h_{ijl}^2. \tag{1.6}$$

2. G-invariant Hypersurface in S^{n+1}

For $G \simeq O(k) \times O(p) \times O(q)$, $\mathbb{R}^{n+2}$ splits into the orthogonal direct sum of irreducible invariant subspaces, namely

$$\mathbb{R}^{n+2} \simeq \mathbb{R}^k \oplus \mathbb{R}^p \oplus \mathbb{R}^q = \{(X, Y, Z)\}$$

where X is a generic k-vector, Y is a generic p-vector and Z is a generic q-vector. Here if we set $x = |X|, y = |Y|$ and $z = |Z|$, then the orbit space $\mathbb{R}^{n+2}/G$ can be parametrized by (x, y, z); $x, y, z \in \mathbb{R}_+$ and the orbital distance metric is given by $ds^2 = dx^2 + dy^2 + dz^2$. By restricting the above G-action to the unit sphere $S^{n+1} \subset \mathbb{R}^{n+2}$, it is easy to see that

$$S^{n+1}/G \simeq \{(x, y, z) : x^2 + y^2 + z^2 = 1;\ x, y, z \geq 0\}$$

which is isometric to a spherical triangle of $S^2(1)$ with $\pi/2$ as its three angles. The orbit labeled by (x, y, z) is exactly $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$.

Analytically, it is more convenient to use the following polar coordinate system of S^{n+1}/G, namely, by performing the coordinate transformation:

$$z = \cos r, \quad x = \sin r \cos\theta, \quad y = \sin r \sin\theta, \qquad 0 \leq r, \theta \leq \frac{\pi}{2}.$$

To investigate those *G-invariant minimal hypersurfaces*, M^n, in S^{n+1} we study their generating curves, $\gamma(s) = M^n/G$, in the orbit space S^{n+1}/G [4, 6].

LEMMA 2.1. *Let M^n be a G-invariant hypersurface in S^{n+1}. Then there is a local orthonormal frame field $e_1, \ldots, e_{n+1}$ in S^{n+1} such that after restriction to M^n, the $e_1, \ldots, e_n$ are tangent to M^n and $h_{ij} = 0$ if $i \neq j$.*

PROOF. Let $(X_0, Y_0, Z_0) \in M^n \subset S^{n+1}$ with $x = |X_0|$, $y = |Y_0|$ and $z = |Z_0|$ and choose a local orthonormal frame field on a neighborhood of (X_0, Y_0, Z_0) as follows.

First, we choose vector fields $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{p-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ on a neighborhood U of (X_0, Y_0, Z_0) in the orbit $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$ such that:

(1) $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}$ are lifts of orthonormal tangent vector fields $u_1, \ldots, u_{k-1}$ on a neighborhood of X_0 in $S^{k-1}(x)$ to $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$ respectively,

(2) $\widetilde{v}_1, \ldots, \widetilde{v}_{p-1}$ are lifts of orthonormal tangent vector fields $v_1, \ldots, v_{p-1}$ on a neighborhood of Y_0 in $S^{p-1}(y)$ to $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$ respectively,

(3) $\widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ are lifts of orthonormal tangent vector fields $w_1, \ldots, w_{q-1}$ on a neighborhood of Z_0 in $S^{q-1}(z)$ to $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$ respectively.

Second, let $c(t) = (c_1(t), c_2(t), c_3(t))$ be the unit speed geodesic in S^{n+1}/G orthogonal to the curve $\gamma(s) = (x(s), y(s), z(s))$. For each

$P = (X, Y, Z) \in U$, let $\widetilde{\gamma}(P, s)$ and $\widetilde{c}(P, t)$ be the horizontal lifts in S^{n+1} of $\gamma(s)$ and $c(t)$ through P respectively. Then we see

$$\widetilde{\gamma}'(P, s) = \Big(x'(s)\frac{X}{x},\, y'(s)\frac{Y}{y},\, z'(s)\frac{Z}{z}\Big),$$

and

$$\widetilde{c}'(P, t) = \Big(c_1'(t)\frac{X}{x},\, c_2'(t)\frac{Y}{y},\, c_3'(t)\frac{Z}{z}\Big).$$

Third, we extend these vector fields over a neighborhood of (X_0, Y_0, Z_0) in S^{n+1} as follows:

(1) we translate $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{p-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ parallel along $\widetilde{\gamma}$ and $\widetilde{c}$.

(2) we extend $\widetilde{\gamma}'$ and $\widetilde{c}'$ in the usual fashion.

Then these extended vector fields $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{p-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}', \widetilde{c}'$ is a local orthonormal frame field in S^{n+1}. After restriction these vector fields to M^n, $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{p-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}'$ are tangent to M^n. For convenience, we write them as $e_1, \ldots, e_{n+1}$, in order.

Let $\bar{\alpha}_i(u) = (\alpha_i(u), Y, Z)$ be a curve in $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$ through P with $\bar{\alpha}_i'(0) = (\alpha_i'(0), 0, 0) = \widetilde{u}_i(P)$. Then,

$$\widetilde{\gamma}(\bar{\alpha}_i(u), s) = \Big(x(s)\frac{\alpha_i(u)}{x},\, y(s)\frac{Y}{y},\, z(s)\frac{Z}{z}\Big),$$

and

$$\widetilde{c}(\bar{\alpha}_i(u), t) = \Big(c_1(t)\frac{\alpha_i(u)}{x},\, c_2(t)\frac{Y}{y},\, c_3(t)\frac{Z}{z}\Big).$$

It implies that

$$\widetilde{\gamma}'(\bar{\alpha}_i(u), s) = \Big(x'(s)\frac{\alpha_i(u)}{x},\, y'(s)\frac{Y}{y},\, z'(s)\frac{Z}{z}\Big),$$

and

$$\widetilde{c}'(\bar{\alpha}_i(u), t) = \Big(c_1'(t)\frac{\alpha_i(u)}{x},\, c_2'(t)\frac{Y}{y},\, c_3'(t)\frac{Z}{z}\Big).$$

Hence, we have

$$\begin{aligned}\overline{\nabla}_{\widetilde{u}_i(P)}\widetilde{\gamma}' &= \{\frac{x'(0)}{x}\,(\alpha_i'(0),\, 0,\, 0)\}^{\top} \\ &= \{\frac{x'(0)}{x}\,\widetilde{u}_i(P)\}^{\top} = \frac{x'(0)}{x}\,\widetilde{u}_i(P),\end{aligned} \tag{2.1}$$

$$\overline{\nabla}_{\widetilde{u}_i(P)}\widetilde{c}' = \left\{\frac{c_1'(0)}{x}\,(\alpha_i'(0),\,0,\,0)\right\}^{\top} = \left\{\frac{c_1'(0)}{x}\,\widetilde{u}_i(P)\right\}^{\top} = \frac{c_1'(0)}{x}\,\widetilde{u}_i(P)$$

and

$$\begin{aligned} h_{ij} &= \langle\,\overline{\nabla}_{\widetilde{u}_i(P)}\widetilde{u}_j,\,\widetilde{c}'(0)\,\rangle = -\langle\,\widetilde{u}_j(P),\,\overline{\nabla}_{\widetilde{u}_i(P)}\widetilde{c}'\,\rangle \\ &= -\left\langle\,\widetilde{u}_j(P),\,\frac{c_1'(0)}{x}\widetilde{u}_i(P)\,\right\rangle = \frac{-c_1'(0)}{x}\delta_{ij}. \end{aligned}$$

In the same way, we have

$$\begin{cases} h_{(k-1+i)(k-1+j)} = \langle\,\overline{\nabla}_{\widetilde{v}_i(P)}\widetilde{v}_j,\,\widetilde{c}'\,\rangle = \dfrac{-c_2'(0)}{y}\delta_{ij}, \\ h_{(k+p-2+i)(k+p-2+j)} = \langle\,\overline{\nabla}_{\widetilde{w}_i(P)}\widetilde{w}_j,\,\widetilde{c}'\,\rangle = \dfrac{-c_3'(0)}{z}\delta_{ij}, \\ h(\widetilde{u}_i,\widetilde{v}_j) = h(\widetilde{u}_i,\widetilde{w}_j) = h(\widetilde{v}_i,\widetilde{w}_j) = 0, \\ h(\widetilde{u}_i,\widetilde{\gamma}') = h(\widetilde{v}_i,\widetilde{\gamma}') = h(\widetilde{w}_i,\widetilde{\gamma}') = 0. \end{cases}$$

And, since $\nabla_{\gamma'(P)}\gamma' = (x''(0), y''(0), z''(0))^{\top}$,

$$\begin{aligned} h_{nn} &= \langle\,\overline{\nabla}_{\widetilde{\gamma}'}\widetilde{\gamma}',\,\widetilde{c}'\,\rangle \\ &= \left\langle\,(x''(0)\frac{X}{x},\,y''(0)\frac{Y}{y},\,z''(0)\frac{Z}{z})^{\top},\,(c_1'(0)\frac{X}{x},\,c_2'(0)\frac{Y}{y},\,c_3'(0)\frac{Z}{z})\,\right\rangle \\ &= x''(0)\,c_1'(0) + y''(0)\,c_2'(0) + z''(0)\,c_3'(0) \\ &= \langle\,(x''(0), y''(0), z''(0)),\,\mathfrak{n}\,\rangle \\ &= \langle\,\nabla_{\gamma'}\gamma',\,\mathfrak{n}\,\rangle = \kappa_g(\gamma), \end{aligned}$$

where $\mathfrak{n} = (c_1{}'(0), c_2{}'(0), c_3{}'(0))$. Recall that

$$\gamma(s) = (\sin r(s)\cos\theta(s), \sin r(s)\sin\theta(s), \cos r(s)).$$

Then, we have

$$\gamma'(s) = \frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta},$$

where $\partial/\partial r = (\cos r\cos\theta,\ \cos r\sin\theta,\ -\sin r)$ and $\partial/\partial\theta = \sin r(-\sin\theta$ $\cos\theta,\ 0)$. Thus, we see

$$\left|\frac{\partial}{\partial r}\right| = 1, \quad \left|\frac{\partial}{\partial\theta}\right|^2 = \sin^2 r, \ \text{ and } \left\langle\frac{\partial}{\partial r},\,\frac{\partial}{\partial\theta}\right\rangle = 0.$$

And we see

$$1 = |\gamma'(s)|^2 = \left(\frac{dr}{ds}\right)^2 + \left(\frac{d\theta}{ds}\right)^2 |\frac{\partial}{\partial\theta}|^2 = \left(\frac{dr}{ds}\right)^2 + \left(\frac{d\theta}{ds}\right)^2 \sin^2 r.$$

Hence, we obtain

$$\cos\alpha = \left\langle \gamma', \frac{\partial}{\partial r}\right\rangle \Big/ |\gamma'| \left|\frac{\partial}{\partial r}\right| = \frac{dr}{ds} \text{ and } \sin\alpha = \frac{d\theta}{ds}\sin r,$$

where α is the angle between the curve γ and the radial direction $\partial/\partial r$.

Suppose S^{n+1}/G is orientated by the frame field $\{(\partial/\partial r),\ 1/\sin r\ (\partial/\partial\theta)\}$ and $U = (\partial/\partial r) \times 1/\sin r\ (\partial/\partial\theta)$. Then we have

$$\begin{aligned}
\mathfrak{n} &= U \times T = U \times \gamma' = U \times \left(\frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta}\right)\\
&= \frac{1}{\sin r}\frac{dr}{ds}\frac{\partial}{\partial\theta} - \sin r\frac{d\theta}{ds}\frac{\partial}{\partial r}\\
&= \frac{dr}{ds}(-\sin\theta, \cos\theta, 0) - \sin r\frac{d\theta}{ds}(\cos r\cos\theta, \cos r\sin\theta, -\sin r)\\
&= ({c_1}'(0), {c_2}'(0), {c_3}'(0)).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\kappa_g(\gamma) &= \langle \nabla_{\gamma'}\gamma', \mathfrak{n}\rangle\\
&= \left\langle \nabla_{\gamma'}\left(\frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta}\right), \left(\frac{1}{\sin r}\frac{dr}{ds}\frac{\partial}{\partial\theta} - \sin r\frac{d\theta}{ds}\frac{\partial}{\partial r}\right)\right\rangle\\
&= \frac{d\alpha}{ds} + \cos r\frac{d\theta}{ds}.
\end{aligned}$$

Therefore, we compute

$$\tag{2.2} \begin{cases}
h_{ii} = -\dfrac{c_1'(0)}{x} = \cos r\dfrac{d\theta}{ds} + \dfrac{\tan\theta}{\sin r}\dfrac{dr}{ds},\\
h_{(k-1+i)(k-1+i)} = -\dfrac{c_2'(0)}{y} = \cos r\dfrac{d\theta}{ds} - \dfrac{\cot\theta}{\sin r}\dfrac{dr}{ds},\\
h_{(k+p-2+i)(k+p-2+i)} = -\dfrac{c_3'(0)}{z} = -\dfrac{\sin^2 r}{\cos r}\dfrac{d\theta}{ds},\\
h_{nn} = \kappa_g(\gamma) = \dfrac{d\alpha}{ds} + \cos r\dfrac{d\theta}{ds},\\
h_{ij} = 0, \quad \text{if} \quad i \neq j.
\end{cases}$$

The proof of Lemma 2.1 is complete. □

LEMMA 2.2. *Let M^n be a G-invariant hypersurface in S^{n+1} and let $\{e_A\}$ be a local orthonormal frame field in S^{n+1} as in Lemma 2.1. Then,*

(a) *all $h_{ijl} = 0$ except when $\{i,j,l\}$ is a permutation of either $\{i,i,n\}$,*

(b) *if $j \neq l$, then $h_{iijl} = h_{jlii} = h_{jjjl} = h_{ljjj} = 0$,*

(c) *if i,j,l,m are distinct, then $h_{ijlm} = 0$.*

PROOF. (a) Since h_{ijl} is symmetric in all indices, it suffices to show that $h_{ijl} = 0$ if $i \leq j \leq l$ and $\{i,j,l\} \neq \{i,i,n\}$.

(a.1) In the case $j \neq i$, Lemma 3.1 implies that $h_{ij} = 0$ and

$$h_{ijl} = e_l(h_{ij}) + \sum_s h_{sj}\,\omega_{si}(e_l) + \sum_s h_{is}\,\omega_{sj}(e_l) = (h_{jj} - h_{ii})\,\omega_{ji}(e_l). \tag{2.3}$$

Since $h_{ii} = h_{jj}$ if $i,j \leq k-1$, (2.3) implies $h_{ijl} = 0$ for all l.

If $k \leq i,j \leq k+p-2$ or $k+p-1 \leq i,j \leq n-1$, then also $h_{ijl} = 0$ for all l.

And, if $i \leq k-1$ and $k \leq j < n$, then for all l we have

$$h_{ijl} = h_{lij} = (h_{ii} - h_{ll})\,\omega_{il}(e_j) = (h_{ii} - h_{ll})\langle \nabla_{e_j} e_i,\, e_l\rangle = 0, \tag{2.4}$$

since $\nabla_{e_j} e_i = 0$ by the Koszul formula. In the similar cases, we also have $h_{ijl} = 0$.

Moreover, if $j = l = n$, then $h_{inn} = h_{nni} = e_i(h_{nn}) = 0$ since h_{nn} is constant on each orbit from (2.2).

(a.2) In the case $j = i$ and $l \neq n$, since h_{ii} is constant on each orbit from (2.2),

$$h_{ijl} = h_{iil} = e_l(h_{ii}) + \sum_s h_{si}\,\omega_{si}(e_l) + \sum_s h_{is}\,\omega_{si}(e_l) = e_l(h_{ii}) = 0. \tag{2.5}$$

Therefore, we see all $h_{ijl} = 0$ except when $\{i,j,l\}$ is a permutation of either $\{i,i,n\}$.

(b) If $j \neq l$, then $e_l(h_{iij}) = e_l\{e_j(h_{ii})\} = e_j\{e_l(h_{ii})\} = 0$ since neither j nor l is n and h_{ii} is constant on each orbit. Hence, we have

$$\begin{aligned} h_{iijl} &= e_l(h_{iij}) + \sum_s h_{sij}\,\omega_{si}(e_l) + \sum_s h_{isj}\,\omega_{si}(e_l) \\ &\qquad + \sum_s h_{iis}\,\omega_{sj}(e_l) \\ &= 2h_{jij}\,\omega_{ji}(e_l) - h_{iin}\,\omega_{nj}(e_l) = 0, \end{aligned} \tag{2.6}$$

since $h_{jji} = 0$ if $i \neq n$ and $\omega_{nj}(e_l) = \langle \nabla_{e_l} e_n, e_j \rangle = 0$ from (2.1).

And since $j \neq l$, from (1.4), Lemma 2.1 and (2.6) we also have

$$\begin{aligned} h_{jlii} = h_{ijli} &= h_{ijil} + \sum_s h_{sj} R_{sili} + \sum_s h_{is} R_{sjli} \\ &= h_{iijl} + h_{jj} R_{jili} + h_{ii} R_{ijli} = 0. \end{aligned} \tag{2.7}$$

Moreover,

$$\begin{aligned} h_{jjjl} &= e_l(h_{jjj}) + \sum_s h_{sjj} \omega_{sj}(e_l) + \sum_s h_{jsj} \omega_{sj}(e_l) \\ &\quad + \sum_s h_{jjs} \omega_{sj}(e_l) \\ &= 3h_{jjn} \omega_{nj}(e_l) = 0, \end{aligned} \tag{2.8}$$

since $e_l(h_{jjj}) = e_l\{e_j(h_{jj})\} = e_j\{e_l(h_{jj})\} = 0$ and $\omega_{nj}(e_l) = 0$.

And so,

$$\begin{aligned} h_{ljjj} &= h_{jjlj} \\ &= h_{jjjl} + \sum_s h_{sj} R_{sjlj} + \sum_s h_{js} R_{sjlj} \\ &= 2h_{jj} R_{jjlj} = 0. \end{aligned} \tag{2.9}$$

(c) Without loss of generality, it suffices to show that $h_{ijln} = h_{ijnl} = 0$ and $h_{ijlm} = 0$ for all i, j, l, m such that i, j, l, $m < n$.

By using (a), we easily see that

$$\begin{aligned} h_{ijln} &= e_n(h_{ijl}) + \sum_s h_{sjl} \omega_{si}(e_n) + \sum_s h_{isl} \omega_{sj}(e_n) \\ &\quad + \sum_s h_{ijs} \omega_{sl}(e_n) = 0, \end{aligned} \tag{2.10}$$

since i, j, $l < n$ and i, j, l are distinct.

And, from (1.4) and Lemma 2.1 we also have

$$\begin{aligned} h_{ijnl} &= h_{ijln} + \sum_s h_{sj} R_{sinl} + \sum_s h_{is} R_{sjnl} \\ &= h_{jj} R_{jinl} + h_{ii} R_{ijnl} = 0. \end{aligned} \tag{2.11}$$

If i, j, l, $m < n$, from (a) we can easily see

$$\begin{aligned} h_{ijlm} &= e_m(h_{ijl}) + \sum_s \{h_{sjl}\,\omega_{sj}(e_m) \\ &\qquad + h_{isl}\,\omega_{sj}(e_m) + h_{ijs}\,\omega_{sl}(e_m)\} \\ &= 0. \end{aligned} \tag{2.12}$$

It completes the proof of Lemma 2.2. □

Under such frame field as Lemma 2.1, we have

$$e_k(h_{ii}) = h_{iik} - \sum_s h_{si}\omega_{si}(e_k) - \sum_s h_{is}\omega_{si}(e_k) = h_{iik}. \tag{2.13}$$

Hence, in the case M^n is minimal, by differentiating $\sum_m h_{mm} = 0$ we have

$$\begin{aligned} 0 &= (e_j\,e_i - \nabla_{e_j}e_i)\Big(\sum_m h_{mm}\Big) \\ &= \sum_m \{e_j(h_{mmi}) - \sum_s \omega_{is}(e_j)h_{mms}\} \\ &= \sum_m h_{mmij} - \sum_{m,s} h_{smi}\,\omega_{sm}(e_j) - \sum_{m,s} h_{msi}\,\omega_{sm}(e_j) \\ &\qquad - \sum_{m,s} h_{mms}\,\omega_{si}(e_j) - \sum_{m,s} h_{mms}\omega_{is}(e_j) \\ &= \sum_m h_{mmij}. \end{aligned} \tag{2.14}$$

Moreover, we have

$$\begin{aligned} e_k\Big(\sum_{i,j} h_{ij}^2\Big) &= 2\sum_{i,j} h_{ij}e_k(h_{ij}) \\ &= 2\sum_{i,j} h_{ij}\{h_{ijk} - \sum_s h_{sj}\omega_{si}(e_k) \\ &\qquad - \sum_s h_{is}\omega_{sj}(e_k)\} \\ &= 2\sum_{i,j} h_{ij}h_{ijk}. \end{aligned} \tag{2.15}$$

Hence, in the case S is constant, by differentiating $\sum_{i,j} h_{ij}^2 = S$ twice, we have

(2.16)

$$
\begin{aligned}
0 &= (e_l\, e_k - \nabla_{e_l} e_k)\Big(\frac{1}{2}\sum_{i,j} h_{ij}^2\Big) \\
&= \sum_{i,j} e_l(h_{ij}h_{ijk}) - \sum_{i,j,s} \omega_{ks}(e_l) h_{ij} h_{ijs} \\
&= \sum_{i,j} \{e_l(h_{ij})h_{ijk} + h_{ij}\, e_l(h_{ijk})\} - \sum_{i,j,s} h_{ij}h_{ijs}\,\omega_{ks}(e_l) \\
&= \sum_{i,j} \Big\{ h_{ijl} - \sum_s h_{sj}\,\omega_{si}(e_l) - \sum_s h_{is}\,\omega_{sj}(e_l) \Big\} h_{ijk} \\
&\quad + \sum_{i,j} h_{ij} \Big\{ h_{ijkl} - \sum_s \Big[h_{sjk}\,\omega_{si}(e_l) \\
&\qquad\qquad + h_{isk}\,\omega_{sj}(e_l) + h_{ijs}\,\omega_{sk}(e_l) \Big] \Big\} - \sum_{i,j,s} h_{ij}h_{ijs}\,\omega_{ks}(e_l) \\
&= \sum_{i,j} h_{ij}h_{ijkl} + h_{ijk}h_{ijl} \\
&\quad - \sum_{i,j,s} \{ h_{sj}h_{ijk}\,\omega_{si}(e_l) + h_{is}h_{ijk}\,\omega_{sj}(e_l) + h_{ij}h_{sjk}\,\omega_{si}(e_l) \\
&\qquad + h_{ij}h_{isk}\,\omega_{sj}(e_l) + h_{ij}h_{ijs}\,\omega_{sk}(e_l) + h_{ij}h_{ijs}\,\omega_{ks}(e_l) \} \\
&= \sum_{i,j} h_{ij}h_{ijkl} + h_{ijk}h_{ijl} \\
&\quad - \sum_{i,j,s} \{ h_{sj}h_{ijk}\,\omega_{si}(e_l) + h_{is}h_{ijk}\,\omega_{sj}(e_l) + h_{sj}h_{ijk}\,\omega_{is}(e_l) \\
&\qquad + h_{is}h_{ijk}\,\omega_{js}(e_l) + h_{ij}h_{ijs}\,\omega_{sk}(e_l) + h_{ij}h_{ijs}\,\omega_{ks}(e_l) \} \\
&= \sum_i h_{ii}h_{iikl} + \sum_{i,j} h_{ijk}\, h_{ijl}.
\end{aligned}
$$

3. G-invariant minimal hypersurface in S^5

Throughout this section, we assume that $G \simeq O(2)\times O(2)\times O(2)$ and M^4 is a closed G-invariant minimal hypersurface with constant scalar curvature in S^5. Let $\{e_A\}$ be a local orthonormal frame field in S^5 as in Lemma 2.1. Then by differentiating $\sum_i h_{ii} = 0$ and $\sum_i h_{ii}^2 = S$ with respect to e_4 respectively, we have

$$h_{114} + h_{224} + h_{334} + h_{444} = 0, \tag{3.1}$$

$$h_{11}h_{114} + h_{22}h_{224} + h_{33}h_{334} + h_{44}h_{444} = 0. \tag{3.2}$$

From (1.5), we also have

$$h_{ii11} + h_{ii22} + h_{ii33} + h_{ii44} = (4 - S)h_{ii}. \tag{3.3}$$

Here, if $i \neq 4$, from (1.3) we know

$$\begin{aligned} h_{ii4} = h_{i4i} &= e_i(h_{i4}) + \sum_s h_{s4}\omega_{si}(e_i) + h_{is}\omega_{s4}(e_i) \\ &= (h_{44} - h_{ii})\,\omega_{4i}(e_i) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} h_{iiii} &= e_i(h_{iii}) + \sum_s \{h_{sii}\omega_{si}(e_i) + h_{isi}\omega_{si}(e_i) + h_{iis}\omega_{si}(e_i)\} \\ &= 3h_{ii4}\,\omega_{4i}(e_i). \end{aligned} \tag{3.5}$$

Moreover, if $i,\ j \neq 4$ and if $i \neq j$,

$$\begin{aligned} h_{iijj} &= e_j(h_{iij}) + \sum_s \{h_{sij}\omega_{si}(e_j) + h_{isj}\omega_{si}(e_j) + h_{iis}\omega_{sj}(e_j)\} \\ &= h_{ii4}\,\omega_{4j}(e_j). \end{aligned} \tag{3.6}$$

Now, to prove our Theorem we need the following two lemmas.

LEMMA 3.1. *Suppose $h_{ii} = h_{44} = \lambda$ at some point p for $i = 1$, 2 or 3. Then,*

$$S = \frac{12\lambda^4 + 4\lambda^2}{5\lambda^2 - 1}. \tag{3.7}$$

PROOF. Without loss of generality, we can assume $h_{33} = h_{44} = \lambda$ at some point p. By using (3.4), we have $h_{334}(p) = 0$. Using (3.5) and (3.6), we have at p

$$h_{3311} = h_{3322} = h_{3333} = 0. \tag{3.8}$$

Hence, (3.3) and (3.8) imply

$$h_{3344} = (4 - S)h_{33} \tag{3.9}$$

and (1.4) implies

$$h_{4433} = h_{3344} + (h_{44} - h_{33})(1 + h_{44}h_{33}) = h_{3344}. \tag{3.10}$$

Since $\sum_{i,j} h_{ij3}^2 = 0$ at p, from (2.16) we have

$$h_{11}h_{1133} + h_{22}\, h_{2233} + h_{33}\, h_{3333} + h_{44}\, h_{4433} = 0. \tag{3.11}$$

Let $h_{ii} = \lambda_i$. Then, by using (1.4) and (3.8) we know

$$h_{1133} = h_{3311} + (h_{11} - \lambda)(1 + h_{11}\,\lambda) = (\lambda_1 - \lambda)(1 + \lambda_1\,\lambda) \tag{3.12}$$

and

$$h_{2233} = h_{3322} + (h_{22} - \lambda)(1 + h_{22}\,\lambda) = (\lambda_2 - \lambda)(1 + \lambda_2\,\lambda).$$

Hence, (3.11) and (3.12) imply

$$\lambda_1\,(\lambda_1 - \lambda)(1 + \lambda_1\,\lambda) + \lambda_2\,(\lambda_2 - \lambda)(1 + \lambda_2\,\lambda) + \lambda\,(4 - S)\lambda = 0, \tag{3.13}$$

that is,

$$\lambda_1^2 + \lambda_2^2 - (\lambda_1 + \lambda_2)\lambda + (\lambda_1^3 + \lambda_2^3)\lambda - (\lambda_1^2 + \lambda_2^2)\lambda^2 + (4 - S)\lambda^2 = 0.$$

Here, since

$$\lambda_1 + \lambda_2 + 2\lambda = 0, \;\; \lambda_1^2 + \lambda_2^2 + 2\lambda^2 = S, \;\; \lambda_1\,\lambda_2 = 3\lambda^2 - \frac{S}{2},$$
$$\lambda_1^3 + \lambda_2^3 = (\lambda_1^2 + \lambda_2^2 - \lambda_1\,\lambda_2)(\lambda_1 + \lambda_2) = 10\lambda^3 - 3S\lambda,$$

(3.13) becomes

$$S - 2\lambda^2 - (-2\lambda)\lambda + (10\lambda^3 - 3S\lambda)\lambda - (S - 2\lambda^2)\lambda^2 + (4 - S)\lambda^2 = 0$$

and so,

$$S = \frac{12\lambda^4 + 4\lambda^2}{5\lambda^2 - 1}.$$

It completes the proof of Lemma 3.1. □

LEMMA 3.2. *If $S > 4$ and $i = 1, 2, 3$, then for each i, there exists a point q_i in M^4 so that $h_{ii}(q_i) = 0$.*

PROOF. Suppose that the conclusion is not valid. Without loss of generality, we can assume that $h_{33} > 0$ everywhere. Consider a point p_0, such that

$$h_{33}(p_0) = \min_{M^4} h_{33} > 0.$$

Then, due to the maximal principle, we have

$$e_4(h_{33})(p_0) = h_{334}(p_0) = 0 \quad \text{and} \tag{3.14}$$

$$Hess.\, h_{33}(e_4, e_4)(p_0) = (e_4\, e_4 - \nabla_{e_4} e_4)(h_{33}) \geq 0.$$

Hence, from (3.14) we have at p_0

$$Hess.\, h_{33}(e_4, e_4) = h_{3344} - \sum_s \omega_{4s}(e_4) h_{33s} = h_{3344} \geq 0. \tag{3.15}$$

Since $h_{334}(p_0) = 0$, using (3.5) and (3.6) as in Lemma 4.1, we have at p_0

$$h_{3311} = h_{3322} = h_{3333} = 0$$

and so,

$$h_{3344} = (4 - S)h_{33}. \tag{3.16}$$

By using (3.15) and (3.16), we have at p_0

$$h_{3344} = (4 - S)h_{33} \geq 0,$$

which is contrary to the hypothesis $S > 4$. It completes the proof. □

We are ready to prove our Theorem:

THEOREM. *Let M^4 be a closed G-invariant minimal hypersurface with constant scalar curvature in S^5, where $G = O(2) \times O(2) \times O(2)$.*

(1) *If M^4 has 2 distinct principal curvatures at some point, then $S = 4$.*
(2) *If $S > 4$, then M^4 does not have simple principal curvatures everywhere.*

PROOF. (1) Suppose M^4 has 2 distinct principal curvatures at some point, say, p. Without loss of generality, we can assume either one of the following three cases for some $\lambda \neq 0$:

Case 1. Suppose $h_{11} = h_{22} = h_{33} = \lambda$ and $h_{44} = -3\lambda$ at the point p. Then from (2.2), we have at p

$$\cos r \frac{d\theta}{ds} + \frac{\tan\theta}{\sin r}\frac{dr}{ds} = \cos r \frac{d\theta}{ds} - \frac{\cot\theta}{\sin r}\frac{dr}{ds} = -\frac{\sin^2 r}{\cos r}\frac{d\theta}{ds}.$$

It implies that

$$\frac{dr}{ds} = 0 \quad \text{and} \quad \frac{d\theta}{ds} = 0,$$

which means that $h_{11} = h_{22} = h_{33} = h_{44} = \lambda = 0$ at p. It is contrary to the hypothesis.

Case 2. Suppose $h_{22} = h_{33} = h_{44} = \lambda$ and $h_{11} = -3\lambda$ at the point p. Then

$$S = h_{11}^2 + h_{22}^2 + h_{33}^2 + h_{44}^2 = 12\lambda^2. \tag{3.13}$$

Hence, (3.7) and (3.13) imply $S = 4$. i.e. $M^4 = S^1(\sqrt{1/4}) \times S^3(\sqrt{3/4})$.

Case 3. Suppose $h_{11} = h_{22} = -\lambda$, $h_{33} = h_{44} = \lambda$ at the point p. Then

$$S = h_{11}^2 + h_{22}^2 + h_{33}^2 + h_{44}^2 = 4\lambda^2. \tag{3.14}$$

Hence, (3.7) and (3.14) imply $S = 4$. i.e. $M^4 = S^2(\sqrt{1/2}) \times S^2(\sqrt{1/2})$. But, it is not G-invariant.

(2) Suppose that M^4 has only simple principal curvatures everywhere. Then since all principal curvatures h_{ii}'s are constant on each orbit, without loss of generality we can assume everywhere either one of the following three cases:

(a) $h_{11} < h_{22} < h_{33} < h_{44}$ or
(b) $h_{11} < h_{22} < h_{44} < h_{33}$ or
(c) $h_{44} < h_{11} < h_{22} < h_{33}$.

But by Lemma 3.2, there exist points q_1 and q_3 in M^4 such that $h_{11}(q_1) = 0$ and $h_{33}(q_3) = 0$ respectively. Hence the above each case is contrary to the fact that

$$h_{11}(q_1) + h_{22}(q_1) + h_{33}(q_1) + h_{44}(q_1) = 0 \text{ or}$$
$$h_{11}(q_3) + h_{22}(q_3) + h_{33}(q_3) + h_{44}(q_3) = 0.$$

Therefore, M^4 does not have simple principal curvatures everywhere. □

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