Commun. Korean Math. Soc. **17** (2002), No. 2, pp. 363–370

AN ASYMPTOTIC FORMULA FOR $\exp(\frac{x}{1-x})$

JUNHO SONG AND CHANGWOO LEE

ABSTRACT. We show that $G(x) = e^{x/(1-x)} - 1$ is the exponential generating function for the labeled digraphs whose weak components are transitive tournaments and derive both a recursive formula and an explicit formula for the number of them on n vertices. Moreover, we investigate the asymptotic behavior for the coefficients of $G(x)$ using Hayman's method.

1. Introduction

When we know the exponential generating function $G(x)$ for a class of graphs, we can easily derive the exponential generating function $C(x)$ for the corresponding connected graphs using the relation

$$1 + G(x) = e^{C(x)}.$$

This is a well-known technique in graph theory [1].

Let us try in the reverse direction, from $C(x)$ to $G(x)$. The most common power series

$$\frac{1}{1-x} - 1 = x + x^2 + x^3 + x^4 + \cdots$$

is, in some sense, meaningless as an ordinary generating function. However, we can make it meaningful as an exponential generating function. This means that the series

$$\begin{aligned} C(x) &= \frac{1}{1-x} - 1 = \frac{x}{1-x} \\ &= x + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + 4!\frac{x^4}{4!} + \cdots \end{aligned}$$

Received November 17, 2001.

2000 Mathematics Subject Classification: 05C30, 05C20.

Key words and phrases: transitive tournament, exponential generating function, recursive formula, asymptotics, admissible, Hayman's method.

This research of the first author was supported by the 2001 Research Fund of the University of Seoul.

could be regarded as the exponential generating function for labeled transitive tournaments in graph theoretical sense [3]. From this fact, we know that the exponential generating function

$$G(x) = e^{C(x)} - 1 = e^{x/(1-x)} - 1$$
$$= x + 3\frac{x^2}{2!} + 13\frac{x^3}{3!} + 73\frac{x^4}{4!} + 501\frac{x^5}{5!} + 4051\frac{x^6}{6!} + \cdots$$

counts labeled digraphs whose weak components are transitive tournaments.

In this paper, we show that a recursive formula for the coefficient a_n of the term $x^n/n!$ in $G(x)$ is

$$a_n = (2n-1)a_{n-1} - (n-1)(n-2)a_{n-2} \quad \text{for} \quad n \geq 3$$

with the initial condition

$$a_1 = 1 \quad \text{and} \quad a_2 = 3,$$

in two different ways and that an explicit formula for a_n is

$$a_n = \sum_{k=0}^{n-1} \binom{n}{k} \langle n-1 \rangle_k,$$

where $\langle n-1 \rangle_k$ means a falling factorial. Moreover, we show that an asymptotics for a_n is

$$a_n \sim \frac{2^n n^{2n} \exp(-n + \frac{1}{2}\sqrt{4n+1} - \frac{1}{2})}{(2n+1-\sqrt{4n+1})^n (4n+1)^{1/4}}.$$

2. Formulas for $\mathbf{a_n}$

In this section we derive a recursive formula for a_n in two different ways and next an explicit formula for a_n.

First, differentiating $y = e^{x/(1-x)} - 1$ and rearranging it, we have a differential equation

$$(1-x)^2 y' = y + 1, \qquad y(0) = 0.$$

Solving this equation for $y = \sum_{n \geq 1} (a_n/n!) x^n$, we get a recursive formula

$$a_n = (2n-1)a_{n-1} - (n-1)(n-2)a_{n-2} \quad \text{for} \quad n \geq 3$$

with the initial condition

$$a_1 = 1 \quad \text{and} \quad a_2 = 3,$$

as is evidenced by enumerating labeled digraphs under consideration for small n.

Another method to derive this recursive formula is as follows. Let

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} = \exp\left(\frac{x}{1-x}\right).$$

Taking the logarithm of both sides of this equation, we have

$$\log\left(\sum_{n\geq 0} a_n \frac{x^n}{n!}\right) = \frac{x}{1-x}.$$

Differentiating both sides and multiplying through by x, we have

$$\frac{\sum_{n\geq 0} n a_n (x^n/n!)}{\sum_{n\geq 0} a_n (x^n/n!)} = \frac{x}{(1-x)^2}.$$

Clear this equation of fractions. For each n, find the coefficients of $x^n/n!$ on both sides of the equation and equate them. Ignoring $a_0 = 1$ from the fact that we do not consider digraphs on zero vertices, we get the same recursive formula.

To find a_n itself, we regard the function $e^{z/(1-z)}$ as a complex function. Let

$$\exp\left(\frac{z}{1-z}\right) = \sum_{n\geq 0} \frac{a_n}{n!} z^n.$$

Then, by Cauchy's formula, we have

$$\begin{aligned}
\frac{a_n}{n!} &= \frac{1}{2\pi i}\int_C \frac{e^{z/(1-z)}}{z^{n+1}}dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{e^w}{w^{n+1}}(1+w)^{n-1}dw \\
&= \frac{1}{2\pi i} \times 2\pi i \times \operatorname{Res}\left[\frac{e^w}{w^{n+1}}(1+w)^{n-1}; 0\right] \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{(n-k)!},
\end{aligned}$$

where $z/(1-z) = w$, $C:\ re^{i\theta}$ with $0 < r < 1$ and $0 \le \theta \le 2\pi$, and Γ is the circle corresponding to C. Therefore, we have

$$\begin{aligned} a_n &= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{n!}{(n-k)!} \\ &= \sum_{k=0}^{n-1} \binom{n}{k} \langle n-1 \rangle_k \quad \text{for} \quad n \ge 1. \end{aligned}$$

THEOREM 1. *Let a_n be the number of labeled digraphs of order n whose weak components are transitive tournaments. Then*

(1) $a_n = (2n-1)a_{n-1} - (n-1)(n-2)a_{n-2}$ *for $n \ge 3$ with the initial condition $a_1 = 1$ and $a_2 = 3$.*
(2) $a_n = \sum_{k=0}^{n-1} \binom{n}{k} \langle n-1 \rangle_k$.

3. Asymptotics for $\mathbf{a_n}$

In this section we want to investigate the asymptotic behavior for the coefficients of $1 + G(z) = \exp(\frac{z}{1-z})$, that is, to find a simple function of n that affords a good approximation to the values of our coefficients when n is large.

To do this, we let $f(z) = \exp(\frac{z}{1-z})$ and apply Hayman's method for this $f(z)$. Now, we introduce the admissibility for Hayman's method and the method itself.

DEFINITION 2. [2, 4] Let $f(z) = \sum_{n \ge 0} a_n z^n$ be regular in $|z| < R$, where $0 < R \le \infty$. Next define two auxiliary functions

$$a(r) = r\frac{f'(r)}{f(r)}$$

and

$$b(r) = ra'(r).$$

We say that $f(z)$ is *admissible* in $|z| < R$ if

(a) there exists an $R_0 < R$ such that $f(r) > 0$ for $R_0 < r < R$,
(b) there exists a function $\delta(r)$ defined for $R_0 < r < R$ such that $0 < \delta(r) < \pi$ for those r, and such that uniformly for $|\theta| \le \delta(r)$, we have

$$f(re^{i\theta}) \sim f(r)e^{i\theta a(r) - \frac{1}{2}\theta^2 b(r)} \quad \text{as} \quad r \to R,$$

(c) uniformly for $\delta(r) \le |\theta| \le \pi$, we have

$$f(re^{i\theta}) = \frac{o(f(r))}{\sqrt{b(r)}} \quad \text{as} \quad r \to R,$$

(d) we have $b(r) \to \infty$ as $r \to R$.

LEMMA 3. ([2]) *Suppose that* $f(z) = \sum_{n\ge 0} a_n z^n$ *is regular in* $|z| < 1$, *positive in some range* $R_0 < z < 1$, *and that there exist constants* $0 < \alpha$, $0 < \beta < 1$, *and a positive function* $C(r)$, $0 < r < 1$, *satisfying*

$$(1 - r)\frac{C'(r)}{C(r)} \to 0 \quad \text{as} \quad r \to 1, \tag{3.1}$$

and such that

$$\log f(z) \sim C(|z|)(1 - z)^{-\alpha} \quad \textit{as} \quad z \to 1, \tag{3.2}$$

uniformly for $|\arg z| \le \beta(1 - r)$.

Suppose further that for r *sufficiently near* 1, *we have*

$$|f(re^{i\theta})| \le |f(re^{i\beta(1-r)})| \quad \textit{for} \quad \beta(1 - r) \le |\theta| \le \pi. \tag{3.3}$$

Then $f(z)$ *is admissible in* $|z| < 1$.

LEMMA 4. ([2, 4]) *Let* $f(z) = \sum_{n\ge 0} a_n z^n$ *be an admissible function in* $|z| < R$ *and let the function* $a(r)$ *be positive increasing in some range* $r_0 \le r < R$. *Let* r_n *be the positive real root of the equation* $a(r_n) = n$ *for each* $n = 1, 2, 3, \ldots$ *such that* $r_0 < r_n < R$. *Then*

$$a_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}}.$$

LEMMA 5. *Let*

$$f(z) = \exp(\frac{z}{1 - z}) = \sum_{n\ge 0} \frac{a_n}{n!} z^n.$$

Then $f(z)$ *is admissible in* $|z| < 1$.

PROOF. Since $f(z) = \exp(\frac{z}{1-z}) = e^{-1} \cdot \exp(\frac{1}{1-z})$, it suffices to show that $g(z) = \exp(\frac{1}{1-z})$ is admissible in $|z| < 1$ [2]. To do this, we apply Lemma 3 for $g(z)$.

We note that $g(z)$ is regular in $|z| < 1$ and that $g(r)$ is positive for $0 < r < 1$. Let us take $\alpha = 1$, β any number in between 0 and 1, and $C(r) = 1$ for $0 < r < 1$. Then, clearly, the conditions (3.1) and (3.2) are satisfied.

We want to check the condition (3.3). Since

$$\left|\frac{g(re^{i\theta})}{g(re^{i\beta(1-r)})}\right| = \left|\frac{\exp(1/(1-re^{i\theta}))}{\exp(1/(1-re^{i\beta(1-r)}))}\right|$$
$$= \left|\exp\left(\frac{1}{1-re^{i\theta}} - \frac{1}{1-re^{i\beta(1-r)}}\right)\right|,$$

it is enough to show that

$$\Re\left(\frac{1}{1-re^{i\theta}} - \frac{1}{1-re^{i\beta(1-r)}}\right) \le 0$$

for $\beta(1-r) \le |\theta| \le \pi$ and r sufficiently near 1. Actually, we have

$$\Re\left(\frac{1}{1-re^{i\theta}} - \frac{1}{1-re^{i\beta(1-r)}}\right)$$
$$= \frac{r(1-r)(1+r)(\cos\theta - \cos\beta(1-r))}{(1-2r\cos\theta + r^2)(1-2r\cos\beta(1-r)+r^2)} \le 0$$

for $\beta(1-r) \le |\theta| \le \pi$ and r sufficiently near 1. Therefore, the condition (3.3) is satisfied. □

Now we want to state an asymptotics for the coefficient a_n in $f(z) = \exp(\frac{z}{1-z}) = \sum_{n\ge 0} \frac{a_n}{n!} z^n$.

THEOREM 6. *Let a_n be the number of labeled digraphs of order n whose weak components are transitive tournaments. Then*

$$a_n \sim \frac{2^n n^{2n} \exp(-n + \frac{1}{2}\sqrt{4n+1} - \frac{1}{2})}{(2n+1-\sqrt{4n+1})^n (4n+1)^{1/4}}.$$

PROOF. Since we already showed in Lemma 5 that $f(z)$ is an admissible function in $|z| < 1$, we may apply Lemma 4 for $f(z)$.

First, we note that $f(z)$ is regular in $|z| < 1$, and have

$$a(r) = \frac{r}{(1-r)^2},$$

$$b(r) = \frac{r(1+r)}{(1-r)^3}.$$

Since $a(r)$ is positive increasing for $-1 \le r < 1$, we let r_n be the solution of the equation $a(r_n) = n$ for positive integer n such that $0 < r_n < 1$. In this case, the equation is

$$\frac{r_n}{(1-r_n)^2} = n$$

and thus our solution is

$$r_n = 1 + \frac{1}{2n} - \sqrt{\frac{1}{n} + \frac{1}{4n^2}}.$$

Therefore, we have

$$f(r_n) = \exp\left(\frac{1}{2}\sqrt{4n+1} - \frac{1}{2}\right)$$

and

$$b(r_n) = n\frac{4n+1-\sqrt{4n+1}}{\sqrt{4n+1}-1} \sim n\sqrt{4n+1}.$$

Using the formula in Lemma 4, we have

$$\frac{a_n}{n!} \sim \frac{(2n)^n \exp\left(\frac{1}{2}\sqrt{4n+1} - \frac{1}{2}\right)}{(2n+1-\sqrt{4n+1})^n\sqrt{2\pi n\sqrt{4n+1}}}.$$

Finally, using Stirling's formula, we have

$$a_n \sim \frac{2^n n^{2n} \exp\left(-n + \frac{1}{2}\sqrt{4n+1} - \frac{1}{2}\right)}{(2n+1-\sqrt{4n+1})^n(4n+1)^{1/4}}. \tag{3.4}$$

□

The last column of the following table shows the speed of convergence for our estimator.

n	(3.4)	a_n	$(3.4)/a_n$
200	4.9013×10^{384}	4.8376×10^{384}	1.0132
400	2.8943×10^{883}	2.8676×10^{883}	1.0093
600	3.3829×10^{1426}	3.3573×10^{1426}	1.0076
800	3.2480×10^{1998}	3.2267×10^{1998}	1.0066
1000	1.1381×10^{2592}	1.1314×10^{2592}	1.0059

References

[1] F. Harary and E. M. Palmer, *Graphical Enumeration*, Academic, New York, 1973.

[2] W. K. Hayman, *A Generalisation of Stirling's Formula*, J. für die reine und angewandte Mathematik **196** (1956), 67–95.

[3] J. W. Moon, *Topics on Tournaments*, Holt, New York, 1968.

[4] H. S. Wilf, *Generatingfunctionology*, Academic, San Diego, 1990.

Department of Mathematics
University of Seoul
Seoul 130–743, Korea
E-mail: jsong@uoscc.uos.ac.kr
chlee@uoscc.uos.ac.kr