Commun. Korean Math. Soc. **17** (2002), No. 2, pp. 309–319

ON STATIONARITY OF NONLINEAR AR PROCESSES WITH NONLINEAR ARCH ERRORS

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ABSTRACT. We consider the nonlinear autoregressive model with nonlinear ARCH errors, and find sufficient conditions for the existence of a strictly stationary process. New results are obtained, and known results are shown to emerge as special cases.

1. Introduction

The typical p-th order nonlinear AR-model is given by

$$y_t = \phi(y_{t-1}, y_{t-2}, \cdots, y_{t-p}) + e_t,$$

where $\{e_t\}$ are independent and identically distributed (i.i.d.). While this model has constant variance, many types of economics and financial data possess the property that their conditional variances depend on the past information. To explain the time series with conditional heteroscedastic variances, ARCH (autoregresive conditional heteroscedasticity) process was introduced by Engle (1982). ARCH model has been proved useful in financial applications and studied by many authors, for instance, Bollerslev *et al* (1992), Guégan and Diebolt (1994), Lu (1996), Li and Li (1996), An *et al* (1997), Wong and Li (1996), Ling (1999), Borkovec (2000) etc. As an extension of ARCH process, a class of autoregressive model with ARCH errors proposed first by Weiss (1984), and Tong (1990) suggested a threshold model with an ARCH error, which is entitled the SETAR-ARCH model. However, some data show that linearity of the squared past disturbances in ARCH models is not adequate and the conditional variance is asymmetric conditional on previous returns (see, e.g., Rabemanjara and Zkoian (1992), Liu *et al* (1997)). In

Received February 19, 2001.

2000 Mathematics Subject Classification: 62M10, 60J10.

Key words and phrases: stationarity, nonlinear AR, nonlinear ARCH, double threshold ARCH.

This work was supported by grant R04-2000-00006 from the Basic Research Program of the Korea Science & Engineering Foundation.

order to accommodate the asymmetric conditionality of conditional variance, double threshold ARCH model was introduced in Li and Li (1996).

In this paper, we consider the nonlinear AR models with nonlinear ARCH innovations, which is a natural extension of double threshold ARCH model. This model combines the advantages of nonlinear AR models which target on the conditional means given the past and nonlinear ARCH model which concentrate on the conditional variances given the past. That is, this model is capable of modeling time series with changing conditional mean and conditional variance via nonlinear manners.

Let $\{y_t\}$ be the nonlinear autoregressive time series with nonlinear ARCH errors given by

$$y_t = \phi(y_{t-1}, y_{t-2}, \cdots, y_{t-p}) + \varepsilon_t, \tag{1.1}$$

$$\varepsilon_t = h_t^{1/2} e_t, \tag{1.2}$$

$$h_t = \alpha_0 + \psi(\varepsilon_{t-1}, \varepsilon_{t-2}, \cdots, \varepsilon_{t-q}), \tag{1.3}$$

where ϕ and ψ are real-valued measurable functions defined on R^p and R^q, respectively, $p \geq 0$, $q \geq 0$, $\alpha_0 > 0$, and $\{e_t\}$ is a sequence of i.i.d random variables with mean zero and unit variance.

The process obtained by (1.1)-(1.3) includes various well known nonlinear models such as nonlinear AR models with constant variance, TAR, ($\beta-$) ARCH, double threshold ARCH models etc. Our aim is to derive sufficient conditions for the stationarity and finiteness of the moments of a model given above. We study the stationarity of y_t by applying the Tweedie's result (1988) to associated Markov process and derive the desired results from that for Markov chain.

For terminologies and relevant results in Markov chain theory, we refer to Meyn and Tweedie (1993).

Section 2 presents main results and their proofs are given in Section 3.

2. Main results

Consider the following nonlinear autoregressive model with nonlinear ARCH errors;

$$y_t = \phi(y_{t-1}, \cdots, y_{t-p}) + \varepsilon_t, \tag{2.4}$$

$$\varepsilon_t = \sqrt{h_t} \cdot e_t, \tag{2.5}$$

$$h_t = \alpha_0 + \psi(\varepsilon_{t-1}, \cdots, \varepsilon_{t-q}), \tag{2.6}$$

where ϕ and ψ are real-valued measurable functions on R^p and R^q respectively, $\psi \geq 0$, $\alpha_0 > 0$, $\{e_t\}$ is a sequence of i.i.d. random variables with mean 0 and variance $\sigma^2 = 1$, and $\{y_0, y_{-1}, \cdots, y_{-p+1}, \varepsilon_0, \cdots, \varepsilon_{-q+1}\}$ are arbitrarily specified real-valued random variables independent of $\{e_t; t \geq 1\}$.

Denote $\mathbf{X}_t = (y_t, \cdots, y_{t-p+1}, \varepsilon_t, \cdots, \varepsilon_{t-q+1})'$, then $\{\mathbf{X}_t; t \geq 0\}$ is a Markov chain with the n-step transition probability function, say $P^{(n)}(\mathbf{x}, d\mathbf{y})$.

We make the following assumptions:

(A-1) There exist constants λ, $0 < \lambda < 1$, and c_1 such that for $\mathbf{u} = (u_1, \cdots, u_p)' \in R^p$,

$$|\phi(\mathbf{u})| \leq \lambda \max\{|u_1|, \cdots, |u_p|\} + c_1. \tag{2.7}$$

(A-2) There exist constants $\gamma_i \geq 0$, $i = 1, \cdots, q$ with $\sum_{i=1}^q \gamma_i < 1$ and c_2 such that for $\mathbf{z} = (z_1, \cdots, z_q)' \in R^q$,

$$\sqrt{\psi(\mathbf{z})} \leq \sum_{i=1}^{q} \gamma_i |z_i| + c_2. \tag{2.8}$$

(A-3) There exist constants $\gamma_i \geq 0$, $i = 1, \cdots, q$ with $\sum_{i=1}^q \gamma_i < 1$ and c_2 such that for $\mathbf{z} = (z_1, \cdots, z_q)' \in R^q$,

$$\psi(\mathbf{z}) \leq \sum_{i=1}^{q} \gamma_i z_i^2 + c_2. \tag{2.9}$$

(A-4) $\{\mathbf{X}_t\}$ is a Feller chain, i.e. for each bounded continuous function g on R^{p+q}, $E[g(\mathbf{X}_{t+1}) \mid \mathbf{X}_t = \mathbf{x}]$ is continuous in $\mathbf{x}$.

Followings are our main results.

THEOREM 1. *Under the assumptions* (A-1), (A-2), *and* (A-4), *there exists a stationary solution* (y_t, ε_t) *satisfying* (2.4)-(2.6), *and* $E_{\pi_1}(|y_t|)$ *and* $E_{\pi_2}(|\varepsilon_t|)$ *are finite where* π_1 *and* π_2 *are stationary distributions of* $\{y_t\}$ *and* $\{\varepsilon_t\}$, *respectively.*

THEOREM 2. *Assume* (A-1), (A-3), *and* (A-4). *Then a stationary solution of* (2.4)-(2.6) *exists and* $E_{\pi_1}(|y_t|) < \infty$ *and* $E_{\pi_2}(\varepsilon_t^2) < \infty$.

THEOREM 3. *In addition to the assumptions* (A-1), (A-2) (*or* (A-3)) *and* (A-4), *suppose that the Markov chain* $\{\mathbf{X}_t\}$ *is aperiodic and* φ-*irreducible. Then* $\{\mathbf{X}_t\}$ *is geometrically ergodic.*

We now give some examples. In each case, corresponding Markov chain is assumed to be a Feller chain, if necessary.

EXAMPLE 1. The classical nonlinear autoregressive model with order p is given by

$$y_t = \phi(y_{t-1}, \cdots, y_{t-p}) + e_t \tag{2.10}$$

with nonlinear measurable function $\phi : R^p \to R$. Above model has been studied by many authors, for example, Bhattacharya and Lee (1995), An *et al* (1997), Tong (1990), Tjøstheim (1990), Lee (2000) etc. Taking $q = 0$ in (2.6) yield (2.10). The existence of strictly stationary solution of (2.10) is ensured provided that for some $\lambda < 1$,

$$|\phi(\mathbf{y})| \leq \lambda \max\{|y_1|, \cdots, |y_p|\}, \quad \mathbf{y} = (y_1, \cdots, y_p). \tag{2.11}$$

Note that if $|\phi(\mathbf{y})| \leq \sum_{i=1}^{p} \lambda_i |y_i|$ and $\sum_{i=1}^{p} \lambda_i < 1$, then (2.11) holds with $\lambda = \sum_{i=1}^{p} \lambda_i$.

EXAMPLE 2 (Threshold β-ARCH). Suppose that $\{\varepsilon_t\}$ is generated by

$$\varepsilon_t = \sqrt{h_t} \cdot e_t, \tag{2.12}$$

$$h_t = \sum_{j=1}^{l} (\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_i^{(j)} \varepsilon_{t-i}^{2\beta}) I_{(\varepsilon_{t-d} \in [b_{j-1}, b_j))}, \tag{2.13}$$

where $-\infty = b_0 < b_1 < \cdots < b_l = \infty$. Squaring both sides of the equation (2.12) and taking logarithm, we obtain that

$$y_t = \log \Big[\sum_{j=1}^{l} \Big(\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_i^{(j)} e^{\beta y_{t-i}} \Big) I_{(\varepsilon_{t-d} \in [b_{j-1}, b_j))} \Big] + \log e_t^2, \tag{2.14}$$

where $y_t = \log \varepsilon_t^2$.

Equation(2.14) is of the form of (2.10) and

$$\begin{aligned} &\Big| \log \Big[\sum_{j=1}^{l} \Big(\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_i^{(j)} e^{\beta y_{t-i}} \Big) I_{(\varepsilon_{t-d} \in [b_{j-1}, b_j))} \Big] \Big| \\ \leq &\Big| \log \Big[\max_j (\alpha_0^{(j)} + \sum_{i=1}^{p} \alpha_i^{(j)} e^{\beta y_{t-i}} \Big] \Big| \\ \leq &\ \log \max\{e^{\beta y_{t-1}}, \cdots, e^{\beta y_{t-q}}\} + \text{constant} \\ \leq &\ \beta \max\{|y_{t-1}|, \cdots, |y_{t-q}|\} + \text{constant}. \end{aligned}$$

Hence if $0 < \beta < 1$ in (2.13), a strictly stationary solution of (2.12) and (2.13) exists.

EXAMPLE 3 (Double threshold ARCH). The process $\{y_t\}$ is said to be a double threshold ARCH model (DTARCH) if it is defined by

$$y_t = \phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i} + \varepsilon_t, \quad a_{j-1} \le y_{t-b} < a_j, \tag{2.15}$$

$$\varepsilon_t = \sqrt{h_t} \cdot e_t, \tag{2.16}$$

$$h_t = \psi_0^{(k)} + \sum_{i=1}^{q} \psi_i^{(k)} \varepsilon_{t-i}^2, \quad b_{k-1} \le \varepsilon_{t-d} < b_k, \tag{2.17}$$

where $j = 1, 2, \cdots, l_1$, $k = 1, 2, \cdots, l_2$, $-\infty = a_0 < a_1 < \cdots < a_{l_1} = \infty$, $-\infty = b_0 < b_1 < \cdots < b_{l_2} = \infty$, $\phi_0^{(j)}, \cdots, \phi_p^{(j)}, \psi_0^{(k)}, \cdots, \psi_q^{(k)}$ are constants with $\psi_0^{(k)} > 0$, $\psi_i^{(k)} \ge 0$, $1 \le i \le q$. This model is studied in Li and Li (1996), Liu *et al* (1997), Ling (1999). Clearly TAR, ARCH, TARCH are special cases of DTARCH model. (2.15) can be rewrite as

$$y_t = \sum_{j=1}^{l_1} (\phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i}) I_{(y_{t-d} \in [a_{j-1}, a_j))} + \varepsilon_t,$$

and

$$\begin{aligned} & \Big| \sum_{j=1}^{l_1} \Big(\phi_0^{(j)} + \sum_{i=1}^{p} \phi_i^{(j)} y_{t-i} \Big) I_{(y_{t-d} \in [a_{j-1}, a_j)} \Big| \\ \le\ & \max_{1 \le j \le l_1} \{ |\phi_0^{(j)}| \} + \max_{1 \le j \le l_1} \sum_{i=1}^{l_1} \Big(\sum_{i=1}^{p} |\phi_i^{(j)}| |y_{t-i}| \Big) I_{(y_{t-d} \in [a_{j-1}, a_j)} \\ \le\ & \max_{1 \le j \le l_1} \{ |\phi_0^{(j)}| \} + \Big(\max_{1 \le j \le l_1} \sum_{i=1}^{p} |\phi_i^{(j)}| \Big) \cdot (\max\{ |y_{t-1}|, \cdots, |y_{t-p}| \}). \end{aligned}$$

Therefore if $\max_{1 \le j \le l_1} \sum_{i=1}^{p} |\phi_i^{(j)}| < 1$, assumption *(A-1)* holds.

On the other hand,

$$h_t \le \sum_{i=1}^{q} \max_k \{ \psi_i^{(k)} \} \varepsilon_{t-i}^2 + \text{constant}.$$

Hence if $\max_j \sum_{i=1}^{p} |\phi_i^{(j)}| < 1$ and $\sum_{i=1}^{q} \max_k \{ \psi_i^{(k)} \} < 1$, then the assumptions *(A-1)* and *(A-3)* hold and hence, by Theorem 2, strictly stationary solution satisfying (2.15)-(2.17) exists. Compare this result with that of Ling (1999).

3. Proofs

To prove Theorem1, we need the following theorem due to Tweedie.

THEOREM 4. (Tweedie (1988)) *Suppose* $\{X_t\}$ *is a Feller chain with transition probability function* $P(x, dy)$.

(a) *If there exists, for some compact set* A, *a nonnegative function* g *and an* $\epsilon > 0$ *such that*

$$\int_{A^c} P(x, dy)g(y) \leq g(x) - \epsilon, \quad x \in A^c, \tag{3.18}$$

then there exists a σ*-finite invariant measure* μ *for* P *with* $0 < \mu(A) < \infty$.

(b) *Further, if*

$$\int_A \mu(dx)\Big[\int_{A^c} P(x, dy)g(y)\Big] < \infty, \tag{3.19}$$

then μ *is finite.*

(c) *Further, if*

$$\int_{A^c} P(x, dy)g(y) \leq g(x) - f(x), \quad x \in A^c, \tag{3.20}$$

then μ *admits a finite* f*-moment.*

The main part of proof of Theorem 1-Theorem 3 is to construct a proper test function g under which (3.18)-(3.20) hold.

PROOF OF THEOREM 1. Define a test function $g : R^{p+q} \to R$ by for any $\mathbf{x} = (u_1, \cdots, u_p, z_1, \cdots, z_q)'$ in R^{p+q},

$$g(\mathbf{x}) = 1 + \max\{\alpha_1|u_1|, \cdots, \alpha_p|u_p|\} + \sum_{i=1}^{q} \beta_i|z_i|, \tag{3.21}$$

where $\alpha_i > 0, \beta_j > 0$, $i = 1, \cdots, p$, $j = 1, \cdots, q$ are to be defined later.

For any $\mathbf{x} = (u_1, \cdots, u_p, z_1, \cdots, z_q)'$, we have that

$$\begin{aligned}
& E[g(\mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] \\
= \; & E[g(\phi(u_1, \cdots, u_p) + \varepsilon_t, \\
& \quad u_1, \cdots, u_{p-1}, \varepsilon_t, z_1, \cdots, z_{q-1})|\mathbf{X}_{t-1} = \mathbf{x}] \\
\leq \; & 1 + E[\max\{\alpha_1|\phi(u_1, \cdots, u_p) + \varepsilon_t|, \\
& \quad \alpha_2|u_1|, \cdots, \alpha_p|u_{p-1}|\}|\mathbf{X}_{t-1} = \mathbf{x}] \\
& + E[\beta_1|\varepsilon_t| + \beta_2|z_1| + \cdots + \beta_q|z_{q-1}||\mathbf{X}_{t-1} = \mathbf{x}] \\
\leq \; & 1 + \max\{\alpha_1|\phi(u_1, \cdots, u_p)|, \alpha_2|u_1|, \cdots, \alpha_p|u_{p-1}|\} \\
& + \alpha_1 E[|\varepsilon_t||\mathbf{X}_{t-1} = \mathbf{x}] + \beta_1 E[|\varepsilon_t||\mathbf{X}_{t-1} = \mathbf{x}] \\
& + \beta_2|z_1| + \cdots + \beta_q|z_{q-1}| \\
\leq \; & 1 + \max\{\lambda\alpha_1 \max\{|u_1|, \cdots, |u_p|\}, \alpha_2|u_1|, \cdots, \alpha_p|u_{p-1}|\} \\
& + \alpha_1 c_1 + \alpha_1\sqrt{\psi}E|e_t| \\
& + \beta_1\sqrt{\psi}E|e_t| + \beta_2|z_1| + \cdots + \beta_q|z_{q-1}| \\
& + (\alpha_1 + \beta_1)\sqrt{\alpha_0}E|e_t| \\
\leq \; & I + II + K, \qquad (3.22)
\end{aligned}$$

where

$$I = \max\{\lambda\alpha_1 \max\{|u_1|, \cdots, |u_p|\}, \alpha_2|u_1|, \cdots, \alpha_p|u_{p-1}|\}, \tag{3.23}$$

$$II = (\alpha_1 + \beta_1)E|e_t|\Big(\sum_{i=1}^{q}\gamma_i|z_i|\Big) + \beta_2|z_1| + \cdots + \beta_q|z_{q-1}|, \tag{3.24}$$

$$K = 1 + \alpha_1 c_1 + (\alpha_1 + \beta_1)E|e_t|(\sqrt{\alpha_0} + c_2) < \infty. \tag{3.25}$$

Note that

$$\lambda < \lambda^{\frac{k}{p}} < \lambda^{\frac{1}{p}} < 1, \quad 1 < k < p. \tag{3.26}$$

Now choose $\alpha_1 > 0$ arbitrarily and define

$$\alpha_k = \lambda^{\frac{1}{p}}\alpha_{k-1}, \quad k = 2, 3, \cdots, p. \tag{3.27}$$

Then

$$\alpha_1 > \alpha_2 > \cdots > \alpha_p = \lambda^{\frac{p-1}{p}}\alpha_1 > \lambda\alpha_1 > 0, \tag{3.28}$$

and hence

$$\begin{aligned} I &= \max\{\lambda\alpha_1 \max\{|u_1|,\cdots,|u_p|\},\alpha_2|u_1|,\cdots,\alpha_p|u_{p-1}|\} \\ &\leq \max\{\lambda\alpha_1 \max\{|u_1|,\cdots,|u_p|\}, \\ &\qquad \lambda^{\frac{1}{p}}\alpha_1|u_1|,\lambda^{\frac{1}{p}}\alpha_2|u_2|,\cdots,\lambda^{\frac{1}{p}}\alpha_{p-1}|u_{p-1}|\} \\ &\leq \lambda^{\frac{1}{p}}\max\{\alpha_p \max\{|u_1|,\cdots,|u_p|\},\alpha_1|u_1|,\cdots,\alpha_{p-1}|u_{p-1}|\} \\ &\leq \lambda^{\frac{1}{p}}\max\{\alpha_p|u_p|,\alpha_1|u_1|,\cdots,\alpha_{p-1}|u_{p-1}|\}. \end{aligned} \tag{3.29}$$

The last inequality in (3.29) follows from the fact that $\alpha_p \leq \alpha_k$ for $1 \leq k \leq p$.

On the other hand, if we take $b_1 = (\alpha_1 + \beta_1)E|e_t|$ for simplicity of the notation,

$$\begin{aligned} II &= b_1\sum_{i=1}^{q}\gamma_i|z_i| + \beta_2|z_1| + \cdots + \beta_q|z_{q-1}| \\ &= (b_1\gamma_1+\beta_2)|z_1| + (b_1\gamma_2+\beta_3)|z_2| + \cdots + (b_1\gamma_{q-1}+\beta_q)|z_{q-1}| \\ &\quad + b_1\gamma_q|z_q| \end{aligned} \tag{3.30}$$

Now choose $\delta > 0$ so that $\sum_{i=1}^{q}\gamma_i + \delta = 1$, and then define

$$\beta_{i+1} = b_1(1-\gamma_1-\cdots-\gamma_i-\frac{i\delta}{q}), \quad i=1,2,\cdots,q-1. \tag{3.31}$$

Then

$$\beta_2 = b_1(1-\gamma_1-\frac{\delta}{q}), \tag{3.32}$$

$$\beta_{i+1} = \beta_i - b_1\gamma_i - b_1\frac{\delta}{q}, \quad i=2,3,\cdots,q-1, \tag{3.33}$$

and

$$b_1 > \beta_i > \beta_{i+1}, \quad i=2,3,\cdots,q-1. \tag{3.34}$$

Since $E|e_t| \leq 1$, we may choose $r > 0$, $\beta_1 > 0$ so that

$$(1-\frac{\delta}{p})E|e_t| < r < 1 \tag{3.35}$$

and

$$\beta_1 \geq \frac{\alpha_1(1-\frac{\delta}{q})E|e_t|}{r-(1-\frac{\delta}{q})E|e_t|}. \tag{3.36}$$

From equations (3.31) and (3.34), $\beta_q = b_1(\gamma_q + \frac{\delta}{q})$ and

$$b_1\gamma_q = \beta_q - b_1\frac{\delta}{q} \leq \beta_q(1 - \frac{\delta}{q}). \tag{3.37}$$

From (3.36), we have $\beta_2 + b_1\gamma_1 \leq r\beta_1$ for some $r, 0 < r < 1$, and using (3.33) and (3.34), we obtain that $\beta_{i+1} + b_1\gamma_i = \beta_i - b_1\frac{\delta}{q} \leq \beta_i(1 - \frac{\delta}{q})$, and therefore

$$II \leq \max\left\{\left(1 - \frac{\delta}{q}\right), r\right\} \sum_{i=1}^{q} \beta_i|z_i|. \tag{3.38}$$

Combining (3.5), (3.12), and (3.21), we have

$$\begin{aligned}
& E[g(\mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] \\
\leq\ & \lambda^{\frac{1}{p}} \max\{\alpha_1|u_1|, \cdots, \alpha_p|u_p|\} \\
& + \max\left\{\left(1 - \frac{\delta}{q}\right), r\right\} \sum_{i=1}^{q} \beta_i|z_i| + K \\
\leq\ & \rho\Big(1 + \max\{\alpha_1|u_1|, \cdots, \alpha_p|u_p|\} + \sum_{i=1}^{q} \beta_i|z_i|\Big) + K - \rho \\
=\ & \rho g(\mathbf{x}) + K - \rho \qquad (3.39)
\end{aligned}$$

where $0 < \rho = \max\{\lambda^{\frac{1}{p}}, (1 - \frac{\delta}{q}), r\} < 1$.

Let $\|\cdot\|$ be denote any norm on R^{p+q}. Then, by equivalence of norms on R^{p+q}, $g(\mathbf{x})$ increases as $\|\mathbf{x}\|$ increases, and hence we can choose $\epsilon > 0$, ρ', $0 < \rho < \rho' < 1$ and $M > 0$ so large that

$$E[g(\mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] \leq \rho' g(\mathbf{x}) - \epsilon, \quad \|\mathbf{x}\| > M. \tag{3.40}$$

Clearly,

$$\sup_{\|\mathbf{x}\| \leq M} E[g(\mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] < \infty. \tag{3.41}$$

(3.40) and (3.41) imply that (3.18)-(3.20) in Theorem 4 hold with a compact set $A = \{\mathbf{x} : \|\mathbf{x}\| \leq M\}$, from which the stationarity of $\{y_t\}$ is obtained.

Let π be its invariant probability measure. Then by part (c) of Theorem 4, we have

$$\int g(\mathbf{x})\pi(dx) < \infty. \tag{3.42}$$

Take $\pi_1(B) = \pi(\mathbf{B})$, where $\mathbf{B} = B \times R^{p+q-1}$, and $\pi_2(C) = \pi(\mathbf{C})$, where $\mathbf{C} = R^p \times C \times R^{q-1}$. Here $B, C \in \mathcal{B}(R)$, and $\mathcal{B}(R)$ denote the Borel sigma field on R. Then by (3.42), $E_{\pi_1}|y_t| < \infty$ and $E_{\pi_2}(|\varepsilon_t|) < \infty$. This completes the proof.

PROOF OF THEOREM 2. To prove Theorem 2, we define $g(\mathbf{x})$ which is slightly different from that in the proof of Theorem 1.

For $\mathbf{x} = (u_1, \cdots, u_p, z_1, \cdots, z_q)$, let

$$g(\mathbf{x}) = 1 + \max\{\alpha_1|u_1|, \cdots, \alpha_p|u_p|\} + \sum_{i=1}^{q} \beta_i z_i^2.$$

Then $E[g(\mathbf{X}_t)|\mathbf{X}_{t-1} = \mathbf{x}] \leq I + II + K$, where I is exactly the same as that of (3.23), and

$$(3.43)\quad II = (\alpha_1 + \beta_1)E(e_t^2)\Big(\sum_{i=1}^{q} \gamma_i z_i^2\Big) + \beta_2 z_1^2 + \cdots + \beta_q z_{q-1}^2$$

and

$$(3.44)\qquad K = 1 + \alpha_1(c_1 + 1) + E(e_t^2)(\alpha_1 + \beta_1)(\alpha_0 + c_2).$$

(3.43) is due to simple inequality $|x| \leq x^2 + 1$. The remaining part of the proof follows the same line as that of Theorem 1, we omit it.

PROOF OF THEOREM 3. Conclusion follows immediately from Theorem 1 and Theorem A1.5 (p. 457) in Tong (1990).

References

[1] H. An, M. Chen, and F. Huang, *The geometric ergodicity and existence of moments for a class of nonlinear time series model*, Stat. and Prob. Letters **31** (1997), 213–224.

[2] R. N. Bhattacharya and C. Lee, *On geometric ergodicity of nonlinear autoregressive models*, Stat. and Prob. Letters **22** (1995), 311–315.

[3] T. Bollerslev, R. T. Chou, and K. F. Kroner, *Arch modeling in finance*, J. Econometrics **52** (1992), 5–59.

[4] M. Borkovec, *Extremal behavior of the autoregressive process with ARCH(1) errors*, Stoch. Proc. and their Appl. **85** (2000), 189–207.

[5] R. F. Engle, *Autoregressive conditional heteroscedasticity with estimates of the variance of the United Kingdom inflation*, Econometrica **50** (1982), 987–1007.

[6] D. Guégan and J. Diebolt, *Probabilistic properties of the β-ARCH model*, Statistica Sinica **4** (1994), 71–87.

[7] O. Lee, *On probabilistic properties of nonlinear ARMA(p,q) models*, Stat. and Prob. Letters **46** (2000), 121–131.

[8] C. W. Li and W. K. Li, *On a double threshold autoregressive conditional heteroskedastic time series model*, J. Appl. Econometrics **11** (1996), 253–274.

[9] S. Ling, *On the probabilistic properties of a double threshold ARMA conditional heteroskedastic model*, J. Appl. Prob. **36** (1999), 688–705.

[10] J. Liu, W. Li, and C. W. Li, *On a threshold autoregression with conditional heteroscedastic variances*, J. Stat. Plann. and Inference **62** (1997), 279–300.
[11] Z. Lu, *A note on geometric ergodicity of autoregressive conditional heteroscedasticity(ARCH) model*, Stat. and Prob. Letters **30** (1996), 305–311.
[12] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer, London, 1993.
[13] R. Rabemanjara and J. M. Zakoian, *Threshold ARCH model and asymmetries in volatility*, J. Appl. Econometrics **8** (1993), 31–49.
[14] D. Tjøstheim, *Nonlinear time series and Markov chains*, Adv. Appl. Prob. **22** (1990), 587–611.
[15] H. Tong, *Nonlinear Time Series: a Dynamical System approach*, Oxford University Press, Oxford, 1990.
[16] R. L. Tweedie, *Invariant measure for Markov chains with no irreducibility assumptions*, J. Appl. Prob. **25A** (1988), 275–285.
[17] H. Wong and W. K. Li, *On a multivariate conditional heteroscedastic model*, Biometrica **84** (1997), 111–123.
[18] A. Weiss, *ARMA models with ARCH errors*, J. Time Series **5** (1984), 129–143.

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