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COUNTABLY APPROXIMATING FRAMES

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ABSTRACT. Using the countably way below relation, we show that the category σ**CFrm** of σ-coherent frames and σ-coherent homomorphisms is coreflective in the category **Frm** of frames and frame homomorphisms. Introducing the concept of stably countably approximating frames which are exactly retracts of σ-coherent frames, it is shown that the category **SCAFrm** of stably countably approximating frames and σ-proper frame homomorphisms is coreflective in **Frm**. Finally we introduce strongly Lindelöf frames and show that they are precisely lax retracts of σ-coherent frames.

0. Introduction

Since Scott ([12]) introduced continuous lattices, continuous lattices have been shown to have many interesting properties and characterizations in various points of view ([6]). Among others, continuous frames are a natural pointfree version of locally compact spaces.

The concept of coherent frames gives rise to an equivalence with the category of distributive lattices and homomorphisms ([9]). Furthermore, stably continuous frames are exactly retracts of coherent frames ([1], [2]).

We have introduced concepts of countably way below relations on a complete lattice and countably approximating lattices which generalize continuous lattices ([10]). We note that countably approximating frames are a pointfree counterpart of locally Lindelöf spaces.

The purpose of this paper is to generalize results on continuous frames to countably approximating frames.

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First, we study basic properties of countably approximating frames and then using the countably way below relation and the frame of σ-ideals, it is shown that the category σ**CFrm** of σ-coherent frames and σ-coherent homomorphisms is coreflective in the category **Frm** of frames and frame homomorphisms.

We introduce a concept of stably countably approximating frames and show that a frame is stably countably approximating iff it is a retract of a σ-coherent frame and that the category **SCAFrm** of stably countably approximating frames and σ-proper frame homomorphisms is also coreflective in **Frm**.

Finally using the concept of strong convergence of generalized filters, we introduce strongly Lindelöf frames and show that they are precisely lax retracts of σ-coherent frames.

In the following, we always assume that a lattice is a bounded lattice i.e., a lattice with the top e and the bottom 0, and that homomorphisms between lattices preserve e and 0. For terminology not introduced in the paper, we refer to [6], [9].

1. Countably approximating frames

Recall that a *frame* is a complete lattice L satisfying the infinite distributive law $x \wedge (\bigvee S) = \bigvee\{x \wedge s | s \in S\}$ for all $x \in L$ and $S \subseteq L$. A *frame homomorphism* is a map between frames preserving arbitrary joins, including the bottom 0 and finitary meets, including the top e. Let **Frm** denote the category of frames and frame homomorphisms. Furthermore, a complete lattice A is said to be a *continuous lattice* if for all $x \in A$,

$$x = \bigvee\{u \in A | u \ll x\},$$

where $x \ll y$ means that for any directed subset D of A with $y \leq \bigvee D$, there is $d \in D$ with $x \leq d$ ([6] for the details).

DEFINITION 1.1 ([10]). A complete lattice A is said to be a *countably approximating lattice* if for all $x \in A$,

$$x = \bigvee\{u \in A | u \ll_c x\},$$

where $\ll_c$ is the countably way below relation, i.e., $x \ll_c y$ means that for any countably directed subset D of A with $y \leq \bigvee D$, there is $d \in D$ with $x < d$.

EXAMPLE 1.2. 1) A continuous lattice is a countably approximating lattice.

2) A countable complete lattice is a countably approximating lattice.

3) If X is a locally Lindelöf space, i.e., each point x of X has a local base consisting of Lindelöf neighborhoods of x, then the open set frame $\Omega(X)$ is a countably approximating frame.

4) A countably approximating frame need not be a continuous frame. For example, the open set frame $\Omega(Q)$, where Q is the rational line with the usual topology, is not a continuous but countably approximating frame ([10]).

In the following, $Fin(X)$ ($Count(X)$, resp.) denotes the set of all finite (countable, resp.) subsets of X.

Recall that a *σ-frame* is a lattice A with countable joins in which $x \wedge (\bigvee K) = \bigvee\{x \wedge k | k \in K\}$ for all $x \in A$ and $K \in Count(A)$ ([7]). A *σ-frame homomorphism* is a map between σ-frames preserving countable joins and finite meets. Let $\sigma\mathbf{Frm}$ be the category of σ-frames and σ-frame homomorphisms.

An *ideal* of a lattice is a down set which is closed under finite joins. A *σ-ideal* of a lattice is a countably directed ideal. For a σ-frame A, an ideal of A is a σ-ideal iff it is closed under countable joins. For a lattice A, the set of all ideals (σ-ideals, resp.) is denoted by $\mathcal{J}A$ ($\mathcal{H}A$, resp.). If A is a distributive lattice (σ-frame, resp.) then $\mathcal{J}A$ ($\mathcal{H}A$, resp.) is a frame. Indeed, for the category $\mathbf{DLatt}$ of distributive lattices and homomorphisms, the functor $\mathcal{J}$: $\mathbf{DLatt} \to \mathbf{Frm}$ ($\mathcal{H}$: $\sigma\mathbf{Frm} \to \mathbf{Frm}$, resp.) gives rise to the left adjoint of the forgetful functor ([9], [11]).

For a complete lattice A, $\downarrow_w x = \{y \in A | y \ll x\}$ is an ideal of A and $\downarrow_c x = \{y \in A | y \ll_c x\}$ is a σ-ideal of A.

REMARK 1.3. A complete lattice L is a countably approximating lattice iff for each $x \in L$, the set $\downarrow_c x$ is the smallest σ-ideal I with $x \leq \bigvee I$ and therefore $\downarrow_c : L \to \mathcal{H}L$ is a left adjoint of the join map $\bigvee : \mathcal{H}L \to L$. Thus we characterize countably approximating lattices via σ-ideals. That is, every countably approximating lattice L is the image of the complete lattice $\mathcal{H}L$ under a map preserving arbitrary meets and joins ([10] for the details).

If $x \ll x$ ($x \ll_c x$, resp.) in a lattice A, then x is called a *compact* (*Lindelöf, resp.*) *element* of A. A complete lattice L is said to be a *compact* (*Lindelöf, resp.*) *lattice* if the top element e of L is a compact (Lindelöf, resp.) element of L. The set of all compact (Lindelöf, resp.) elements of L will be denoted by $\mathcal{K}(L)$ ($\mathcal{L}(L)$, resp.).

PROPOSITION 1.4. *If X is a regular space and $\Omega(X)$ is a countably approximating frame, then X is a locally Lindelöf space.*

PROOF. Let $x \in X$ and V an open neighborhood of x. Since $\Omega(X)$ is a countably approximating frame, there is $U \in \Omega(X)$ with $x \in U \ll_c V$. Since X is regular, there is an open neighborhood W of x with $\mathrm{cl}W \subseteq U$, where $\mathrm{cl}W$ denotes the closure of W. If $\mathcal{G} = \{G_\alpha | \alpha \in \Lambda\}$ is an open cover of $\mathrm{cl}W$, then $\mathcal{G} \cup \{X - \mathrm{cl}W\}$ is also an open cover of V. Since $U \ll_c V$, there is $\mathcal{G}' \in Count(\mathcal{G} \cup \{X - \mathrm{cl}W\})$ such that $\mathcal{G}'$ covers U. Thus $\mathcal{G}'$ is a cover of $\mathrm{cl}W$; hence $\mathrm{cl}W$ is a Lindelöf neighborhood of x contained in V. □

COROLLARY 1.5. *A regular space X is a locally Lindelöf space iff the open set frame $\Omega(X)$ is a countably approximating frame.*

A frame L is said to be *regular* if $a = \bigvee\{t \in L | t \prec a\}$ for all $a \in L$, where $t \prec a$ iff $t \wedge x = 0$ and $a \vee x = e$ for some $x \in L$, or equivalently, $a \vee t^* = e$ for the pseudocomplement $t^* = \bigvee\{s \in L | t \wedge s = 0\}$ of $t \in L$.

In a frame L and $x_n \in L$ $(n \in N)$, $x_n \prec a$ does not imply $\bigvee x_n \prec a$.

In a compact regular frame, $x \ll y$ iff $x \prec y$. But in a regular Lindelöf frame, $x \ll_c y$ does not imply $x \prec y$ in general. In the regular Lindelöf frame $\Omega(R)$, $(0,3) \ll_c (0,3)$ but $(0,3) \not\prec (0,3)$.

A *δ-frame* is a lattice A with countable meets satisfying the property $x \vee (\bigwedge K) = \bigwedge\{x \vee k | k \in K\}$ for all $x \in A$ and $K \in Count(A)$, or equivalently, the dual A^{op} of A is a σ-frame.

If a frame L is also a δ-frame, then $x_n \prec a$ implies $\bigvee x_n \prec a$; hence $\{t \in L | t \prec a\}$ is a σ-ideal of L. So we have the following:

PROPOSITION 1.6. *Let L be a frame, then for any $x, y \in L$ we have:*
1) *If L is a Lindelöf frame, then $x \prec y$ implies $x \ll_c y$.*
2) *If L is a regular δ-frame, then $x \ll_c y$ implies $x \prec y$.*
3) *If L is a regular Lindelöf δ-frame, then $x \prec y$ iff $x \ll_c y$.*
4) *Every regular Lindelöf frame is countably approximating.*

PROOF. 1) Suppose that $x \prec y$ and $y \leq \bigvee S$ for any $S \subseteq L$. Then $x^* \vee y = e$ implies $x^* \vee (\bigvee S) = e$. Since L is a Lindelöf frame, there is $K \in Count(S)$ with $x^* \vee (\bigvee K) = e$; hence $x \leq \bigvee K$. So $x \ll_c y$.

2) Let $x \ll_c y$. Since L is regular, $y = \bigvee\{t \in L | t \prec y\}$; hence $x \in \{t \in L | t \prec y\}$, because $\{t \in L | t \prec y\}$ is a σ-ideal of L. So $x \prec y$.

3) It follows from 1) and 2).

4) For any $a \in L$, $a = \bigvee\{x \in L | x \prec a\} \leq \bigvee \downarrow_c a \leq \bigvee \downarrow a \leq a$; therefore $a = \bigvee \downarrow_c a$. □

PROPOSITION 1.7. *Let L be a countably approximating frame, then $\ll_c$ interpolates, i.e., if $x \ll_c y$, then there is $z \in L$ with $x \ll_c z \ll_c y$.*

PROOF. Let $x \ll_c y$, then $y = \bigvee\{a | a \ll_c y\} = \bigvee\{\bigvee\{b | b \ll_c a\} | a \ll_c y\} = \bigvee\{b | b \ll_c a \ll_c y$ for some $a \in L\}$. Since $x \ll_c y$, there are sequences (a_n), (b_n) in L such that $b_n \ll_c a_n \ll_c y$ for any $n \in N$, and $x \leq \bigvee b_n$. Since $\bigvee b_n \ll_c \bigvee a_n, x \leq \bigvee b_n \ll_c \bigvee a_n \ll_c y$. Let $\bigvee a_n = z$, then $x \ll_c z \ll_c y$. □

REMARK 1.8. Suppose that a countably approximating frame L is generated by the set $\mathcal{L}(L)$ of Lindelöf elements, then $x \ll_c y$ in L iff there is $z \in \mathcal{L}(L)$ with $x \leq z \leq y$.

PROOF. Let $x \ll_c y = \bigvee(\downarrow y \cap \mathcal{L}(L))$, then there is $K \subseteq Count(\downarrow y \cap \mathcal{L}(L))$ with $x \leq \bigvee K$. Let $z = \bigvee K$, then $z \in \mathcal{L}(L)$ and $x \leq z \leq y$. Conversely, suppose $y \leq \bigvee S$, then $x \leq z \leq y \leq \bigvee S$ for some $z \in \mathcal{L}(L)$. Since $z \leq \bigvee S$, there is $K \in Count(S)$ with $z \leq \bigvee K$; hence $x \leq \bigvee K$. Thus $x \ll_c y$. □

2. σ-coherent frames

In this section, we establish the equivalence between the category σ**Frm** and the category σ**CFrm** of σ-coherent frames and σ-coherent homomorphisms, and then show that σ**CFrm** is coreflective in **Frm**.

We recall that a frame L is said to be *coherent* if $\mathcal{K}(L)$ is a sublattice of L and generates L.

DEFINITION 2.1 ([11]). A frame L is said to be *σ-coherent* if $\mathcal{L}(L)$ is a subσ-frame of L and generates L.

In a frame L, $\mathcal{L}(L)$ is closed under countable joins. Thus $\mathcal{L}(L)$ is a subσ-frame of L iff $e \in \mathcal{L}(L)$ and for $x, y \in \mathcal{L}(L)$, $x \wedge y \in \mathcal{L}(L)$.

REMARK 2.2. 1) For a lattice A, $\downarrow a$ is a Lindelöf element of $\mathcal{H}A$ for any $a \in A$, because for any countably directed subset $\mathcal{E}$ of $\mathcal{H}A$, $\downarrow a \subseteq \bigvee \mathcal{E}$ iff $a \in \bigcup \mathcal{E}$ iff $a \in S$ for some $S \in \mathcal{E}$ iff $\downarrow a \subseteq S$ for some $S \in \mathcal{E}$; hence $\downarrow a$ is a Lindelöf element of $\mathcal{H}A$. Thus $\{\downarrow a | a \in A\} \subseteq \mathcal{L}(\mathcal{H}A)$.

2) A σ-coherent frame need not be a coherent frame. For example, the complete chain $[0,1]$ with the usual order $\leq$ is a σ-coherent frame but not a coherent frame.

3) Every σ-coherent frame L is a countably approximating frame, because for any $a \in L$, $a = \bigvee(\downarrow a \cap \mathcal{L}(L)) \leq \bigvee \downarrow_c a \leq a$.

4) Let $T = [0, \Omega) \cup \{z_1, z_2\}$, where Ω is the first uncountable ordinal, $x \leq z_1, z_2$ for all $x \in [0, \Omega)$, and $[0, \Omega)$ is a chain with the ordinal order $\leq$. Then $\mathcal{D}T = \{U \subseteq T | \phi \neq U = \downarrow U\}$ is a countably approximating frame and $\mathcal{L}(\mathcal{D}T) = \{T, \downarrow z_1, \downarrow z_2\} \cup \{\downarrow x | x \in [0, \Omega)\}$. Since $\downarrow z_1 \cap \downarrow z_2 = [0, \Omega)$ is not a Lindelöf element of $\mathcal{D}T$, $\mathcal{D}T$ is not a σ-coherent frame, although $\mathcal{L}(\mathcal{D}T)$ generates $\mathcal{D}T$.

DEFINITION 2.3. A frame homomorphism $h : L \to M$ is said to be *σ-coherent* if $h(\mathcal{L}(L)) \subseteq \mathcal{L}(M)$.

The class of all σ-coherent frames and σ-coherent homomorphisms between them form a category which will be denoted by σ**CFrm**.

PROPOSITION 2.4. *Let A be a σ-frame, then we have:*
1) *Lindelöf elements of $\mathcal{H}A$ are precisely principal ideals.*
2) *$\mathcal{H}A$ is a σ-coherent frame.*
3) *The down map $\downarrow: A \to \mathcal{L}(\mathcal{H}A)$ ($\downarrow (a) = \downarrow a$) is an isomorphism.*

PROOF. 1) Take any Lindelöf element I of $\mathcal{H}A$, then $I = \bigvee\{\downarrow x | x \in I\}$. Since $I \ll_c I$, there is $K \in Count(I)$ such that $I \leq \bigvee\{\downarrow x | x \in K\}$ in $\mathcal{H}A$. Let $a = \bigvee K$, then $\bigvee\{\downarrow x | x \in K\} = \downarrow a$; hence $I = \downarrow a$. Conversely, $\downarrow a$ is a Lindelöf element of $\mathcal{H}A$ by 1) of Remark 2.2.

2) $\mathcal{L}(\mathcal{H}A) = \{\downarrow a | a \in A\}$ by 1). Note that $\downarrow a \wedge \downarrow b = \downarrow (a \wedge b) \in \mathcal{L}(\mathcal{H}A)$ and $\mathcal{H}A$ is a Lindelöf frame. Moreover, for any $I \in \mathcal{H}A$, $I = \bigvee\{\downarrow x | x \in I\}$. Thus $\mathcal{H}A$ is σ-coherent.

3) By 1), the down map is a $1-1$ onto map which preserves finite meets. Moreover $\downarrow(\bigvee K) = \bigvee\{\downarrow k | k \in K\}$ for any $K \in Count(A)$, so $\downarrow$ is a σ-frame homomorphism. Hence $\downarrow$ is an isomorphism. □

REMARK 2.5. Let L be a σ-coherent frame, then we have:
1) $\downarrow(\downarrow a \cap \mathcal{L}(L)) = \downarrow_c a$ for all $a \in L$.
2) $\downarrow(\downarrow a \cap \mathcal{L}(L)) = \downarrow a$ for all $a \in \mathcal{L}(L)$.

PROOF. Since L is σ-coherent, L is a countably approximating frame which is generated by $\mathcal{L}(L)$. Thus by Remark 1.8, we have 1).
2) follows from 1) together with the fact that for $a \in \mathcal{L}(L)$, $\downarrow_c a = \downarrow a$. □

PROPOSITION 2.6. *A frame is σ-coherent if and only if it is isomorphic to the frame of σ-ideals of a σ-frame.*

PROOF. ($\Rightarrow$) Let L be a σ-coherent frame, then $\mathcal{L}(L)$ is a σ-frame and $\mathcal{HL}(L)$ is a σ-coherent frame. Since the inclusion map $i : \mathcal{L}(L) \to L$ is a σ-frame homomorphism, there is a unique frame homomorphism $f : \mathcal{HL}(L) \to L$ with $f \circ \downarrow = i$ for the down map $\downarrow : \mathcal{L}(L) \to \mathcal{HL}(L)$. Indeed, $f(I) = \bigvee I$. Define $g : L \to \mathcal{HL}(L)$ by $g(a) = \downarrow a \cap \mathcal{L}(L)$. By the above remark, $g(a) = \downarrow_c a \cap \mathcal{L}(L)$; hence g is well defined. For any $I \in \mathcal{HL}(L)$, $x \in I$ iff $x \leq \bigvee I$, for x is a Lindelöf element; therefore $g(f(I)) = I$. Since L is σ-coherent, $f(g(a)) = a$ for all $a \in L$. Thus f is an isomorphism.
($\Leftarrow$) It is immediate from 2) of Proposition 2.4. □

For a σ-frame homomorphism $h : A \to B$, we have a frame homomorphism $\mathcal{H}h : \mathcal{H}A \to \mathcal{H}B$ ($\mathcal{H}h(I) = \downarrow h(I)$). $\mathcal{H}h$ is σ-coherent, because for any $\downarrow a$ ($a \in A$), $\mathcal{H}h(\downarrow a) = \downarrow h(a) \in \mathcal{L}(\mathcal{H}B)$. Thus $\mathcal{H} : \sigma\mathbf{Frm} \to \sigma\mathbf{CFrm}$ is a functor.
For any σ-coherent homomorphism $f : L \to M$, $\mathcal{L}(f) : \mathcal{L}(L) \to \mathcal{L}(M)$ ($\mathcal{L}(f)(x) = f(x)$) is a σ-frame homomorphism. Thus $\mathcal{L} : \sigma\mathbf{CFrm} \to \sigma\mathbf{Frm}$ is a functor.
Using these, we have the following:

THEOREM 2.7. *$\sigma\mathbf{Frm}$ and $\sigma\mathbf{CFrm}$ are equivalent.*

PROOF. For any $A \in \sigma\mathbf{Frm}$, let $\eta_A : A \to \mathcal{L}(\mathcal{H}A)$ be the map given by $\eta_A(a) = \downarrow a (a \in A)$. Then by Proposition 2.4, η_A is an isomorphism. For any $h : A \to B$ in $\sigma\mathbf{Frm}$, $\eta_B \circ h = \mathcal{L}(\mathcal{H}h) \circ \eta_A$; therefore $(\eta_A)_{A \in \sigma\mathbf{Frm}} : 1_{\sigma\mathbf{Frm}} \to \mathcal{L} \circ \mathcal{H}$ is a natural isomorphism. For any $L \in \sigma\mathbf{CFrm}$, let $\epsilon_L : \mathcal{H}\mathcal{L}(L) \to L$ be the map given by $\epsilon_L(I) = \bigvee I$, then it is an isomorphism by Proposition 2.6. Furthermore, for any $g : L \to M$ in $\sigma\mathbf{CFrm}$, $g \circ \epsilon_L = \epsilon_M \circ \mathcal{H}\mathcal{L}(g)$; hence $(\epsilon_L)_{L \in \sigma\mathbf{CFrm}} : \mathcal{H} \circ \mathcal{L} \to 1_{\sigma\mathbf{CFrm}}$ is a natural isomorphism. In all, $\sigma\mathbf{Frm}$ and $\sigma\mathbf{CFrm}$ are equivalent. □

PROPOSITION 2.8. *Let L be a σ-coherent frame, then there is a frame homomorphism $k : L \to \mathcal{H}L$ with $\bigvee \circ k = 1_L$, where $\bigvee : \mathcal{H}L \to L$ is the join map.*

PROOF. Define $k : L \to \mathcal{H}L$ by $k(a) = \downarrow_c a$. Then k preserves finite meets by Remark 1.8 and the fact that $\mathcal{L}(L)$ is closed under finite meets. Since L is countably approximating, k is a left adjoint of $\bigvee : \mathcal{H}L \to L$; hence k preserves arbitrary joins. In all, k is a frame homomorphism and $\bigvee(k(a)) = \bigvee(\downarrow_c a) = a = 1_L(a)$. □

THEOREM 2.9. *$\sigma\mathbf{CFrm}$ is coreflective in $\mathbf{Frm}$. Indeed, for any $L \in \mathbf{Frm}$, the join map $\bigvee_L : \mathcal{H}L \to L$ is the coreflection of L.*

PROOF. Since L is a σ-frame, $\mathcal{H}L$ is a σ-coherent frame and $\bigvee_L : \mathcal{H}L \to L$ is a frame homomorphism. Take any frame homomorphism $h : M \to L$, where M is a σ-coherent frame. Then by the above proposition, there is a frame homomorphism $\downarrow_c : M \to \mathcal{H}M$ with $\bigvee_M \circ \downarrow_c = 1_M$. For any $u \in \mathcal{L}(M)$, $\downarrow_c u = \downarrow u \in \mathcal{L}(\mathcal{H}M)$; hence $\downarrow_c$ is σ-coherent. Since $\mathbf{Frm}$ is a subcategory of $\sigma\mathbf{Frm}$, $\mathcal{H}h$ is σ-coherent and $\mathcal{H}h \circ \downarrow_c : M \to \mathcal{H}L$ is also σ-coherent. Let $\mathcal{H}h \circ \downarrow_c = f$, then $\bigvee_L \circ f = \bigvee_L \circ \mathcal{H}h \circ \downarrow_c = h \circ \bigvee_M \circ \downarrow_c = h$ by Remark 2.2. To show the uniqueness of f, suppose that $g : M \to \mathcal{H}L$ is a σ-coherent homomorphism with $\bigvee_L \circ g = h$. Take any $b \in M$ and $x \in \downarrow b \cap \mathcal{L}(M)$, then $g(x) = \downarrow a \subseteq g(b)$ for some $a \in L$, because g is σ-coherent. So $h(x) = (\bigvee_L \circ g)(x) = a \in g(b)$; hence $(\bigvee_L \circ f)(x) = h(x) \in g(b)$. Since $f(b) = f(\bigvee_M(\downarrow b \cap \mathcal{L}(M))) = \bigvee\{f(x) | x \in \downarrow b \cap \mathcal{L}(M)\}$, $f(b) \subseteq g(b)$ for all $b \in M$. Hence $f \leq g$. Interchanging the role of f and g, we have $g \leq f$. This completes the proof. □

3. Stably countably approximating frames

We note that the relation $\ll_c$ in a frame need not be closed under finite meets as 4) of Remark 2.2 shows.

In this section, we introduce a concept of stably countably approximating frames and study the relations between σ-coherent frames and stably countably approximating frames.

DEFINITION 3.1. A frame L is called:

1) *stably continuous* if L is continuous and the relation $\ll$ is closed under finite meets.

2) *stably countably approximating* if L is countably approximating and the relation $\ll_c$ is closed under finite meets.

By the definition, a countably approximating frame L is stably countably approximating iff $e \ll_c e$, and $x \ll_c a$, $y \ll_c b$ imply $x \wedge y \ll_c a \wedge b$, or equivalently, L is a Lindelöf frame, and $x \ll_c a, x \ll_c b$ imply $x \ll_c a \wedge b$.

REMARK 3.2. 1) A regular Lindelöf frame which is also a δ-frame, is stably countably approximating by 3) of Remark 1.6, because the relation $\prec$ is closed under finite meets.

2) Every σ-coherent frame L is stably countably approximating.

3) A countably approximating frame L is stably countably approximating iff $\downarrow_c: L \to \mathcal{H}L$ is a frame homomorphism, because $\downarrow_c$ preserves arbitrary joins by Remark 1.3.

LEMMA 3.3. 1) *Every retract of a countably approximating frame is again countably approximating.*

2) *Every retract of a stably countably approximating frame is again stably countably approximating.*

PROOF. 1) Let L be a countably approximating frame and M a retract of L, i.e., M is a subframe of L and there is a frame homomorphism $r : L \to M$ with $r|_M = 1_M$. If for any $b \in M$ and $x \in L$ $x \ll_c b$ in L, then $r(x) \ll_c b$ in M, because for any $S \subseteq M$ with $b \leq \bigvee_M S = \bigvee_L S$, there is $K \in Count(S)$ with $x \leq \bigvee_L K$. So $r(x) \leq r(\bigvee_L K) = \bigvee_M r(K)$. Since $r|_M = 1_M$, $r(x) \leq \bigvee_M K$.

For any $b \in M$, $b = \bigvee_L \{x | x \ll_c b \text{ in } L\}$, because L is countably approximating. So $r(b) = b = \bigvee_M \{r(x) | x \ll_c b \text{ in } L\} \leq \bigvee_M \{y | y \ll_c b \text{ in } M\} \leq b$. Thus M is countably approximating.

2) Let L be a stably countably approximating frame and M a retract of L. By 1), M is countably approximating. Since L is a Lindelöf frame, M is also a Lindelöf frame, for M is a subframe of L. Suppose $b \ll_c x$ and $b \ll_c y$ in M. Since $b \ll_c x = \bigvee_M \{r(p) | p \ll_c x \text{ in } L\}$, there is $p \in L$ such that $p \ll_c x$ in L and $b \le r(p)$. Similarly, there is $q \in L$ such that $q \ll_c y$ in L and $b \le r(q)$. Since L is stably countably approximating, $p \wedge q \ll_c x \wedge y$ in L. Since $x \wedge y \in M$, $r(p \wedge q) \ll_c x \wedge y$ in M. Thus $b \ll_c x \wedge y$ in M, because $b \le r(p) \wedge r(q) = r(p \wedge q)$. □

THEOREM 3.4. *A frame L is stably countably approximating iff L is a retract of a σ-coherent frame.*

PROOF. Suppose that L is stably countably approximating, then $\downarrow_c : L \to \mathcal{H}L$ is a frame homomorphism by Remark 3.2. Since $\bigvee : \mathcal{H}L \to L$ is a frame homomorphism and $(\bigvee \circ \downarrow_c)(a) = a = 1_L(a)$, $\bigvee : \mathcal{H}L \to L$ is a retraction. The converse is immediate from Remark 3.2 and Lemma 3.3. □

DEFINITION 3.5. A frame homomorphism $h : L \to M$ is said to be *σ-proper* if whenever $x \ll_c y$ in L, $h(x) \ll_c h(y)$ in M.

The class of all stably countably approximating frames and σ-proper homomorphisms between them forms a category which will be denoted by **SCAFrm**.

REMARK 3.6. 1) σ**CFrm** is a full subcategory of **SCAFrm**.
2) A stably countably approximating frame L is a σ-coherent frame iff $x \ll_c y$ in L implies that there is $z \in \mathcal{L}(L)$ with $x \le z \le y$.

PROPOSITION 3.7. **SCAFrm** *is coreflective in* **Frm**. *Indeed, for any $L \in$* **Frm**, *$\bigvee_L : \mathcal{H}L \to L$ is the coreflection of L.*

PROOF. Let L be a frame, then $\mathcal{H}L$ is a σ-coherent frame and hence a stably countably approximating frame. Take any frame homomorphism $h : M \to L$, where M is a stably countably approximating frame. Then $\downarrow_c : M \to \mathcal{H}M$ is a frame homomorphism with $\bigvee_M \circ \downarrow_c = 1_M$. For any $x \ll_c y$ in M, $\downarrow_c x \subseteq \downarrow x \subseteq \downarrow_c y$, so $\downarrow_c x \ll_c \downarrow_c y$ in $\mathcal{H}L$. Hence $\downarrow_c$ is a σ-proper homomorphism. Since $\mathcal{H}h : \mathcal{H}M \to \mathcal{H}L$ is σ-proper,

$\mathcal{H}h\circ\downarrow_c : M \to \mathcal{H}L$ is also σ-proper. Let $\mathcal{H}h\circ\downarrow_c = f$, then $\bigvee_L \circ f = h$. To show the uniqueness of f, suppose that $g : M \to \mathcal{H}L$ is a σ-proper homomorphism with $\bigvee_L \circ g = h$. Take any $b \in M$ and $x \in \downarrow_c b$, then $g(x) \subseteq \downarrow a \subseteq g(b)$ for some $a \in L$, because $\mathcal{H}L$ is σ-coherent. So $h(x) = (\bigvee_L \circ g)(x) \leq a \in g(b)$; hence $(\bigvee_L \circ f)(x) = h(x) \in g(b)$. Thus by the exactly same argument in Theorem 2.9, we have $f = g$. □

4. Strongly Lindelöf frames

In this section, we introduce a concept of strongly Lindelöf frames as a generalization of strongly compact frames and study the relations between σ-coherent frames and strongly Lindelöf frames.

In the following, bounded meet-semilattice homomorphisms on a frame L to *any* frame T, will be called *filters* on L. A filter $\varphi : L \to T$ is called a *prime filter (σ-prime filter, resp.)* if φ is a lattice (σ-frame, resp.) homomorphism ([3, 4, 8] for the details). A filter $\varphi : L \to T$ is said to be *convergent* if φ sends a cover of L to a cover of T ([8]) and to be *strongly convergent* if there is a frame homomorphism $h : L \to T$ with $h \leq \varphi$ ([5]).

DEFINITION 4.1. A frame L is said to be a *strongly compact* (*strongly Lindelöf, resp.*) frame if every prime (σ-prime, resp.) filter $\varphi : L \to T$ is strongly convergent.

The category of strongly Lindelöf frames and frame homomorphisms will be denoted by **SLFrm**.

REMARK 4.2. 1) Every strongly compact frame is a strongly Lindelöf frame, but the converse need not be true. The open set frame $\Omega(R_c)$, where R_c is the real line endowed with the cocountable topology, is a strongly Lindelöf frame but not a strongly compact frame.

2) Every σ-prime filter $\varphi : L \to T$ preserves countable covers; hence every σ-prime filter on a Lindelöf frame L is convergent.

3) Every Lindelöf regular frame is a strongly Lindelöf frame, but the converse need not be true. The open set frame $\Omega(R_c)$ in 1) is not regular.

PROPOSITION 4.3. *Let L be a frame. Then we have:*

1) *If the frame homomorphism $\bigvee : \mathcal{H}L \to L$ has a right inverse, then L is a Lindelöf frame*

2) *If L is a regular Lindelöf δ-frame, then the frame homomorphism $\bigvee : \mathcal{H}L \to L$ has a right inverse.*

PROOF. 1) is immediate from the fact that L is isomorphic with a subframe of the Lindelöf frame $\mathcal{H}L$. For 2), by 1) of Remark 3.2, L is a stably countably approximating frame; hence the map $\downarrow_c : L \to \mathcal{H}L$ is a frame homomorphism and $\bigvee \circ \downarrow_c = 1_L$. Thus $\downarrow_c$ is a right inverse of $\bigvee$. □

The following is due to Banaschewski and Hong [5].

DEFINITION 4.4. Let L and M be frames, then M is said to be a *lax retract* of L if there are frame homomorphisms $f : L \to M$ and $g : M \to L$ with $f \circ g \leq 1_M$.

Clearly every retract is a lax retract.

LEMMA 4.5. 1) *Every σ-coherent frame is a strongly Lindelöf frame.*

2) *A lax retract of a strongly Lindelöf frame is also a strongly Lindelöf frame.*

PROOF. 1) Let L be a σ-coherent frame, then there is a σ-frame A with $L = \mathcal{H}A$. Take any σ-prime filter $\varphi : \mathcal{H}A \to T$, then $\varphi \circ \downarrow$ is a σ-frame homomorphism, for the map $\downarrow : A \to \mathcal{H}A$ is a σ-frame homomorphism. So there is a unique frame homomorphism $h : \mathcal{H}A \to T$ with $h \circ \downarrow = \varphi \circ \downarrow$, and $h(I) = h(\bigvee\{\downarrow x | x \in I\}) = \bigvee\{(h \circ \downarrow)(x) | x \in I\} = \bigvee\{(\varphi \circ \downarrow)(x) | x \in I\} \leq \varphi(I)$ for any $I \in \mathcal{H}A$; hence $h \leq \varphi$. Thus φ is strongly convergent.

2) Let L be a strongly Lindelöf frame and M a lax retract of L. Then there are frame homomorphisms $f : L \to M$ and $g : M \to L$ with $f \circ g \leq 1_M$. Take any σ-prime filter $\varphi : M \to T$, then $\varphi \circ f : L \to T$ is a σ-prime filter. Since L is a strongly Lindelöf frame, there is a frame homomorphism $h : L \to T$, with $h \leq \varphi \circ f$. So $h \circ g : M \to T$ is a frame homomorphism and $h \circ g \leq \varphi$. Thus M is a strongly Lindelöf frame. □

THEOREM 4.6. *For a frame L, the following are equivalent:*
1) *L is a strongly Lindelöf frame.*
2) *L is a lax retract of $\mathcal{H}L$.*
3) *L is a lax retract of a σ-coherent frame.*

PROOF. 1) $\Rightarrow$ 2) Since the map $\downarrow : L \to \mathcal{H}L$ is a σ-prime filter, there is a frame homomorphism $h : L \to \mathcal{H}L$ with $h \leq \downarrow$ and $\bigvee \circ h \leq \bigvee \circ \downarrow = 1_L$.

2) $\Rightarrow$ 3) It follows from the fact that $\mathcal{H}L$ is σ-coherent.

3) $\Rightarrow$ 1) It follows from Lemma 4.5. □

COROLLARY 4.7. 1) *Every stably countably approximating frame is a strongly Lindelöf frame.*
2) *Every strongly Lindelöf frame is a Lindelöf frame.*
3) **SLFrm** *is coproductive.*

Collecting the previous results, we have:

PROPOSITION 4.8. *Suppose that L is a regular frame which is also a δ-frame, then the following are equivalent:*
1) *The frame homomorphism $\bigvee : \mathcal{H}L \to L$ has a right inverse.*
2) *L is a Lindelöf frame.*
3) *Every σ-prime filter on L is convergent.*
4) *L is a strongly Lindelöf frame.*
5) *L is a stably countably approximating frame.*
6) *The map $\downarrow_c : L \to \mathcal{H}L$ is a frame homomorphism.*

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